

Extreme Points of $(0, \delta)$ -LDP Polytope with Small Input and Arbitrary Output Sizes.

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Intro to Differential Privacy.

Release of seemingly harmless functions of private data set could leak identities of individuals.

- Robust De-anonymization of Large Sparse Datasets, Arvind Narayanan, Vitaly Shmatikov
IEEE Security & Privacy 2008.

"Netflix Prize" released anonymized data
the authors showed possible de-anonymization
using publicly available IMDB dataset

- On Taxis and Rainbows, V. Pandurangan

NYC taxis trip and fare data was anonymized and released. Authors show de-anonymization using prior info about taxi license numbers to run MD5 hash over smaller input set.

Intro to Differential Privacy.

Formal framework of differential Privacy introduced by Dwork et al.

Given neighbouring databases $\mathcal{D}, \mathcal{D}'$ (that don't match in one row)
 (ϵ, δ) -DP mechanism M needs to ensure that

$$P(M(\mathcal{D}) \in S) \leq e^\epsilon P(M(\mathcal{D}') \in S) + \delta.$$

for every measurable set S .

"In this framework the server implements DP mechanism to ensure differential Privacy"

Dwork, McSherry, Nissim and Smith: Calibrating Noise to Sensitivity in Private Data Analysis, TCC 06.

Intro to Differential Privacy.

An example : Consider the following database

<u>User</u>	<u>Income</u>
Alice	10L
Bob	5L
John	12L
Dave	25L
Jesse	30L

Suppose you want to query average income in a DP way.

If no noise is added, a person with side information about incomes of Alice, Bob, John, Dave can determine Jesse's income.

$$\frac{x_u + 10 + 5 + 12 + 25}{5} = f(\underline{x}).$$

Intro to Differential Privacy.

An example : Consider the following database

User	x Income	x'
Alice	10L	10L
Bob	5L	5L
John	12L	12L
Dave	25L	25L
Jesse	30L	x L.

* Suppose we know incomes can vary in 5L - 85L interval.

* Single user's data may impact the average value by $\frac{85L - 5L}{5} = 16L$.

$$f(x) = \frac{\sum_{i=1}^5 x_i}{5}$$

$$\Delta f = \max_{x, x' \text{ neighbours}} |f(x) - f(x')|_1 = 16L.$$

Define $M(x) = f(x) + Y$ $Y \sim \text{Laplace}\left(\frac{\Delta f}{\epsilon}\right)$.

Intro to Differential Privacy.

Laplace Mechanism: $M(\underline{x}) = f(\underline{x}) + \Upsilon$.

$\Upsilon \sim \text{Laplace}\left(\frac{\Delta f}{\epsilon}\right)$ and $f_{\Upsilon}(y) = \frac{\epsilon}{2\Delta f} e^{-\frac{|y|\epsilon}{\Delta f}}$ $y \in \mathbb{R}$

$P_{\underline{x}}(z)$, $P_{\underline{x}'}(z)$ be the density functions that describe distributions of $M_{\underline{x}}(z)$, $M_{\underline{x}'}(z)$.

$$\frac{P_{\underline{x}}(z)}{P_{\underline{x}'}(z)} = \frac{f_{\Upsilon}(y - f(\underline{x}))}{f_{\Upsilon}(y - f(\underline{x}'))} = e^{\frac{\epsilon}{\Delta f} [|y - f(\underline{x})| - |y - f(\underline{x}')|]}$$

$$\leq e^{\frac{\epsilon}{\Delta f} |f(\underline{x}) - f(\underline{x}')|} \leq e^{\epsilon}.$$

$$\Rightarrow P(M(\underline{x}) \in \mathcal{Y}) \leq e^{\epsilon} P(M(\underline{x}') \in \mathcal{Y}).$$

Local Differential Privacy

- * End user adds noise to their data.

