

1.)

b lies in the sub space spanned by columns of matrix A ;

if solution exists for $Ax=b$;

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$$A^T = \begin{bmatrix} -4 & 5 & -2 \\ 2 & -2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$x = A^T b$$

$$b = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

$$x = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

$$x_1 = -2, x_2 = 3, x_3 = 1$$

$\therefore b$ lies in Column space of A .

2) given matrix;

$$\begin{bmatrix} 2 & 4 & 8 & 16 \\ 3 & 9 & 27 & 81 \\ 4 & 8 & 16 & 64 \end{bmatrix}$$

for Column Space:

$$R_2 \rightarrow -\frac{3}{2}R_1 + R_2$$

$$\begin{bmatrix} 2 & 4 & 8 & 16 \\ 0 & 3 & 15 & 57 \\ 4 & 8 & 16 & 64 \end{bmatrix}$$

$$R_3 \rightarrow -2R_1 + R_3$$

$$\begin{bmatrix} \boxed{2} & 4 & 8 & 16 \\ 0 & \boxed{3} & 15 & 57 \\ 0 & 0 & 0 & \boxed{32} \end{bmatrix}$$

$\therefore$ Column Space is $\lambda_1 \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + \lambda_2 \begin{bmatrix} 4 \\ 9 \\ 8 \end{bmatrix} + \lambda_3 \begin{bmatrix} 16 \\ 81 \\ 64 \end{bmatrix}$

$\lambda_1, \lambda_2, \lambda_3$ are any real numbers.

for Calculating row space: (1)

$$\begin{bmatrix} 2 & 4 & 8 & 16 \\ 3 & 9 & 27 & 81 \\ 4 & 8 & 16 & 64 \end{bmatrix}$$

$$R_1 \rightarrow \frac{R_1}{2}$$

$$R_2 \rightarrow -3R_1 + R_2$$

$$\begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 3 & 15 & 57 \\ 4 & 8 & 16 & 64 \end{bmatrix}$$

$$R_3 \rightarrow -4R_1 + R_3$$

$$\begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 3 & 15 & 57 \\ 0 & 0 & 0 & 32 \end{bmatrix}$$

$$R_2 \rightarrow \frac{R_2}{3}$$

$$R_3 \rightarrow \frac{R_3}{32}$$

$$\begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 1 & 5 & 19 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \rightarrow -19R_3 + R_2$$

$$\begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \rightarrow -8R_3 + R_1$$

$$\begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \rightarrow -2R_2 + R_1$$

$$\begin{bmatrix} 1 & 0 & -6 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So all rows form basis for row space of our matrix.

Their linear combinations will form row space.

for calculating null space: —

$$\begin{bmatrix} 2 & 4 & 8 & 16 \\ 3 & 9 & 27 & 81 \\ 4 & 8 & 16 & 64 \end{bmatrix}$$

$$R_1 \rightarrow \frac{R_1}{2}$$

$$R_2 \rightarrow -3R_1 + R_2$$

$$\begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 3 & 15 & 57 \\ 4 & 8 & 16 & 64 \end{bmatrix}$$

$$R_3 \rightarrow -4R_1 + R_3$$

$$\begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 3 & 15 & 57 \\ 0 & 0 & 0 & 32 \end{bmatrix}$$

$$R_2 \rightarrow \frac{R_2}{3}$$

$$R_3 \rightarrow \frac{R_3}{32}$$

$$\begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 1 & 5 & 19 \\ 0 & 0 & 0 & 32 \end{bmatrix}$$

$$R_2 \rightarrow -19R_3 + R_2$$

$$\begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \rightarrow -8R_3 + R_1$$

$$\begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \rightarrow -2R_2 + R_1$$

$$\begin{bmatrix} 1 & 0 & -6 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Null space} \Rightarrow Ax = 0$$

$$\begin{bmatrix} 1 & 0 & -6 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - 6x_3 = 0$$

$$\Rightarrow x_1 = 6x_3$$

$$x_2 + 5x_3 = 0 \Rightarrow x_2 = -5x_3$$

$$x_4 = 0$$

$x_3 \begin{bmatrix} 6 \\ -5 \\ 1 \\ 0 \end{bmatrix}$ is null space of our matrix.

x_3 is any real number.

