

$$1) \text{ (i). } T((x_1, x_2)) = x_1^2 + x_2^2.$$

$$T(x+y) = T((x_1, x_2) + (y_1, y_2))$$

$$= T((x_1 + y_1, x_2 + y_2))$$

$$= (x_1 + y_1)^2 + (x_2 + y_2)^2$$

$$= x_1^2 + y_1^2 + 2x_1y_1 + x_2^2 + y_2^2 + 2x_2y_2.$$

$$T(x) + T(y) = T((x_1, x_2)) + T((y_1, y_2))$$

$$= x_1^2 + x_2^2 + y_1^2 + y_2^2.$$

$$T(x+y) \neq T(x) + T(y).$$

T is not linear.

$$\text{(ii). } T((z_1, z_2)) = (z_1 + z_2, z_1 - 2z_2).$$

$$T((z_1, z_2) + (u_1, u_2)) = T((z_1 + u_1, z_2 + u_2))$$

$$= (z_1 + u_1 + z_2 + u_2, z_1 + u_1 - 2z_2 - 2u_2)$$

$$= (z_1 + z_2, z_1 - 2z_2) + (u_1 + u_2, u_1 - 2u_2)$$

$$= T(z_1, z_2) + T(u_1, u_2)$$

$$\begin{aligned}
 T(C(z_1, z_2)) &= T(Cz_1, Cz_2) \\
 &= (Cz_1 + Cz_2, Cz_1 - 2Cz_2) \\
 &= C(z_1 + z_2, z_1 - 2z_2) \\
 &= CT(z_1, z_2).
 \end{aligned}$$

$\therefore T$ is linear transformation.

2) given: $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$; $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ 3x_2 \\ 4x_1 + 5x_2 \end{bmatrix}$

$$(T(x)) = Ax \quad \forall x \in \mathbb{R}^2;$$

Standard basis vectors for $\mathbb{R}^2$:

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$x = x_1 e_1 + x_2 e_2$$

$$T(x) = T(x_1 e_1 + x_2 e_2)$$

$$= x_1 T(e_1) + x_2 T(e_2)$$

$$= \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

for $T(x)$ to be 3×1 , A must be

3×2 .

$$A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}$$

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$$

$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 1 & -1 \\ 0 & 3 \\ 4 & 5 \end{bmatrix}$$

(1)

$$A = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} \right\}$$

$$B = \left\{ \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \right\}$$

matrix for change of basis from B to A.

$$M_{B \rightarrow A} = \begin{bmatrix} (b_1)_A & (b_2)_A & (b_3)_A \end{bmatrix}$$

$$(b_1)_A \Rightarrow \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & -3 & 1 \end{array} \right)$$

$$R_1 \rightarrow R_1 - R_2 \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & -3 & 1 \end{array} \right)$$

$$R_3 \rightarrow R_3 - R_1 \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -3 & -1 \end{array} \right)$$

$$a_3 = 1/3, \quad a_1 = 2, \quad a_2 = 1/3.$$

$$(b_1)_A = \begin{pmatrix} 2 \\ 1/3 \\ 1/3 \end{pmatrix}$$

similarly,

$$(b_2)_A = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$$

$\Rightarrow$ solving for a_1, a_2, a_3 :

$$(b_2)_A = \begin{pmatrix} 0 \\ 2/3 \\ -1/3 \end{pmatrix}$$

and

$$(b_3)_A = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$$

solving for a_1, a_2, a_3 :

$$(b_3)_A = \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix}$$

$$\Rightarrow M_{B \rightarrow A} = \begin{bmatrix} 2 & 0 & -3 \\ 1/3 & 2/3 & 4 \\ 1/3 & -1/3 & -1 \end{bmatrix}$$

is the change of basis from $B \rightarrow A$.

④

$$A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 2 & 3 & -1 \\ 4 & 1 & -2 & 1 \\ 1 & 2 & -1 & 0 \end{bmatrix}; B = \begin{bmatrix} -1 & 2 & 4 & 2 \\ 3 & 5 & 1 & 0 \\ -2 & 0 & -2 & 0 \\ 1 & 3 & 2 & -1 \end{bmatrix}$$

$$\text{trace}(A) = 1 + 2 - 2 + 0 = 1$$

$$\text{trace}(B) = -1 + 5 - 2 - 1 = 1 \Rightarrow \text{tr}(A) = \text{tr}(B).$$