- * (ϵ, δ) -LDP mechanism $M: \mathcal{X} \rightarrow \mathcal{Y}$
needs to satisfy

$$P(M(x) \in S) \leq P(M(x') \in S)e^\epsilon + \delta$$

for any $x, x' \in \mathcal{X}$ and $S \subseteq \mathcal{Y}$.

- * M can be thought of as a random function.

Duchi et al. Local Privacy and Statistical minimax rates, FOCs 2013

(ε, δ) -LDP polytope

* We assume $|\mathcal{X}| = k$, $|\mathcal{Y}| = m$

$$\mathcal{X} = [k], \quad \mathcal{Y} = [m]$$

* Every mechanism can be described through a $k \times m$ stochastic matrix T such that

$$T(x, S) \leq T(x', S) e^{\varepsilon} + \delta$$

$$\text{where } T(x, S) = \sum_{y \in S} T(x, y).$$

(ϵ, δ) - LDP polytope

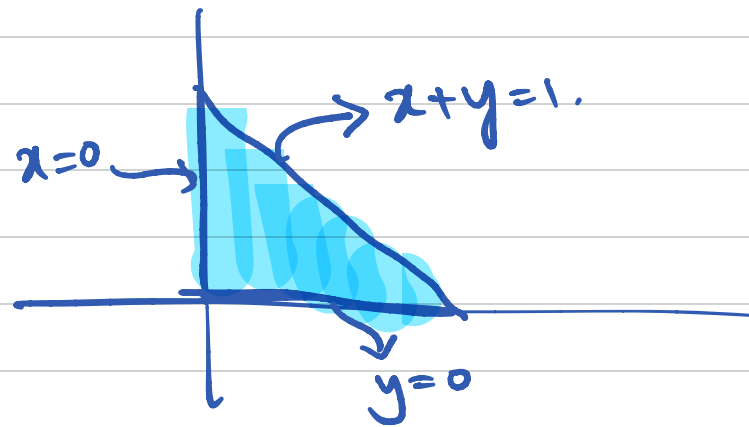
$$\star P_{\epsilon, \delta} = \left\{ T \in \mathbb{R}_+^{k \times m} \mid \begin{array}{l} T(x, S) \leq e^\epsilon T(x', S) + \delta. \\ \text{for any } x, x' \in [k], \\ S \subseteq [m], \quad T(x, [m]) = 1 \end{array} \right\}$$

$\star$ Polytope :

A d -dimensional polytope is specified by set of solutions to linear inequalities.

$$\underbrace{A}_{m \times d} \underbrace{x}_{d \times 1} \leq \underbrace{b}_{m \times 1}.$$

* Example of a polytope in 2-dimensions.



$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$* P_{\varepsilon, \delta} = \left\{ T \in R_+^{k \times m} \mid \begin{array}{l} T(x, S) \leq e^\varepsilon T(x', S) + \delta. \\ \text{for any } x, x' \in [k], \\ S \subseteq [m], \quad T(x, [m]) = 1 \end{array} \right\}$$

• k equations

• $\binom{k}{2} 2^m + km$ inequalities.

$$* P_\varepsilon = P_{\varepsilon, 0} \quad \text{and} \quad P_\delta = P_{0, \delta}.$$

$(\mathcal{E}, \mathcal{S})$ - LDP Polytope

* Convex set : A set S is said to be convex if for every $x_1, x_2 \in S$, $\lambda \in [0, 1]$.

$$x_1 \lambda + x_2 (1 - \lambda) \in S$$

* Polytope is a convex set.

$$Ax_1 \leq b, \quad Ax_2 \leq b \implies A(\lambda x_1 + (1 - \lambda)x_2) \leq b.$$