$R_1 \rightarrow -8R_2 + R_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

$R_1 \rightarrow -5R_2 + R_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

3) If the product of two matrices
a) is the zero matrix. $AB=0$.

then for each column vector
 b_i of B

$$Ab_i = 0$$

Since any vector v in the Column

space of B is a linear combination
of the columns of B

$$v = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

then

$$Av = a_1 Ab_1 + a_2 Ab_2 + \dots + a_n Ab_n = 0$$

Therefore, every vector in the Column

space of B is in the null space of A

which means that the Column space of

A is contained in null space of B .

b) If $AB=0$ then taking the transpose
of both sides of the equation,
 $0=B^T A^T$

We have;

$$B^T A^T = 0$$

Therefore, similar to what we found
~~above~~ above; it follows that Column

Space of A^T (row space of A) is

Contained in null space of B^T ,

(the left null space of B).

Therefore if $AB=0$, then row space

of A is contained in left null
space of B .

A is contained in null space of B

④ $v_1 = (2, 0, -1) ; v_2 = (3, 1, 0) ; v_3 = (1, -1, c)$

The set $\{v_1, v_2, v_3\}$ forms a basis for $\mathbb{R}^3$.

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & c \end{bmatrix}$$

$$R_3 \rightarrow 2R_3 + R_1 \quad \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2c+1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 3R_2 \quad \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2c+4 \end{bmatrix}$$

These vectors should be linearly independent to form

a basis.

$$\begin{cases} 2a_1 + 3a_2 + a_3 = 0 \\ a_2 - a_3 = 0 \\ (2c+4)a_3 = 0 \end{cases} \Rightarrow \begin{cases} a_2 = a_3 \\ a_1 = -2a_2 \\ (2c+4)a_3 = 0 \end{cases}$$

$$a_1 = a_2 = a_3 = 0 \quad (\because \text{linearly independent})$$

$$\text{if } a_3 = 0, \text{ and } (2c+4) \cdot a_3 = 0$$

$$\Rightarrow (2c+4) \neq 0$$

$$\Rightarrow 2c \neq -4$$

$$\Rightarrow \boxed{c \neq -2}$$

$$c \in \mathbb{R} - \{-2\}$$

⑤
$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 3 & -1 & 1 \\ 5 & -4 & 2 & 0 \end{bmatrix} \quad (1, 0, 5) = N$$

$$R_3 \rightarrow R_3 - 5R_1 : \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 3 & -1 & 1 \\ 0 & -14 & -3 & -15 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + \frac{14}{3}R_2 : \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 3 & -1 & 1 \\ 0 & 0 & \frac{-23}{3} & \frac{-31}{3} \end{bmatrix}$$

This has 3 pivots.
Hence rank = 3.

⑥ i) $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$. $(a_1, a_2, a_3, a_0) \in \mathbb{R}$
 $\Rightarrow (a_0, a_1, a_2, a_3) \in \mathbb{R}$

$$p(3) = 0.$$

$$\Rightarrow a_0 + 3a_1 + 9a_2 + 27a_3 = 0.$$

$$\Rightarrow a_0 = -3a_1 - 9a_2 - 27a_3.$$

$$\Rightarrow p(x) = (a_0, a_1, a_2, a_3) = (\text{vector form})$$

$$= (-3a_1 - 9a_2 - 27a_3, a_1, a_2, a_3)$$

$$= (-3a_1 - 9a_2 - 27a_3) + a_1x + a_2x^2 + a_3x^3.$$

$$= a_1(-3+x) + a_2(-9+x^2) + a_3(-27+x^3)$$

$$\Rightarrow \text{span of } \{(-3+x), (-9+x^2), (-27+x^3)\}$$

$$\Rightarrow \text{Basis} = \{(-3+x), (-9+x^2), (-27+x^3)\}$$

$$= \{(-3, 1, 0, 0), (-9, 0, 1, 0), (-27, 0, 0, 1)\}$$

$$\left\{ \begin{pmatrix} 1, 0, 0, 1 \end{pmatrix}, \begin{pmatrix} 0, 1, 0, 1 \end{pmatrix}, \begin{pmatrix} 0, 0, 1, 1 \end{pmatrix} \right\} = \text{Basis}$$

$$\text{Dimension} = 3.$$

(ii) $p(3) = 0$ and $p(5) = 0$.

