

INDIAN INSTITUTE OF TECHNOLOGY HYDERABAD

DEPARTMENT OF MATHEMATICS

ASSIGNMENT 2

MA 4010 : Real Analysis

Max Marks: 30

The Euclidean metric on $\mathbb{R}^n$ is given by $d(\bar{x}, \bar{y}) = \sqrt{\sum_{i=1}^n |x_i - y_i|^2}$ for $\bar{x} = (x_1, \dots, x_n) \in \mathbb{R}^n, \bar{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$. **Answer all parts of the same question at the same place.**

1. Prove or disprove the following statements:
 - (a) Let $\mathbf{A}$ be any set and $f : \mathbf{A} \rightarrow \mathbb{N}$ is onto. Then $\mathbf{A}$ is countable. [1]
 - (b) Let $\mathcal{K} = \{\text{functions } f : \mathbb{N} \rightarrow \{0, 1\}\}$. Then $\mathcal{K}$ is countable. [2]
2. Find out the limit points of the following sets:
 - (a) $E_1 = \left\{ \frac{(-1)^n}{n} : n \in \mathbb{N} \right\}$. [1]
 - (b) $E_2 = \left\{ \frac{1}{k^{1+\frac{1}{n}}} : n, k \in \mathbb{N} \right\}$. [1]
 - (c) $E_3 = \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{Q} \text{ and } y \in \mathbb{Q}^c\}$. [2]
3. Determine whether the following sets are compact or not, and explain the details in either case.
 - (a) $F_1 = \left\{ \lambda x + (1 - \lambda)y : \lambda \in \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\} \right\}$, where $x, y \in \mathbb{R}$ are fixed. [1]
 - (b) $F_2 = \{2 + 3x + x^2 : x \in [0, 1]\}$. [2]
4. Which of the following d_i 's are metrics on $\mathbb{R}$? For $x, y \in \mathbb{R}$,
 - (a) $d_1(x, y) = |x^2 - y^2|$. [1]
 - (b) $d_2(x, y) = \min\{|x - y|, 1\}$. [2]
5. Let $\{A_n\}$ be a sequence of open sets in $\mathbb{R}$ such that A_n^c are bounded. If $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, then prove that there exist $x \in \mathbb{R}$ such that $x \notin A_n$ for all n . [2]
6. Find out the closures and interiors of the following sets with respect to the Euclidean topology.
 - (a) $\mathbf{A}_1 = [0, 1) \cup (1, 2]$. [1]
 - (b) $\mathbf{A}_2 = \{(x, y) \in \mathbb{R}^2 : |x|^2 + |y|^2 < 1\}$. [2]
 - (c) $\mathbf{A}_3 = \{(x, 0) \in \mathbb{R}^2 : x \in \mathbb{R}\}$. [2]
7. Let $\mathbf{X}$ be an infinite set. Construct a metric on $\mathbf{X}$ such that $\mathbf{X}$ is bounded, but $\mathbf{X}$ is not compact. [3]
8. Is every point of an open set $E \subseteq \mathbb{R}^2$ is a limit point of E ? Answer the same question for closed sets, as well. [3]
9. Prove that for every real $x > 0$ and every integer $n > 0$ there exists a unique real number $y \in \mathbb{R}$ such that $y^n = x$. [4]