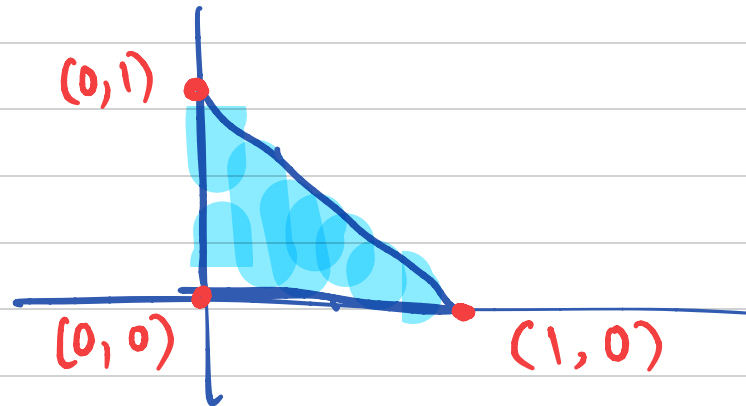
* Extreme Points $\in(S)$.

$x \in S$ is said to be an extreme point if x can't be written in the form $x = \lambda x_1 + (1 - \lambda)x_2$ for any $x_1, x_2 \in S$ st $x_1 \neq x_2$ & $\lambda \in (0, 1)$

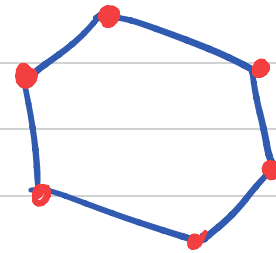
Why Look at Extreme Points

* Examples:

①



②



• extreme points

* Every point in a convex polytope can be expressed as convex combination of extreme points.

$$\left. \begin{array}{l} x = \lambda_1 x_1 + \dots + \lambda_E x_E \\ \text{st } \lambda_1 + \dots + \lambda_E = 1, \lambda_i \geq 0 \end{array} \right\} \begin{array}{l} x_1, \dots, x_E \\ \text{finite} \\ \text{extreme points of } S \end{array}$$

(Minkowski's theorem).

