Streaming Codes for Three-Node Relay Networks With Burst Erasures

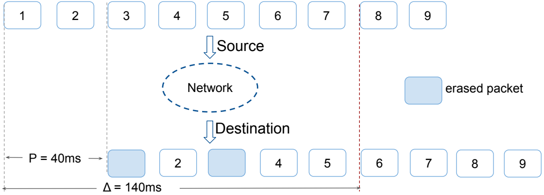
Vinayak Ramkumar * Myna Vajha ‡ M. Nikhil Krishnan †

*Tel Aviv University, ‡IIT Hyderabad, †IIT Palakkad

IEEE ISIT 2024, July 10

Motivation: Latency Sensitive Applications

- Several latency sensitive applications (real-time audio/video, AR/VR etc) that have end-to-end (E2E) latency requirements under packet erasures



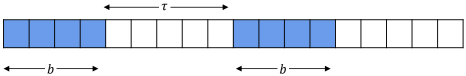
Δ: E2E delay, $\tau = 4$: Delay in packet count

- E2E delay modeled through count of packets accessed in future.
- Goal: Design packet level FECs that can recover erasures within delay τ.

Literature: Settings Considered

Point to Point (PP), Burst Erasures

- Martinian and Sundberg, TIT 2004
  - (b, τ)-streaming codes can recover from any burst of size b within delay τ
  - Correctable erasure pattern:



  - Rate upper bounded by:

$$R \leq \frac{\tau}{\tau + b} \triangleq R_{\text{opt}}^{(\text{PP})}(b, \tau)$$

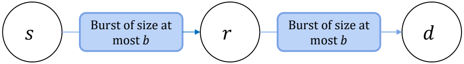
  - $(n = \tau, k = \tau - b)$ wrap-around burst correcting code (WA-BCC) $\implies$ rate-optimal (b, τ) streaming code

- Hollmann and Tolhuizen, TIT 2008
  - binary WA-BCC construction implying rate-optimal binary (b, τ) streaming code

Literature: Settings Considered

- Extensions to either burst b or random erasures a under delay constraint τ.
- Extensions to simultaneous bursts and random erasures. Bhatnagar et al. ISIT 24

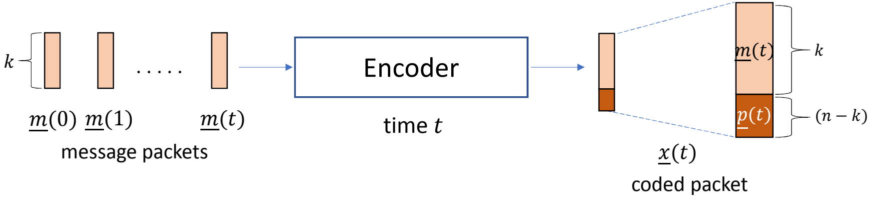
- Current work: 3-node relay, burst erasures only, delay constraint τ
  - Rate upper bound
  - Constructions that achieve rate arbitrarily close to upper bound for any b, τ



- 3 node relay with random erasures: Fong et al. TIT 2020,, Facenda et al. TIT 23, Kaleem et al. ISIT 24
  - Extensions to multiple-node relay network: Domanovitz et al. TIT 2022

- 3 node relay with random and burst erasures: S. Singhvi et al. ISIT 2022.
  - Results in optimal codes for our setting for the special case $b|\tau$.

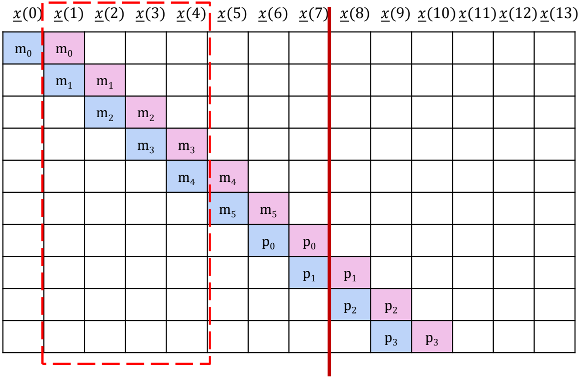
Redundancy through Packet Expansion Framework



- Can use scalar block codes to come up with streaming codes.

Diagonal Embedding (DE)

- Codewords of $[n, k]$ scalar block code are diagonally placed in the packet stream.



