

13) $A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} = I + A$

∴ $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is $I + A$ for $\lambda = -1$
 eigen values of A are $8, -1, -1$.

algebraic multiplicity of 8 is 1

and for -1 it is 2 .

for geometric multiplicity:

$$(A - 8I) = \begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix}$$

Null space of $A - 8I$ is $t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$; $t \in \mathbb{R}$

It's dimension is 1 ; So geometric multiplicity 8 is 1

For $\lambda = -1$;

$$A + I = \begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 5 & 0 & 5 \\ 4 & 2 & 4 \\ 5 & 5 & 5 \end{pmatrix} = A$$

Null Space of $A + I$ is $p \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

1 is a scalar multiple of t , $t \in \mathbb{R}$.

Null Space dimension = 2; b.c.

$\therefore$ geometric multiplicity for $\lambda = -1$; is 2

$\therefore$ geometric multiplicity

$$\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 5 & 0 & 5 \\ 4 & 2 & 4 \\ 5 & 5 & 5 \end{pmatrix} = (I\mathbb{R} - A)$$

Null Space of $I\mathbb{R} - A$ is $\begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $t \in \mathbb{R}$

It's dimension is 2 as given

1 is a scalar multiple

$$2) \quad A = \frac{80}{25} \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} = d; \quad \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} = A$$

eigen values are 3, 2.

Corresponding eigen vectors are $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$$P = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} -2 & 2 \\ 2 & -1 \end{pmatrix}$$

$$A = P D P^{-1} \quad D = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

$$A^5 = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3^5 & 0 \\ 0 & 2^5 \end{pmatrix} \begin{pmatrix} -2 & 2 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 243 & 0 \\ 0 & 32 \end{pmatrix} \begin{pmatrix} -2 & 2 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1215 & 32 \\ 243 & 32 \end{pmatrix} \begin{pmatrix} -2 & 2 \\ 2 & -1 \end{pmatrix}$$

$$A^5 = \begin{pmatrix} -179 & 211 \\ -423 & 454 \end{pmatrix}$$

$$(3) \quad \frac{dx}{dt} = 4x - 2y \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad x(0) = 3 \quad y(0) = 5$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \Rightarrow (A - \lambda I) \mathbf{v} = \mathbf{0}$$

$$\begin{pmatrix} 4 \\ 1 \end{pmatrix} \cdot x \Rightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

is of form $\dot{\mathbf{v}} = A\mathbf{v}$

Then we know that the solution for $\mathbf{v}$ is:

$$\mathbf{v}(t) = e^{At} \cdot \mathbf{v}(0)$$

if A is diagonalisable then e^{At} can be written as:

$$(i.e. S^{-1}AS = D \Rightarrow A = SDS^{-1})$$

$$e^{At} = Se^{Dt}S^{-1}$$

$$\Rightarrow \mathbf{v}(t) = Se^{Dt}S^{-1} \mathbf{v}(0)$$

Finding eigen values and eigen vectors:

$$A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \Rightarrow |A - \lambda I| = 0$$

$$\Rightarrow \lambda = 2, \lambda = 3$$

$$N(A - 2I) \Rightarrow \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \text{eigen vector} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{for } \lambda = 3$$

$$\text{For } \lambda = 3, \quad N(A - 3I) \Rightarrow \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \text{eigen vector} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{for } \lambda = 3$$

$$S = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\Rightarrow S^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\underline{v}(t) = S e^{Dt} S^{-1} \underline{v}(0) \quad \text{where } \underline{v}(0) = \begin{Bmatrix} 3 \\ 5 \end{Bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{Bmatrix} 3 \\ 5 \end{Bmatrix}$$

$$= \begin{bmatrix} e^{2t} & 2e^{3t} \\ e^{2t} & e^{3t} \end{bmatrix} \begin{Bmatrix} 7 \\ -2 \end{Bmatrix}$$

$$\Rightarrow \underline{v}(t) = \begin{Bmatrix} 7e^{2t} - 4e^{3t} \\ 7e^{2t} - 2e^{3t} \end{Bmatrix} \Rightarrow \begin{cases} x(t) \\ y(t) \end{cases} = \begin{cases} 7e^{2t} - 4e^{3t} \\ 7e^{2t} - 2e^{3t} \end{cases}$$

$$x(t) = 7e^{2t} - 4e^{3t}$$

$$\text{and } y(t) = 7e^{2t} - 2e^{3t}$$

is the solution

$$w) (i). \langle u+v, u-v \rangle =$$

$$= \langle u, u-v \rangle + \langle v, u-v \rangle$$

$$= \langle u, u \rangle - \langle u, v \rangle + \langle v, u \rangle - \langle v, v \rangle$$