Why look at Extreme Points

* Convex Optimization

Let $f(\cdot)$ be a convex function i.e.,

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2).$$

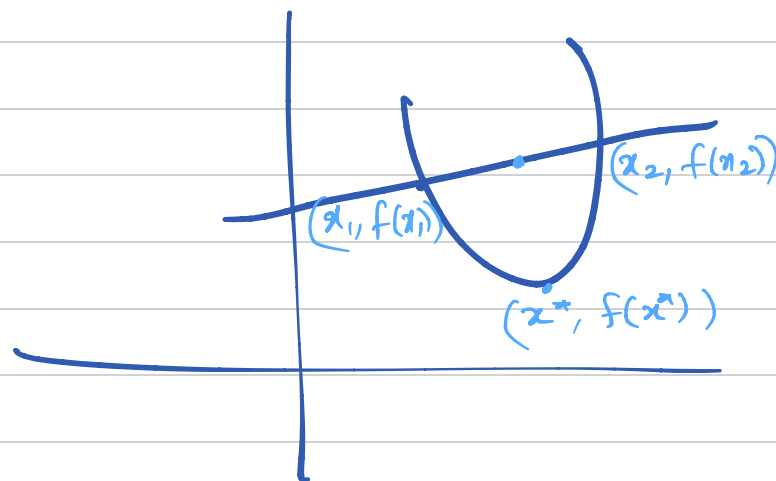
$$* \max_{x \in S} f(x) = f^*.$$

$$\text{let } x = \sum_{i=1}^E \lambda_i x_i$$

$$f(x) \leq \sum_{i=1}^E \lambda_i f(x_i)$$

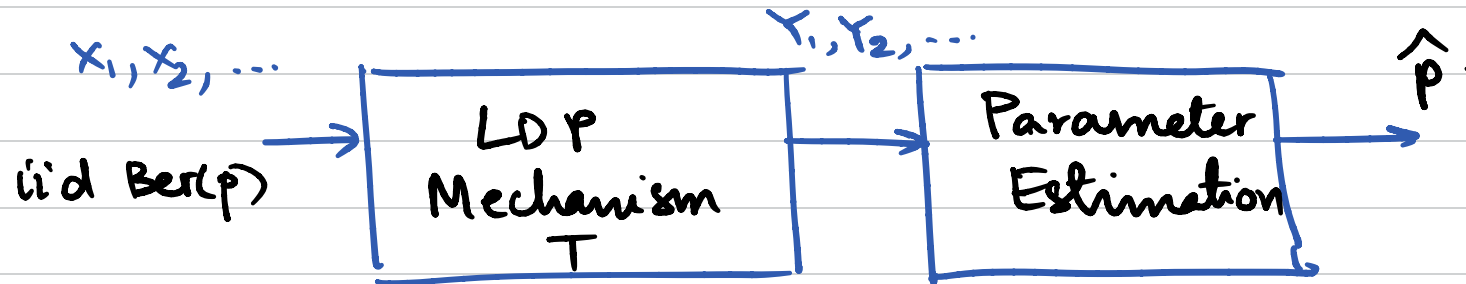
$$\Rightarrow \max_{x \in S} f(x) \leq \sum_{i=1}^E \lambda_i f(x_i).$$

$$\& \max_{x \in S} f(x) = \max_{i \in [E]} f(x_i)$$



Why look at Extreme Points.

* Private Parameter Estimation.



$$\inf_{\hat{p}} \sup_{p \in \mathcal{P}} E[\ell(\hat{p}(y_1, \dots, y_n), p)] \quad \text{s.t. } T \in \mathcal{P}_{\epsilon, \delta}.$$

* Private Hypothesis testing

$$H_0 : x_i \sim P_0.$$

$$H_1 : x_i \sim P_1.$$

Error exponent : $D_{KL}(P_0 \parallel P_1).$

$$\max_{T \in \mathcal{P}_{\epsilon, \delta}} D_{KL}(Q_0 \parallel Q_1)$$

Q_i : marginal of P_i on T

Why look at Extreme Points

* Information Preservation setting

$$\max_{T \in \mathcal{P}_{\epsilon, \delta}} I(X; Y).$$

" There is a utility function that we want to maximize under privacy constraints".

Prior Literature

- * Kairouz, Oh, Viswanadh : External Mechanisms for Local Differential Privacy, JMLR 2016.
 - For sublinear utility functions, it is enough to look at staircase mechanism that have $\log(T(y|x)/T(y|x')) \in \{0, \epsilon\}$ & have $|y| \leq |x|$
- * Holohan, Leith, Mason : Extreme Points of the Local Differential Privacy polytope, Linear Algebra & applications, 2017.
 - Characterized $E(P_\epsilon)$ for $m=1, 2, k$.
- * Elangovan, Jog : On the Extreme Points of the $(0, \delta)$ Differential Privacy Polytope, ISIT 2024.
 - Characterized $E(P_\delta)$ for $m=2, 3$
 - Family of extreme points for general k, m .
- * Our Work
 - Characterize $E(P_\epsilon)$ for $k=2, 3$
 - Conjecture on tight points for $k=3$.
 - General families of extreme points.

P_δ Polytope.

$T \in P_\delta \Rightarrow T$ is stochastic matrix

$$\text{st } T(x, S) \leq T(x', S) + \delta \\ \forall x, x'.$$

$$\Rightarrow \sup_S T(x, S) - T(x', S) \leq \delta.$$

$$\Rightarrow d_{TV}(T_x, T_{x'}) \leq \delta.$$

Total Variation
distance

Row of LDP matrices
are at distance almost δ .

* For general $P_{\epsilon, \delta}$ we get $\underbrace{\sup_S T(x, S) - e^\epsilon T(x', S)}_{E_{\gamma''}(T_x, T_{x'}) \text{ for } \gamma = e^\epsilon} \leq \delta$

" E_γ divergence".

