

INDIAN INSTITUTE OF TECHNOLOGY HYDERABAD

DEPARTMENT OF MATHEMATICS

ASSIGNMENT 7

MA 4070 : Elements of Group and Rings

Marks: 25

1. In a ring with unity, the set of all non-units form an ideal if and only if R has a unique maximal ideal. [2]
2. Suppose I, J are ideals of R . Show that the map $\eta_{IJ} : R/I \rightarrow R/J$ is well-defined iff $I \subseteq J$. If $R = \mathbb{Z}$, $I = m\mathbb{Z}$, and $J = n\mathbb{Z}$, then interpret the above in terms of m and n . [2]
3. Produce a ring R and an irreducible element of R , which is not a prime. (Caution: Don't give the example from the book.) [2]
4. R is a commutative integral domain with unity. Show that a is a prime element of R iff (a) is a prime ideal of R . [2]
5. Produce a ring R where gcd and lcm of two numbers does not exist. Even if gcd and lcm exists in R , does the following relation hold always, i.e.,
$$\gcd(a, b)\text{lcm}(a, b) = ab, \text{ for all } a, b \in R.$$
 [2]
6. Show that $\mathbb{Z}[i\sqrt{3}]$ is not an Euclidean domain wrt $d(r) = r\bar{r}$? Tell me where the argument fails if you try to argue as in showing that $\mathbb{Z}[i]$ is an Euclidean domain. Can $\mathbb{Z}[i\sqrt{3}]$ be an Euclidean domain for any other d ? [3]
7. Write 3 different kind of example of PID's which are not Euclidean domains. [3]
8. Given any $n \in \mathbb{N}$, produce a ring which has exactly n maximal ideals. [3]
9. Without using any theorem, show that $\mathbb{Z}[i][X]$ is not a PID, i.e., produce an ideal which is not a principal ideal. [3]
10. For any ring R with unity, is $R[X_1, X_2]$ a PID? How about $R[X_1, \dots, X_n]$? If you further assume that R is a field, then what can you say about the above statements? [3]