$$= \|u\|^2 - \langle u, v \rangle + \langle u, v \rangle - \|v\|^2$$

$$= \|u\|^2 - \|v\|^2$$

$$(ii). \|u+v\|^2 + \|u-v\|^2 = \langle u+v, u+v \rangle + \langle u-v, u-v \rangle$$

$$= \|u\|^2 + \|v\|^2 + \langle u, v \rangle + \langle v, u \rangle + \|u\|^2 + \|v\|^2 - \langle u, v \rangle - \langle v, u \rangle$$

$$= 2(\|u\|^2 + \|v\|^2).$$

⑤ Span $\{ (1, 1, 1, 1), (-1, 4, 4, 1), (4, -2, 2, 0) \}$
 $x_1 \quad x_2 \quad x_3$

$$u_1 = \frac{x_1}{\|x_1\|} = \frac{1}{2} (1, 1, 1, 1) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)$$

$$y_2 = x_2 - (x_2 \cdot u_1) u_1 =$$

$$= (-1, 4, 4, 1) - \langle (-1, 4, 4, 1), \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \rangle \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 3/2 \\ 3/2 \\ 3/2 \\ 3/2 \end{pmatrix} = \begin{pmatrix} -5/2 \\ 5/2 \\ 5/2 \\ 5/2 \end{pmatrix}$$

$$u_2 = \frac{y_2}{\|y_2\|} = \frac{1}{5} \left(-\frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2} \right)$$

$$\Rightarrow u_2 = \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)$$

$$y_3 = x_3 - (x_3 \cdot u_1) u_1 - (x_3 \cdot u_2) u_2$$

$$= (4, -2, 2, 0) - \langle (4, -2, 2, 0), \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \rangle \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)$$

$$= (4, -2, 2, 0) - \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)$$

$$\begin{aligned} \|v\| + \|w\| &= \langle v, v+w \rangle + \langle w, v+w \rangle \\ &= (4, -2, 2, 0) - (1, 1, 1, 1) = (1, -1, -1, 1) \\ &= \langle v+w, v+w \rangle = (2, -2, 2, -2) \end{aligned}$$

$$u_3 = \frac{y_3}{\|y_3\|} \Rightarrow u_3 = \frac{1}{4} (2, -2, 2, -2)^T = \frac{1}{4} \begin{pmatrix} 2 \\ -2 \\ 2 \\ -2 \end{pmatrix}$$

$$\Rightarrow u_3 = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right)^T$$

The set $\{u_1, u_2, u_3\}$ forms the orthonormal basis.

$$\Rightarrow \text{Orthonormal basis} = \left\{ \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix} \right\}$$

⑥ a) If u, v are orthogonal vectors, then

$$\langle u, v \rangle = 0 \Rightarrow 2 \langle u, v \rangle = 0$$

Adding $\|u\|^2 + \|v\|^2$ on both sides:

$$\|u\|^2 + \|v\|^2 + 2 \langle u, v \rangle = \|u\|^2 + \|v\|^2$$

$$\Rightarrow \langle u, u \rangle + \langle v, v \rangle + 2 \langle u, v \rangle = \|u\|^2 + \|v\|^2$$

$$\Rightarrow \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle = \|u\|^2 + \|v\|^2$$

$$\Rightarrow \langle u, u+v \rangle + \langle v+u, v \rangle = \|u\|^2 + \|v\|^2$$

$$\Rightarrow \langle u+v, u \rangle + \langle u+v, v \rangle = \|u\|^2 + \|v\|^2$$

$$\Rightarrow \langle u+v, u+v \rangle = \|u\|^2 + \|v\|^2$$

Orthogonal projection

$$\Rightarrow \|u+v\|^2 = \|u\|^2 + \|v\|^2 \quad (i) \quad (2)$$

Similarly, we get:

$$\|u-v\|^2 = \|u\|^2 + \|v\|^2 \quad (ii)$$

From (i) & (ii), $\|u+v\|^2 = \|u-v\|^2$

$$\Rightarrow \|u+v\| = \|u-v\|$$

Thus, u, v are orthogonal $\Rightarrow \|u+v\| = \|u-v\| \quad (1)$

b) If $\|u+v\| = \|u-v\|$ then

$$\Rightarrow \|u+v\|^2 = \|u-v\|^2 \Rightarrow \frac{1}{2v} = \frac{1}{2v} = 1$$

$$\Rightarrow \|u\|^2 + \|v\|^2 + 2\langle u, v \rangle = \|u\|^2 + \|v\|^2 - 2\langle u, v \rangle$$

$$\Rightarrow 4\langle u, v \rangle = 0 \Rightarrow \langle u, v \rangle = 0 \Rightarrow u, v \text{ are orthogonal}$$