Properties of Total Variation Distance

$$d_{TV}(P, Q) = \sup_{S \subseteq [m]} P(S) - Q(S).$$

$$S^* = \{ i \in [m] \mid P(i) \geq Q(i) \}.$$

Can see that $P(S) - Q(S) \leq P(S^*) - Q(S^*)$.
 $\forall S \subseteq [m].$

$$\begin{aligned} P(S) - Q(S) &= P(S \cap S^*) - Q(S \cap S^*) \\ &\quad + P(S \setminus S^*) - Q(S \setminus S^*). \end{aligned}$$

$$\Rightarrow d_{TV}(P, Q) = P(S^*) - Q(S^*).$$

Properties of Total Variation Distance

$$d_{TV}(P, Q) = \frac{1}{2} \sum_{i \in [m]} |P(i) - Q(i)|$$

$$d_{TV}(P, Q) = P(S^*) - Q(S^*).$$

$$= (1 - P(S^{*c})) - (1 - Q(S^{*c}))$$

$$= Q(S^{*c}) - P(S^{*c}).$$

$$= \frac{1}{2} [P(S^*) - Q(S^*) + Q(S^{*c}) - P(S^{*c})]$$

$$\Rightarrow \sum_{i=1}^u (P(i) - Q(i)) = \sum_{i=u+1}^m (Q(i) - P(i)) = d_{TV}(P, Q)$$

$$= \frac{1}{2} \left\{ \left[\sum_{i \in S^*} P(i) - Q(i) \right] + \left[\sum_{i \in S^{*c}} Q(i) - P(i) \right] \right\}$$

$$= \frac{1}{2} \sum_{i \in [m]} |P(i) - Q(i)|.$$

Perturbation Theorem

* $T \in S$ is not an extreme point iff $\exists$ a non-zero perturbation matrix E such that

$$T+E, T-E \in S$$

Proof: $\leftarrow$ clear to see!

$$T = \frac{1}{2}(T+E) + \frac{1}{2}(T-E)$$

$\rightarrow$ Suppose T not extreme

$$T = \lambda T_1 + (1-\lambda) T_2 \quad \text{for some } T_1 \neq T_2 \in \mathcal{P}_S$$

$$\text{let } E = \lambda^* (T_1 - T_2) \quad \lambda \in (0, 1)$$

$$\text{where } \lambda^* = \min(\lambda, 1-\lambda).$$

Can see that $T+E, T-E \in \mathcal{P}_S$.

Perturbation Theorem

Let $T \in \mathcal{P}_\delta$ and E be of same dimensions as T .
with $\|E\|_0$ small. Then $T \pm E \in \mathcal{P}_\delta$ iff E satisfies

$$\textcircled{1} \sum_{j=1}^m \epsilon_{ij} = 0 \quad \forall i \in [K]$$

$$\text{Let } E = \begin{bmatrix} \epsilon_{11} & \dots & \epsilon_{1m} \\ \epsilon_{K1} & \dots & \epsilon_{Km} \end{bmatrix}$$

$$\textcircled{2} \quad \epsilon_{ij} = 0 \quad \text{whenever } T_{ij} = 0.$$

$$\textcircled{3} \quad \epsilon_{ij} = \epsilon_{i'j} \quad \text{whenever } d_{TV}(T_i, T_{i'}) = \delta$$

and $T_{ij} = T_{i'j}$

$$\textcircled{4} \quad \sum_{j=1}^m (\epsilon_{ij} - \epsilon_{i'j}) \operatorname{sgn}(T_{ij} - T_{i'j}) = 0$$

whenever $d_{TV}(T_i, T_{i'}) = \delta$.