DE of $[10, 6]$ scalar code where $b = 4, \tau = 6$

Delay Profile Of a Block Code

- Need $(n = \tau + b, \tau)$ streaming code with delay profile

$$(\underbrace{\tau, \cdots, \tau}_{b \text{ of them}}, \tau - 1, \cdots, b)$$

- Hollmann and Tolhuizen (HT) code has parity check matrix:

$$H_{\mathrm{HT}} = [P_{b,\tau} \;\; I_b] \text{ and}$$

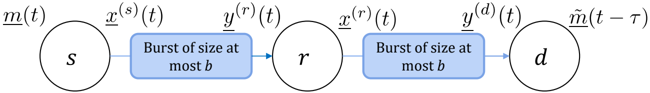
- Recursive construction of $(u \times v)$ matrix $P_{u,v}$ that appends identity matrices column-wise or row-wise.

$$P_{u,v} = \begin{cases} [I_u \;\; P_{u,v-u}] & v > u \\ \begin{bmatrix} I_v \\ P_{u-v,v} \end{bmatrix} & \text{otherwise} \end{cases}$$

- E. Martinian and M. Trott, "Delay-Optimal Burst Erasure Code Construction," ISIT 2007
- H. D. L. Hollmann and L. M. G. M. Tolhuizen, "Optimal Codes for Correcting a Single (Wrap-Around) Burst of Erasures," in TIT 2008

3 Node Relay, Burst Erasures

Rate Bound



- An example, permissible erasure pattern in the R-D link

$x^r(t)$	…	$x^r(t+\tau-b)$	$x^r(t+\tau-b+1)$	…	$x^r(t+\tau)$

- $m(t)$ should be recovered at relay by $t+\tau-b$

- Rate in the S-R link upper bounded by rate of $(b, \tau-b)$ streaming code

$$R \leq R_{\text{opt}}^{(\text{PP})}(b, \tau-b) = \frac{\tau-b}{\tau} \triangleq R_{\text{opt}}^{\text{rel}}(b, \tau)$$

- We construct streaming codes that achieve rates arbitrarily close to the upper bound.

Source-Relay Link

- Transmit at Source
  - $(n = \tau, k = \tau - b)$ HT code is used at the source.
  - Example: $\tau = 10, b = 4$, (4×10) pc matrix

$$H_{HT} = \left[\; P_{4,6} \; \middle| \; I_4 \; \right] = \left[\begin{array}{c|c|c} I_4 & \begin{array}{c} I_2 \\ \hline I_2 \end{array} & I_4 \end{array} \right]$$

★ $p_0 = m_0 + m_4, p_1 = m_1 + m_5, p_2 = m_2 + m_4, p_3 = m_3 + m_5$

★ m_i can be recovered by p_i for $i = 0, \cdots, 3$.

t	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$x^s(t)$	m_0	m_1	m_2	m_3	m_4	m_5	p_0	p_1	p_2	p_3				

- Receive at Relay: Burst starting at time 0/1 in S-R link

$y^r(t)$					m_4	m_5	p_0	p_1	p_2	p_3				
$x^r(t)$					m_4	m_5	m_0	m_1	m_2	m_3	$\hat{p}_0$	$\hat{p}_1$	$\hat{p}_2$	$\hat{p}_3$
$y^r(t)$	m_0					m_5	p_0	p_1	p_2	p_3				
$x^r(t)$					m_0	m_5	p_0	m_1	m_2	m_3	$\hat{p}_0$	$\hat{p}_1$	$\hat{p}_2$	$\hat{p}_3$

Relay-Destination Link

t	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$x^s(t)$	m_0	m_1	m_2	m_3	m_4	m_5	p_0	p_1	p_2	p_3				
$y^r(t)$	m_0	(erased)	(erased)	(erased)	(erased)	m_5	p_0	p_1	p_2	p_3				
$x^r(t)$					m_0	m_5	p_0	m_1	m_2	m_3	$\hat{p}_0$	$\hat{p}_1$	$\hat{p}_2$	$\hat{p}_3$

Burst at 1 in S-R link

- Information set needs to be preserved.
  - Can resolve m_4 from $p_0 = m_0 + m_4$, and m_0
  - $\alpha = \min\{\beta, b-1\}$ where β is burst start and set $(\hat{m}_0, \cdots, \hat{m}_{k-1})$ as

$$\Big(\underbrace{m_0, \cdots, m_{\alpha-1}}_{\alpha \text{ urgent symbols}}, \underbrace{x^s(\alpha), \cdots, x^s(k-b+\alpha-1)}_{(k-\alpha) \text{ non-urgent symbols}}, \underbrace{m_\alpha, \cdots, m_{b-1}}_{(b-\alpha) \text{ urgent symbols}}\Big)$$