$$\det(A) = -401 (\neq 0) \Rightarrow \text{rank}(A) = 4.$$

$$\det(B) = -102 (\neq 0) \Rightarrow \text{rank}(B) = 4.$$

$$\begin{pmatrix} 1 \\ 0 \\ 4 \\ 1 \end{pmatrix} \cdot 1 + \begin{pmatrix} 0 \\ 2 \\ 1 \\ 2 \end{pmatrix} \cdot 0 + \begin{pmatrix} -1 \\ 3 \\ -2 \\ 3 \end{pmatrix} \cdot (-1) + \begin{pmatrix} 2 \\ -1 \\ 1 \\ -1 \end{pmatrix} \cdot 0 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \neq I(4)$$

$$\text{rank}(A) = \text{rank}(B)$$

$$\text{But } \det(A) \neq \det(B).$$

For A, B to be similar matrices, $\det(A) = \det(B)$.

$\Rightarrow$ A, B are not similar matrices.

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 2 & 3 & -1 \\ 4 & 1 & -2 & 1 \\ 1 & 2 & -1 & 0 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 - 4R_1} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 2 & 3 & -1 \\ 0 & 1 & 2 & -7 \\ 1 & 2 & -1 & 0 \end{pmatrix} \xrightarrow{R_4 \rightarrow R_4 - R_1} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 2 & 3 & -1 \\ 0 & 1 & 2 & -7 \\ 0 & 2 & 0 & -2 \end{pmatrix}$$

$$R_2 \leftrightarrow R_3 \Rightarrow \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -7 \\ 0 & 2 & 3 & -1 \\ 0 & 2 & 0 & -2 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 - 2R_2} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -7 \\ 0 & 0 & -1 & 15 \\ 0 & 0 & -2 & 12 \end{pmatrix} \xrightarrow{R_4 \rightarrow R_4 - 2R_3} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -7 \\ 0 & 0 & -1 & 15 \\ 0 & 0 & 0 & -18 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I(4)$$

5.)

$$\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix}$$

$$R_2 \rightarrow R_2 - R_1$$

$$R_3 \rightarrow R_3 - R_1$$

$$= \begin{vmatrix} a+b+c & -c & -b \\ a+b & a+b & -(a+b) \\ a+c & \cancel{a+b} & a+c \\ & -(a+c) & \end{vmatrix}$$

Taking $(a+b)$ and $(c+a)$ common

$$= (a+b)(c+a) \begin{vmatrix} a+b+c & -c & -b \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$C_3 \rightarrow C_3 + C_2$$

$$= (a+b)(c+a) \begin{vmatrix} a+b+c & -c & -(b+c) \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix}$$

Expanding from C_3 .

$$= (a+b)(c+a) [-(b+c)(-1-1)]$$

$$= 2(a+b)(c+a)(b+c)$$

$$6) |A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 & 2 \\ 2 & 3-\lambda & 0 \\ 0 & 4 & 5-\lambda \end{vmatrix} = 0$$

$$-\lambda(\lambda-5)(\lambda-3) - 1 \times (-2(\lambda-5)) + 2 \times 8 = 0$$

Simplifying

$$-(\lambda-6)(\lambda-1)^2 = 0$$

$\Rightarrow$ eigen values are 6, 1.

Case ①: $\lambda = 1$

$$AX = \lambda X$$

$$\begin{pmatrix} 0 & 1 & 2 \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{pmatrix} X = X$$

Simplifying we get $x \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, x \in \mathbb{R}, x \neq 0$

as eigen vectors for eigen value 1.

Case ②: $\lambda = 6$

$$Ax = 6x.$$

Simplifying we get $t \begin{bmatrix} 3 \\ 2 \\ 8 \end{bmatrix}; t \in \mathbb{R}, t \neq 0$

as eigen vectors for eigen value 6.

⑦

$$A = \begin{bmatrix} -1 & 4 \\ 0 & 3 \end{bmatrix}$$

calculating eigen values & vectors:

$$\begin{vmatrix} -1-\lambda & 4 \\ 0 & 3-\lambda \end{vmatrix} = 0 \Rightarrow -(1+\lambda)(3-\lambda) = 0.$$

$$\text{for } \lambda = -1: \begin{bmatrix} 0 & 4 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\Rightarrow x_2 = 0.$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$\text{for } \lambda = 3: \begin{bmatrix} -4 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\Rightarrow -4x_1 + 4x_2 = 0.$$

$$\Rightarrow x_1 = x_2$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$\lambda = -1 \Rightarrow \text{eigen vector: } \begin{pmatrix} 1 \\ 0 \end{pmatrix} = x_1$$

$$\lambda = 3 \Rightarrow \text{eigen vector: } \begin{pmatrix} 1 \\ 1 \end{pmatrix} = x_2$$