Elangovan, Jog : On the Extreme Points of the $(0, \delta)$ -Differential Privacy Polytope, ISIT 2024.

$$\text{Suppose } d_{TV}(T_i, T_{i'}) = \delta \quad \& \quad d_{TV}(T_i + E_i, T_{i'} + E_{i'}) \leq \delta$$

$$\Rightarrow \frac{1}{2} \sum_{j=1}^m |T_{ij} - T_{i'j} + E_{ij} - E_{i'j}| \leq \delta.$$

$$|T_{ij} - T_{i'j} + E_{ij} - E_{i'j}| = \begin{cases} |T_{ij} - T_{i'j}| + \operatorname{sgn}(T_{ij} - T_{i'j}) (E_{ij} - E_{i'j}) & \text{if } T_{ij} \neq T_{i'j} \\ |E_{ij} - E_{i'j}| & \text{otherwise.} \end{cases}$$

$$\Rightarrow \sum_{j: T_{ij} = T_{i'j}} |E_{ij} - E_{i'j}| + \sum_{j: T_{ij} \neq T_{i'j}} \operatorname{sgn}(T_{ij} - T_{i'j}) (E_{ij} - E_{i'j}).$$

$$+ \underbrace{\sum_{j: T_{ij} \neq T_{i'j}} |T_{ij} - T_{i'j}|}_{\ll 2\delta} \leq 2\delta. \quad \rightarrow \textcircled{1}$$

Similarly, $d_{TV}(T_i - E_i, T_{i'} - E_{i'}) \leq \delta$

$$\Rightarrow \sum_{j: T_{ij} = T_{i'j}} |\epsilon_{ij} - \epsilon_{i'j}| - \sum_{j: T_{ij} \neq T_{i'j}} \text{sgn}(T_{ij} - T_{i'j}) (\epsilon_{ij} - \epsilon_{i'j}) \leq 0.$$

From ① + ② we get.

$$2 \sum_{j: T_{ij} = T_{i'j}} |\epsilon_{ij} - \epsilon_{i'j}| = 0.$$

$$\Rightarrow \epsilon_{ij} = \epsilon_{i'j} \text{ for all } j \text{ st } T_{ij} = T_{i'j}. \rightarrow \textcircled{3}$$

①, ②, ③

$$\Rightarrow \sum_{j=1}^m \text{sgn}(T_{ij} - T_{i'j}) (\epsilon_{ij} - \epsilon_{i'j}) = 0.$$

Localization Lemma.

Let $T \in E(P_\delta)$. Then $\exists$ an unique $j_0 \in [m]$ st
 $T_{ij} \leq \delta \quad \forall j \in [m] \setminus \{j_0\}, i \in [k]$. The matrix
is said to be localized in column j_0 . Assume $\delta < \frac{1}{m+1}$.

Proof: Suppose a row has two entries $> \delta$.

W.l.o.g let $T_{i1} > \delta, T_{i2} > \delta$.

$\Rightarrow T_{i1}, T_{i2} > 0 \quad \forall i \in [k]$

$$\text{Let } E = \begin{bmatrix} \epsilon & -\epsilon & 0 & \dots & 0 \\ \epsilon & -\epsilon & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \epsilon & -\epsilon & 0 & \dots & 0 \end{bmatrix}$$

Can check that

$$T+E, T-E \in P_\delta$$

$\Rightarrow$ every row has localization at a column

• Elangovan, J. g. : On the Extreme Points of the $(0, \delta)$ -
Differential Privacy Polytope, ISIT 2024.

Localization Lemma.

Localized column is same for all rows. W.l.o.g

Suppose rows 1 and 2 are localized at columns

$$\begin{array}{ccccccc} 1, 2 & & & & & & \\ & T_{11} & \overset{\leq \delta}{T_{12}} & \overset{\leq \delta}{T_{13}} & \dots & \overset{\leq \delta}{T_{1m}} & \Rightarrow T_{11} \geq 1 - (m-1)\delta \\ & T_{21} & T_{22} & T_{23} & \dots & T_{2m} & T_{22} \geq 1 - (m-1)\delta \\ & \leq \delta & & \leq \delta & & \leq \delta & \end{array}$$

$$\begin{aligned} d_{TV}(T_1, T_2) &\geq \frac{1}{2} (T_{11} - T_{21}) + \frac{1}{2} (T_{22} - T_{12}) \\ &\geq 1 - m\delta > \delta \quad \left[\text{as } \delta < \frac{1}{m+1} \right] \end{aligned}$$

Contradiction!