Thus $\|u+v\| = \|u-v\| \Rightarrow u, v$ are orthogonal $\quad (2)$

From statements (1) & (2)

$$u, v \text{ are orthogonal} \Leftrightarrow \|u+v\| = \|u-v\|$$

$$\frac{1}{2v} = \frac{1}{2v} = 1$$

$$(1, 1, 1, 0) \frac{1}{2v} = 1$$

$$\{u, v\} = \text{orthogonal basis}$$

$$7) A = \begin{pmatrix} a & 1 \\ -2 & d \end{pmatrix}; B = \begin{pmatrix} \frac{114}{25} & \frac{48}{25} \\ \frac{48}{25} & \frac{86}{25} \end{pmatrix}$$

$\therefore$ eigen values of B are $2, 6$.

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} a - \lambda & 1 \\ -2 & d - \lambda \end{vmatrix} = 0$$

$$ad - a\lambda - d\lambda + \lambda^2 + 2 = 0$$

$2, 6$ satisfy above equation.

$$ad - 2a - 2d + 6 = 0 \quad \text{--- (1)}$$

$$ad - 6a - 6d + 38 = 0 \quad \text{--- (2)}$$

from eq - (2);

$$a(d - 6) = 6d - 38$$

$$a = \frac{6d - 38}{d - 6}$$

Substitute $a = \frac{6d - 38}{d - 6}$ in eq (1).

and solving for d , gives $d = 4 \pm 6$

$$\Rightarrow (a, d) = (4 + 6, 4 - 6) \text{ or } (4 - 6, 4 + 6)$$

⑧

$$\text{span} \left\{ \underset{x_1}{(0, 1, 1, 1)}, \underset{x_2}{(-1, 1, 0, 1)} \right\}$$

$$\text{vector } v = (2, 0, -1, 3)$$

$$\langle x_1, x_2 \rangle = 0 + 1 + 0 + 1 = 2 \neq 0$$

x_1, x_2 are not orthogonal.

① — Hence we need to determine orthonormal basis for that subspace.

$$u_1 = \frac{x_1}{\|x_1\|} = \frac{1}{\sqrt{3}} (0, 1, 1, 1)$$

$$y_2 = x_2 - (x_2 \cdot u_1) u_1$$

$$= (-1, 1, 0, 1) - \langle (-1, 1, 0, 1), \frac{1}{\sqrt{3}} (0, 1, 1, 1) \rangle \cdot \frac{1}{\sqrt{3}} (0, 1, 1, 1)$$

$$= (-1, 1, 0, 1) - \frac{2}{3} (0, 1, 1, 1)$$

$$= (-1, \frac{1}{3}, -\frac{2}{3}, \frac{1}{3})$$

$$u_2 = \frac{1}{\|y_2\|} y_2$$

$$\Rightarrow u_2 = \frac{1}{\sqrt{15}} (-3, 1, -2, 1)$$

$$\text{orthonormal basis} = \{u_1, u_2\}$$

$$= \left\{ \frac{1}{\sqrt{3}} (0, 1, 1, 1), \frac{1}{\sqrt{15}} (-3, 1, -2, 1) \right\}$$

Orthogonal projection of vector 'v' (2, 0, -1, 3) on this plane is

$$= \langle v, u_1 \rangle u_1 + \langle v, u_2 \rangle u_2 - (0, 1, 1, 1)$$

$$= \langle (2, 0, -1, 3), \frac{1}{\sqrt{3}} (0, 1, 1, 1) \rangle \frac{1}{\sqrt{3}} (0, 1, 1, 1) +$$

$$\langle (2, 0, -1, 3), \frac{1}{\sqrt{15}} (-3, 1, -2, 1) \rangle \frac{1}{\sqrt{15}} (-3, 1, -2, 1)$$

$$= \frac{2}{3} (0, 1, 1, 1) - \frac{1}{15} (-3, 1, -2, 1)$$

$$= \frac{1}{15} \left(\frac{3}{15}, \frac{9}{15}, \frac{12}{15}, \frac{9}{15} \right) - (1, 1, 0) =$$

$$= \frac{1}{5} (1, 3, 4, 3) - (1, 1, 0)$$

$$(1, 1, 0) - (1, 0, 0) = (1, 1, 0)$$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

column vectors $\Rightarrow$

$$x_1 = (1, 0, -1), x_2 = (2, 1, 0), x_3 = (0, 2, 1)$$

Gram-Schmidt process $\Rightarrow [u] = 0$

$$y_1 = x_1 = (1, 0, -1)$$

$$u_1 = \frac{y_1}{\|y_1\|} = \frac{1}{\sqrt{2}} (1, 0, -1)$$

$$y_2 = x_2 - (x_2 \cdot u_1) u_1$$