  - HT code guarantees the information set property

Relay-Destination Link

t	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$x^s(t)$	m_0	m_1	m_2	m_3	m_4	m_5	p_0	p_1	p_2	p_3				
$y^r(t)$	m_0					m_5	p_0	p_1	p_2	p_3				
$x^r(t)$					m_0	m_5	p_0	m_1	m_2	m_3	$\hat{p}_0$	$\hat{p}_1$	$\hat{p}_2$	$\hat{p}_3$

Burst at 1 in S-R link

- Urgent messages:
  - For $i \in [0 : \alpha - 1]$ $\hat{m}_i$ requires delay $\leq \tau - b$ (until $\hat{p}_i$ available)
  - For $i \in [0 : b - \alpha - 1]$ $\hat{m}_{k-1-i}$ requires delay $\leq b$ (until $\hat{p}_{b-1-i}$ is available)
  - Delay profile of the form:

$$(\underbrace{\tau - b, \cdots, \tau - b}_{\alpha \text{ urgent symbols}}, \underbrace{\tau - \alpha - 1, \cdots, 2b - \alpha}_{(\tau - 2b) \text{ non-urgent symbols}}, \underbrace{b, \cdots, b}_{(b-\alpha) \text{ urgent symbols}})$$

- We construct generalized HT (GHT) code to handle this delay profile
- GHT code used at relay is dependent on $\alpha \implies$ overhead in communicating the α at destination.
  - Overhead can be made negligible by ammortization.

Relay-Destination Link

t	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$x^s(t)$	m_0	m_1	m_2	m_3	m_4	m_5	p_0	p_1	p_2	p_3				
$y^r(t)$	m_0					m_5	p_0	p_1	p_2	p_3				
$x^r(t)$					m_0	m_5	p_0	m_1	m_2	m_3	$\hat{p}_0$	$\hat{p}_1$	$\hat{p}_2$	$\hat{p}_3$

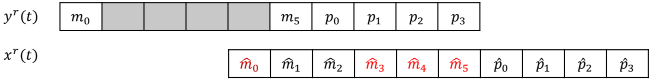
Burst at 1 in S-R link

- Urgent messages:
  - For $i \in [0 : \alpha - 1]$ $\hat{m}_i$ requires delay $\leq \tau - b$ (until $\hat{p}_i$ available)
  - For $i \in [0 : b - \alpha - 1]$ $\hat{m}_{k-1-i}$ requires delay $\leq b$ (until $\hat{p}_{b-1-i}$ is available)
  - Delay profile of the form:

$$(\underbrace{\tau - b, \cdots, \tau - b}_{\alpha \text{ urgent symbols}}, \underbrace{\tau - \alpha - 1, \cdots, 2b - \alpha}_{(\tau - 2b) \text{ non-urgent symbols}}, \underbrace{b, \cdots, b}_{(b-\alpha) \text{ urgent symbols}})$$

- We construct generalized HT (GHT) code to handle this delay profile
- GHT code used at relay is dependent on $\alpha \implies$ overhead in communicating the α at destination.
  - Overhead can be made negligible by ammortization.

GHT construction: An Example



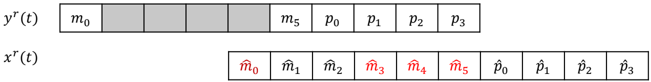
Burst at 1 in S-R link

	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	1	0	1			
1	0	1	0	0	0	1		1		
2	0	0	1	0	1	0			1	
3	0	0	0	1	0	1				1

pc matrix of $(b = 4, \tau - b = 6)$ HT code

- Retain row 0, ensures in time recovery of $\hat{m}_0$

GHT Construction: An Example



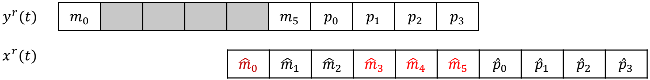
Burst at 1 in S-R link

	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	1	0	1			
1	*	*	*	1	*	*		1		
2	*	*	*	*	1	*			1	
3	*	*	*	*	*	1				1

pc matrix of $(b = 4, \tau - b = 6)$ HT code

- Get 1 in the diagonal of bottom-right $(b - \alpha) \times (b - \alpha)$ sub-matrix through row permutations

