

INDIAN INSTITUTE OF TECHNOLOGY HYDERABAD

DEPARTMENT OF MATHEMATICS

ASSIGNMENT 3

MA 1140 : Linear Algebra

Max Marks: 50

Each question carries 5 marks

1. Let V be a vector space over $\mathbb{R}$.

(a) For $x \in V$, define $T(x) = mx + b$, where $m, b \in \mathbb{R}$ and $b \neq 0$. Is T a linear transformation? [2]

(b) If $R_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined as follows: if $(x, y) \in \mathbb{R}^2$, then the axes are rotated in counterclockwise direction by angle θ to get the new coordinates (x_1, y_1) in $\mathbb{R}^2$. Find the transformation matrix of R_θ . [3]

2. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the linear transformation defined by [5]

$$T(x_1, x_2) = (x_1 - x_2, -x_2, 3x_1 - 5x_2).$$

Find the matrix A such that $T(x) = Ax$ for all $x \in \mathbb{R}^2$.

3. Calculate the change of basis matrix from A to B , where A and B are given by: [5]

$$A^T = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B^T = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & -1 \end{bmatrix}.$$

4. (a) Can there be matrices A and B similar, but A^2 and B^2 are not similar? If there exists such a pair, then produce them. [1]

(b) Prove that if two matrices are similar and if one is invertible then so is the other. [2]

(c) Show that similarity is an equivalence relation. [2]

5. (a) Let [3]

$$A = \begin{bmatrix} 1 & a & bc \\ 1 & b & ac \\ 1 & c & ab \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}.$$

Calculate the determinants of A and B and determine whether they are equal or not.

(b) Find the determinant of matrix [2]

$$A = \begin{bmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{bmatrix}.$$

6. Let $A := \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix}$. Use the eigenvalue method to derive an explicit formula for A^n and also solve:

$$dx/dt = 2x - 3y$$

$$dy/dt = 4x - 5y$$

given that $x = 7, y = 13$, when $t = 0$.

7. Find the eigenvalues and eigenvectors of the linear transformation T defined by $T : P_2 \rightarrow P_2$ [5]

$$a_0 + a_1x + a_2x^2 \rightarrow (5a_0 + 6a_1 + 2a_2) - (a_1 + 8a_2)x + (a_0 - 2a_2)x^2.$$

where P_2 represents all polynomials in $\mathbb{R}[x]$ with degree ≤ 2 .

8. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation. Suppose $x, y \in \mathbb{R}^2$ are linearly independent eigenvectors of T , but they have the same eigenvalue λ . [5]

- (a) Show that every vector in $\mathbb{R}^2$ is an eigenvector of T (associated to the same eigenvalue) and also that the characteristic polynomial of the matrix associated to T has a double-root λ . (Consider the example $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.) [3]

- (b) Is it true that matrices with equal characteristic polynomials are necessarily similar? [2]

9. Determine the minimal polynomial of the following 4×4 matrix : [5]

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}.$$

10. Find the inverse of the matrix A using Cayley-Hamilton theorem. [5]

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{bmatrix}.$$