$$p(3) = 0 \Rightarrow p(x) = a_1(-3+x) + a_2(-9+x^2) + a_3(-27+x^3)$$

$$a_0 = -3a_1 - 9a_2 - 27a_3.$$

$$p(5) = 0 \Rightarrow a_0 + 5a_1 + 25a_2 + 125a_3 = 0.$$

$$\Rightarrow -3a_1 - 9a_2 - 27a_3 + 5a_1 + 25a_2 + 125a_3 = 0.$$

$$\Rightarrow 2a_1 + 16a_2 + 98a_3 = 0.$$

$$\Rightarrow a_1 = -8a_2 - 49a_3.$$

$$a_0 = -3(-8a_2 - 49a_3) - 9a_2 - 27a_3.$$

$$= 24a_2 + 147a_3 - 9a_2 - 27a_3$$

$$= 15a_2 + 120a_3.$$

$$\Rightarrow p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$$

$$\Rightarrow p(x) = (15a_2 + 120a_3) + (-8a_2 - 49a_3)x + a_2x^2 + a_3x^3$$

$$\Rightarrow p(x) = a_2(15 - 8x + x^2) + a_3(120 - 49x + x^3)$$

$$\text{Basis} = \{ (15-8x+x^2), (120-49x+x^3) \}$$

$$\frac{1}{13} \frac{1}{13} = \left\{ \frac{2}{13} \frac{1}{13}, \frac{1}{13} \frac{1}{13} \right\} = \{ (15, -8, 1, 0), (120, -49, 0, 1) \}$$

$$\text{Dimension} = 2.$$

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$$A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 3 & -7 & -5 \\ 0 & 1 & -5 & -3 \\ 0 & 7 & 5 & -1 \end{bmatrix} \begin{matrix} R_1 \leftarrow R_1 \\ R_2 \leftarrow R_2 \\ R_3 \leftarrow R_3 \\ R_4 \leftarrow R_4 \end{matrix}$$

$$\begin{matrix} R_2 \rightarrow -3R_1 + R_2 \\ R_3 \rightarrow -R_1 + R_3 \\ R_4 \rightarrow -7R_1 + R_4 \end{matrix} \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & -13 & -5 & -1 & -2 \\ 0 & -7 & -3 & 0 & -1 \\ 0 & -9 & -1 & -5 & -2 \end{bmatrix}$$

$$R_2 \rightarrow \frac{R_2}{-13} \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 1 & 5/13 & 1/13 & 2/13 \\ 0 & -7 & -3 & 0 & -1 \\ 0 & -9 & -1 & -5 & -2 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + 7R_2 \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 1 & 5/13 & 1/13 & 2/13 \\ 0 & 0 & 4/13 & 7/13 & 1/13 \\ 0 & -9 & -1 & -5 & -2 \end{bmatrix}$$

$$R_4 \rightarrow 9R_2 + R_4$$

$$R_3 \rightarrow R_3 \times \left(-\frac{13}{4}\right)$$

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 1 & \frac{5}{13} & \frac{1}{13} & \frac{2}{13} \\ 0 & 0 & 1 & -\frac{7}{4} & -\frac{1}{4} \\ 0 & 0 & \frac{32}{13} & -\frac{56}{13} & -\frac{8}{13} \end{bmatrix}$$

$$R_4 \rightarrow R_4 + \left(-\frac{32}{13}\right)R_3$$

$$R_2 \rightarrow R_2 + \left(-\frac{5}{13}\right)R_3$$

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 1 & 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 1 & -\frac{7}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \rightarrow R_1 - 2R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 1 & -\frac{7}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

No. of pivots = 3 $\Rightarrow$ rank(A) = 3.

Nullspace $\Rightarrow A\underline{x} = 0$.

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 1 & -\frac{7}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 + \frac{1}{2}x_4 + \frac{1}{2}x_5 = 0 \Rightarrow x_1 = -\frac{1}{2}x_4 - \frac{1}{2}x_5$$

$$x_2 + \frac{3}{4}x_4 + \frac{1}{4}x_5 = 0 \Rightarrow x_2 = -\frac{3}{4}x_4 - \frac{1}{4}x_5$$

$$x_3 + \frac{7}{4}x_4 + \frac{1}{4}x_5 = 0 \Rightarrow x_3 = -\frac{7}{4}x_4 - \frac{1}{4}x_5$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x_4 - \frac{1}{2}x_5 \\ -\frac{3}{4}x_4 - \frac{1}{4}x_5 \\ -\frac{7}{4}x_4 - \frac{1}{4}x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} \\ -\frac{3}{4} \\ -\frac{7}{4} \\ 1 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{4} \\ -\frac{1}{4} \\ 0 \\ 1 \end{bmatrix} x_5$$