GHT Construction: An Example

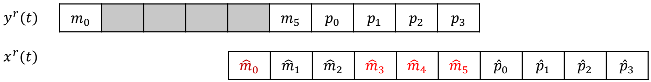


Burst at 1 in S-R link

	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	1	0	1			
1	0	0	0	1	0	1		1		
2	0	0	1	0	1	0			1	
3	0	1	0	0	0	1				1

- Set the upper triangular elements to 0

GHT Construction: An Example



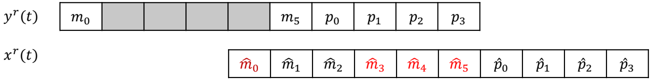
Burst at 1 in S-R link

$$
\begin{array}{c|cccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & & & \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & & 1 & & \\
2 & 0 & 0 & 1 & 0 & 1 & 0 & & & 1 & \\
3 & 0 & 1 & 0 & 0 & 0 & 1 & & & & 1
\end{array}
$$

pc matrix of $(b = 4, \tau - b = 6)$ GHT code for $\alpha = 1$.

- $\hat{m}_3, \hat{m}_5$ can be recovered in delay
- $\hat{m}_1$ can be recovered from any burst

GHT Construction: An Example



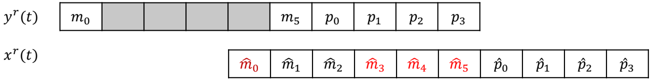
Burst at 1 in S-R link

	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	1	0	1			
1	0	0	0	1	0	0		1		
2	0	0	1	0	1	0			1	
3	0	1	0	0	0	1				1

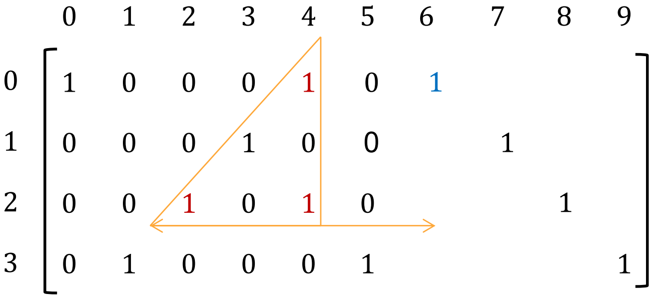
pc matrix of $(b = 4, \tau - b = 6)$ GHT code for $\alpha = 1$.

- $\hat{m}_4$ recovery: part of parities $\hat{p}_0, \hat{p}_2$

GHT Construction: An Example



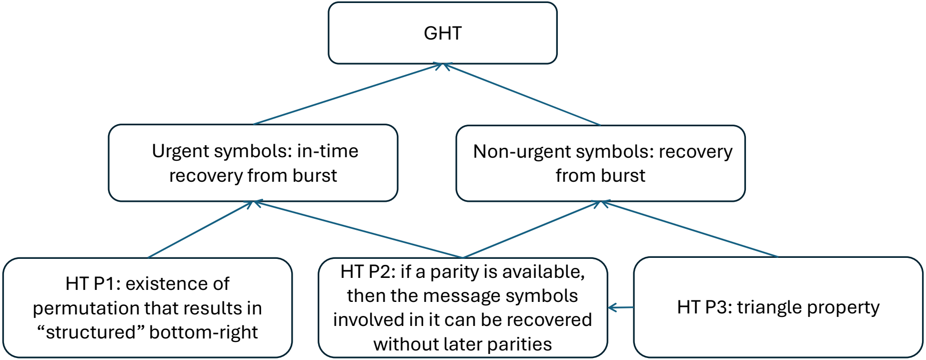
Burst at 1 in S-R link



pc matrix of $(b = 4, \tau - b = 6)$ GHT code for $\alpha = 1$.

- $\hat{m}_4$ recovery: if $\hat{m}_2$ is available can recover it using $\hat{p}_2$ otherwise from $\hat{p}_0$.
- Triangle property: $\hat{p}_0$ and $\hat{m}_2$ are spaced by b.
- $\hat{m}_2$ recovery follows.
- Can recover m_i within delay.

Key Ideas in General Proof



Open Questions

- Construction of codes for any ordering of the urgent symbols $(m_0, \cdots, m_{b-1})$ that are "allowed".

$$(m_0, *, *, m_1, m_2, m_3, p_0, p_1, p_2, p_3) \quad \checkmark$$
$$(m_1, *, *, m_0, m_2, m_3, p_0, p_1, p_2, p_3) \quad ✗$$

  - leads to streaming codes for m-node relay settings

- 3-node relay settings with different size bursts b_1, b_2.

- Characterize the delay-profiles for which constructions are possible

Thank You!