

INDIAN INSTITUTE OF TECHNOLOGY HYDERABAD

DEPARTMENT OF MATHEMATICS

ASSIGNMENT 6

MA 4010 : Real Analysis

Max Marks: 25

1. Suppose f is a non-negative continuous function on $[a, b]$ such that $\int_a^b f(x)dx = 0$. Then prove that $f(x) = 0$ for all $x \in [a, b]$. [2]

2. If f is continuous on $[a, b]$, then there exists $c \in [a, b]$ such that [2]

$$\int_a^b f(x)dx = f(c)(b - a).$$

3. Prove the following series are convergent by using the theory of integration:

(a) $\sum_{n=1}^{\infty} \frac{(n+1)^{\frac{1}{3}} - n^{\frac{1}{3}}}{n^{\frac{2}{3}}(n+1)^{\frac{1}{3}}}.$ [1]

(b) $\sum_{n=1}^{\infty} \frac{\sin \frac{1}{n}}{n}.$ [1]

4. Suppose f is a bounded real function on $[-1, 1]$, and $f^2 \in \mathcal{R}$ on $[-1, 1]$. Does it follow that $f \in \mathcal{R}$? Does the answer change if we assume that $f^3 \in \mathcal{R}$? [2]

5. Suppose f is a bounded real function on $[a, b]$ and continuous except some countable points in $[a, b]$. Then prove that $f \in \mathcal{R}$. [2]

6. Let f be a bounded real valued function on $[0, 1]$ and continuous outside of $\mathcal{C}$, where $\mathcal{C}$ is the Cantor set. Then prove that $f \in \mathcal{R}$. [3]

Definition 0.1. Suppose $f \in \mathcal{R}$ on $[a, b]$ for every $b > a$ where a is fixed. Define $\int_a^{\infty} f(x)dx = \lim_{b \rightarrow \infty} \int_a^b f(x)dx$ if the limit exists, then we say the integral on the left converges.

7. Assume that $f(x) \geq 0$ and that f is a monotonically decreasing function on $[1, \infty)$. Prove that $\int_1^{\infty} f(x)dx$ converges if and only if $\sum_{n=1}^{\infty} f(n)$ converges. [3]

8. If $f : [0, 1] \rightarrow \mathbb{R}$ is a continuously differentiable function and $f(0) = 0$, then prove that the series $\sum_{n \in \mathbb{N}} f(\frac{1}{n}) \sin n$ converges. [3]

9. If $f : [a, b] \rightarrow \mathbb{R}$ is a continuous function and $\int_x^b f(y)dy = \int_a^x f(y)dy$ for all $x \in [a, b]$, then $f = 0$. [3]

10. Prove that there does not exist a continuously differentiable function on $[0, 2]$ such that $f(0) = -1, f(2) = 4$, and $f'(x) \leq 2$ for $0 \leq x \leq 2$. [3]