$$N(A) = \{ Ax = 0 \mid x = c_1 \begin{pmatrix} -\frac{1}{2} \\ -\frac{3}{4} \\ -\frac{7}{4} \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{4} \\ -\frac{1}{4} \\ 0 \\ 1 \end{pmatrix}; c_1, c_2 \in \mathbb{R} \}$$

$$\Rightarrow \text{Nullity}(A) = 2, \text{ Now:}$$

$$\text{Rank}(A) + \text{Nullity}(A) = 3 + 2 = 5.$$

for $A_{4 \times 5}$, Rank-Nullity theorem is verified.

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$$2x \frac{1}{p} - 13x \frac{1}{p} = 13 \quad \Rightarrow \quad 0 = 2x \frac{1}{p} + 13x \frac{1}{p} + 13$$

$$A^T = \begin{bmatrix} 1 & 3 & 1 & 7 \\ 2 & -4 & 5 & 5 \\ 0 & -5 & -3 & -1 \\ 2 & 5 & 2 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \rightarrow -2R_1 + R_2$$

$$R_4 \rightarrow -2R_1 + R_4$$

$$R_5 \rightarrow -R_1 + R_5$$

$$\begin{bmatrix} 1 & 3 & 1 & 7 \\ 0 & -10 & -1 & -7 \\ 0 & -5 & -3 & -1 \\ 0 & -1 & 0 & -5 \\ 0 & -2 & -1 & -2 \end{bmatrix}$$

$$R_2 \rightarrow \frac{R_2}{-10}$$

$$\begin{bmatrix} 1 & 3 & 1 & 7 \\ 0 & 1 & \frac{1}{10} & \frac{7}{10} \\ 0 & -5 & -3 & -1 \\ 0 & -1 & 0 & -5 \\ 0 & -2 & -1 & -2 \end{bmatrix}$$

$$R_3 \rightarrow 5R_2 + R_3$$

$$R_4 \rightarrow R_2 + R_4$$

$$R_5 \rightarrow 2R_2 + R_5$$

$$\begin{bmatrix} 1 & 3 & 1 & 7 \\ 0 & 1 & \frac{1}{10} & \frac{7}{10} \\ 0 & 0 & -\frac{4}{10} & -\frac{32}{10} \\ 0 & 0 & \frac{7}{10} & -\frac{56}{10} \\ 0 & 0 & \frac{7}{10} & -\frac{56}{10} \end{bmatrix}$$

$$R_5 \rightarrow R_5 - R_4$$

$$\begin{bmatrix} 1 & 3 & 1 & 7 \\ 0 & 1 & \frac{1}{10} & \frac{7}{10} \\ 0 & 0 & -\frac{4}{10} & -\frac{32}{10} \\ 0 & 0 & \frac{7}{10} & -\frac{56}{10} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank(A) = 3 + 1 = 4

Rank(A) + Rank(B) = Rank(A+B)

$$\text{Nullspace} \Rightarrow (A^T)\underline{x} = \underline{0} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x_1 &= 0, \quad x_2 + 5x_4 = 0 \Rightarrow x_2 = -5x_4 \\ x_3 - 8x_4 &= 0 \Rightarrow x_3 = 8x_4 \end{aligned}$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -5x_4 \\ 8x_4 \\ x_4 \end{bmatrix} \Rightarrow \underline{x} = \begin{bmatrix} 0 \\ -5 \\ 8 \\ 1 \end{bmatrix} x_4$$

$$\Rightarrow N(A^T) = \left\{ A^T \underline{x} = \underline{0} \mid \underline{x} = c \cdot \begin{bmatrix} 0 \\ -5 \\ 8 \\ 1 \end{bmatrix}, c \in \mathbb{R} \right\}$$

$$\text{Nullity}(A^T) = 1.$$

$$\text{Now, } \text{rank}(A^T) + \text{Nullity}(A^T) = 3 + 1 = 4.$$

for $(A^T)_{5 \times 4}$, rank-nullity theorem is verified.

$$\text{Rank}(A) = \text{Rank}(A^T), \text{ But } \text{Nullity}(A) \neq \text{Nullity}(A^T)$$