

INDIAN INSTITUTE OF TECHNOLOGY HYDERABAD

DEPARTMENT OF MATHEMATICS

ASSIGNMENT 3

MA 4010 : Real Analysis

Max Marks: 45

Through out, let X, Y denote metric spaces. **Answer all parts of the same question at the same place.**

1. Take $s_1 = 2$, and define $s_{n+1} = \sqrt{2 + \sqrt{s_n}}$ for $\forall n \geq 1$. Prove that the sequence $\{s_n\}$ converges. [2]
2. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function such that $f(x+p) = f(x)$, $\forall x \in \mathbb{R}$ and some fixed $p > 0$. Prove that f is bounded and uniformly continuous. [2]
3. For any two real sequences $\{a_n\}, \{b_n\}$, prove that [2]

$$\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n.$$

4. Let f be a continuous real valued function on X . Prove that $Z(f)$ is a closed set, where $Z(f) := \{p \in X : f(p) = 0\}$. [2]
5. If $f : X \rightarrow Y$ is a uniformly continuous function, then prove that every Cauchy sequence in X is mapped to a Cauchy sequence in Y . [2]
6. Suppose $f : [0, 1] \rightarrow [0, 1]$ is a continuous mapping. Prove that $f(x) = x$ for at least one $x \in [0, 1]$. [2]
7. Let $\{x_n\}$ be a sequence in $\mathbb{R}$ such that it tend to $+\infty$ or $-\infty$ and let $\{y_n\}$ be such that $\lim x_n y_n \in \mathbb{R}$. Then, show that y_n tends to 0 as $n \rightarrow \infty$. [2]
8. If f is defined on $E \subseteq \mathbb{R}$, the graph of f is defined by $G(f) = \{(x, f(x)) : x \in E\}$. If E is compact, then show that f is continuous if and only if $G(f)$ is compact. [3]
9. Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X , and some subsequence $\{p_{n_i}\}$ converges to a point $p \in X$. Prove that the sequence $\{p_n\}$ converges to p . [3]
10. Investigate the convergence or divergence of $\sum a_n$, if
 - (a) $a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n}$. [1]
 - (b) $a_n = \frac{1}{1+z^n}$, where $z \in \mathbb{C}^*$. [1]
 - (c) $a_n = \frac{1}{n(\log n)^c}$, where $c > 1$. [1]
11. Find the radius of the convergence of the following power series:
 - (a) $\sum \frac{2^n}{n!} z^n$. [1]
 - (b) $\sum \frac{2^n}{n^2} z^n$. [1]
12. If $\sum a_n$ with $a_n > 0$ is convergent and if $b_n = \frac{a_1 + a_2 + \dots + a_n}{n}$ then show that $\sum b_n$ is always divergent. [3]
13. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function that does not takes on any of its values twice. If $f(0) < f(1)$, then f is a strictly increasing function on $[0, 1]$. [3]
14. Show that the function $f(x) = \frac{1}{1+x^2}$, for $x \in \mathbb{R}$, is uniformly continuous. [3]
15. If f is a continuous mapping of X into Y , then show that $f(\overline{E}) \subseteq \overline{f(E)}$. Give an example such that $f(\overline{E})$ is a proper subset of $\overline{f(E)}$. [3]

16. Show that a real valued function f is not uniformly continuous on $A \subseteq \mathbb{R}$ iff there exist $\epsilon_0 > 0$ and two sequence $\{x_n\}$ and $\{y_n\}$ in A such that $\lim_{n \rightarrow \infty} (x_n - y_n) = 0$ and $|f(x_n) - f(y_n)| \geq \epsilon_0$ for infinitely many n . [4]
17. Suppose $A \subseteq \mathbb{R}$ and $f, g : A \rightarrow \mathbb{R}$ are uniformly continuous functions. Prove or disprove the following:
- (a) $f + g$ is uniformly continuous. [2]
- (b) $f \circ g$ is uniformly continuous, here we assume $A = \mathbb{R}$. [2]