**Department of Mathematics**
**IIT Kharagpur**
**MA20103 Partial Differential Equations**
**Mid-Autumn 2016, Time: 2 hrs.; Max. Marks: 30, Number of students: 424**

**Note**: Please follow the notations and instructions carefully. Answer all the questions. prime ($'$): denotes derivative with respect to $x$. No queries will be entertained during the examination.

1. **[4 marks]** Find two linearly independent power series solutions of the ODE: $(1-x^2)y'' - xy' + 4y = 0$ about $x = 0$.

2. **[2 marks]** Express the polynomial $3x^4 + 6x^2 - 2$ in terms of Legendre polynomials $P_n(x)$.

3. **[2 marks]** Let $f(x)$ be a polynomial of degree $n \geq 1$ such that
$$\int_{-1}^{1} x^k f(x)dx = 0,$$
for $k = 0, 1, \ldots, (n-1)$. Using orthogonality of Legendre polynomials, show that $f(x) = cP_n(x)$ for some constant $c$.

4. **[2 marks]** Using the recurrence relation $(2n-1)xP_{n-1}(x) = nP_n(x) + (n-1)P_{n-2}(x)$, evaluate
$$\int_{-1}^{1} x^2 P_{n+1}(x)P_{n-1}(x)dx.$$
Express the final answer in the form $\frac{f(n)}{g(n)}$ for some functions $f(n)$ and $g(n)$.

5. **[6 marks]** Using the series representation of $n^{th}$ order Bessel functions $J_n(x)$ given by
$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k}$$
(a) find the value of $Lim_{x\to 0}\frac{J_n(x)}{x^n}$, (b). Evaluate $x^{-n}J_{n+1}(x) + \frac{d}{dx}\left(x^{-n}J_n(x)\right)$, and (c) Compute $J_{-\frac{1}{2}}(x)$ and express in terms of trigonometric functions.

6. **[4 marks]** Form a second order PDE by eliminating the arbitrary functions $f$ and $g$ from the relation $z = xf(x+y) + g(x+y)$, ( $z$: dependent variable; $x, y$: independent variables).

7. **[5 marks]** Consider the linear second order ODE with variable coefficients given by $xy'' + 4y' - xy = 0$, for which we seek series solution using Frobenius method, about $x = 0$. Let $f(r) = 0$ denote the indicial equation whose roots are $r_1, r_2$ such that $r_1 > r_2$. Then (a). Compute $r_1$ and $r_2$. (b). Derive the recurrence relation for the coefficients of the series solution corresponding to $r_1$ and hence obtain the series solution. (c). Derive the recurrence relation for the coefficients of the series solution corresponding to $r_2$ and obtain the series solution. (d). Write down explicitly the general solution as a linear combination of two independent solutions.

8. **[5 marks]** Use Lagrange's method to find the general solution of $x(y-z)\frac{\partial z}{\partial x} + y(x+z)\frac{\partial z}{\partial y} = (x+y)z$. Hence, find a particular integral passing through $z = x^2 - 1$ on $y = x$. ( $z$: dependent variable; $x, y$: independent variables).

[End of QP]