Hence Modal matrix $P = \begin{bmatrix} 1 & -1 \\ \epsilon & 0 \end{bmatrix} = A$

$$P = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$$

$$0 \Rightarrow (A - P\lambda) \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & \lambda - 1 \\ \lambda - \epsilon & 0 \end{bmatrix} = 0$$

$$P^{-1} = \frac{1}{\det(P)} \text{Adj } P \Rightarrow P^{-1} = \frac{1}{1} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda & \lambda - 1 \\ \lambda - \epsilon & 0 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$\text{Now } P^{-1}AP = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow \text{Diagonal matrix.}$$

We can observe that $P^{-1}AP$ is a diagonal matrix with eigen values of A as its respective entries.

$$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \text{eigen vector } \leftarrow \lambda = \epsilon$$

8)

$$|A - \lambda I|$$

$$\left| \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right|$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{vmatrix}$$

$$(1-\lambda)(\lambda+1)^2 - 2(-6(\lambda+1)) + 1(-\lambda-13)$$

Simplifying,

$$-(\lambda-3)\lambda(\lambda+4)$$

So $-(\lambda-3)\lambda(\lambda+4)$ is characteristic polynomial.

Its roots are $-4, 0$ and 3 .

9

$$A = \begin{bmatrix} 8 & -6 & 12 \\ -18 & 11 & 18 \\ -6 & -3 & 26 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 8-\lambda & -6 & 12 \\ 0 & 11-\lambda & 18 \\ 0 & -6 & 26-\lambda \end{vmatrix}$$

$$R_1 \rightarrow R_1 - 2R_2$$

$$R_2 \rightarrow R_2 - 3R_3$$

$$= \begin{vmatrix} 20-\lambda & 0 & -40+2\lambda \\ 0 & 20-\lambda & 3\lambda-60 \\ -6 & -3 & 26-\lambda \end{vmatrix}$$

$$= (20-\lambda)(20-\lambda) \begin{vmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ -6 & -3 & 26-\lambda \end{vmatrix}$$

$$= (20-\lambda)^2 \cdot [26-\lambda-9-2(6)]$$

$$= (20-\lambda)^2 (5-\lambda)$$

$$= -(\lambda-20)^2 (\lambda-5)$$

Hence, characteristic polynomial is

$$p(\lambda) = -(\lambda-5)(\lambda-20)^2$$

To calculate minimal polynomial, we have to find least possible degree polynomial such that $p(A) = 0$.

$$i) (\lambda - 20)^0 (\lambda - 5)(-1) = -(\lambda - 5).$$

$$\text{if } m(\lambda) = -(\lambda - 5).$$

$-(A - 5I) = 0$ should be satisfied.

$$A - 5I = \begin{bmatrix} 8 & -6 & 12 \\ -18 & 11 & 18 \\ -6 & -3 & 26 \end{bmatrix} \neq 0.$$

$$ii) (\lambda - 20)^1 (\lambda - 5)(-1) = -(\lambda - 5)(\lambda - 20).$$

$$\text{if } m(\lambda) = -(\lambda - 5)(\lambda - 20), (A - 5I)(A - 20I) = 0.$$

should be satisfied

$$(A - 5I)(A - 20I) = \begin{bmatrix} 3 & -6 & 12 \\ -18 & 6 & 18 \\ -6 & -3 & 21 \end{bmatrix} \begin{bmatrix} -12 & -6 & 12 \\ -18 & -9 & 18 \\ -6 & -3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow -(A - 5I)(A - 20I) = 0.$$

Hence, minimal polynomial of A is

$$m(\lambda) = -(\lambda - 5)(\lambda - 20).$$

(10)

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$\Rightarrow A^3 = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

Now, $A^3 - 6A^2 + 9A - 4I$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} + (-6) \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + (9) \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$= -4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow A^3 - 6A^2 + 9A - 4I = 0$ (satisfied)
while $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$.

Hence Cayley Hamilton theorem verified.

Now, $A^3 - 6A^2 + 9A - 4I = 0$

$\Rightarrow A^2 - 6A + 9I - 4A^{-1} = 0$

$\Rightarrow 4A^{-1} = A^2 - 6A + 9I$

$$\Rightarrow 4A^{-1} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$\Rightarrow A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$