- Elangovan, J. g. : On the Extreme Points of the $(0, \delta)$ -Differential Privacy Polytope, ISIT 2024.

Extreme Points With Singleton Row.

$T \in E(P_S)$ with a singleton row is of form

$$T = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1-\delta & \delta & & \\ 1-\delta & \delta & & \\ \vdots & & & \\ 1-\delta & & & \delta \end{bmatrix}$$

We'll focus on extreme matrices without repeated rows. Referred to as extreme configurations.

upto row/column permutations.

Proof: W.l.o.g assume first row $T_1 = [1 \ 0 \ \cdots \ 0]$.

$$\Rightarrow T_{i1} \geq 1-\delta, \quad \sum_{j=2}^m T_{ij} \leq \delta \quad \text{for } d_{TV}(T, T_i) \leq \delta$$

Rows have support at almost one position in $[2:m]$.

W.l.o.g suppose $T_{22}, T_{23} > 0$.

$$E_{22} = E, \quad E_{23} = -E$$

$$\Rightarrow d_{TV}(T, T_2) = d_{TV}(T, T_2 + E_2)$$

Extreme Points With Singleton Row.

Now we want to check that $d_{TV}(T_2 + E_2, T_j) \leq \delta$.

Case 1: $d_{TV}(T_2, T_j) = d < \delta$

$$\Rightarrow d_{TV}(T_2 + E_2, T_j) \leq d + \varepsilon \leq \delta.$$

by choosing a small ε .

Case 2: $d_{TV}(T_2, T_j) = \delta$

Claim: Rows T_2, T_j have disjoint support in columns $[2:m]$

$$\begin{bmatrix} T_{21} & T_{22} & \dots & T_{2u} & \boxed{T_{2u+1} & \dots & T_{2m}} \\ \vee & \vee & & \vee & \wedge & & \wedge \\ T_{j1} & T_{j2} & & & \boxed{T_{ju+1} & \dots & T_{jm}} \\ & & & & & & \leq \delta \end{bmatrix} \begin{matrix} \leftarrow \delta \\ \\ \leq \delta \end{matrix}$$

$$\Rightarrow \sum_{j'=u+1}^m (T_{j,j'} - T_{2,j'}) = \delta \Rightarrow T_{2,j'} = 0 \quad \forall j' \in [u+1:m],$$

$$T_{j,j'} = 0 \text{ for } j' \in [1:u] \Leftarrow \sum_{j'=u+1}^m T_{j,j'} = \delta.$$

Extreme Points With Singleton Row.

Continuation of case 2: $d_{TV}(T_2, T_j) = \delta$

$$\begin{bmatrix} T_{21} & T_{22} & \dots & T_{2n} & 0 & \dots & 0 \\ T_{j1} & 0 & \dots & 0 & T_{j, j+1} & \dots & T_{jm} \end{bmatrix} \quad E_2 = \begin{bmatrix} 0 & \epsilon & -\epsilon & \dots \end{bmatrix}$$

$$d_{TV}(T_2 + E_2, T_j) = \delta.$$

$\Rightarrow$ Every row has support at exactly one position in $[2:m]$

$$T = \begin{bmatrix} x & & & & \\ x & x & & & \\ x & x & x & & \\ \vdots & & x & & \\ x & & x & x & x \end{bmatrix}$$

Claim: Two rows cannot have common support in $[2:m]$

Extreme Points With Singleton Row.

W.l.o.g assume $T_{22}, T_{32} > 0$ and $T_{21} > T_{31}$.

$$\Rightarrow T_{21} > 1-\delta \quad \text{as } T_{31} \geq 1-\delta.$$

$$\text{Also } T_2 < 1 \Rightarrow d_{TV}(T_2, T_3) < \delta$$

Consider the perturbation st $E_{21} = \epsilon$, $E_{22} = -\epsilon$

$$\begin{array}{cc} x & \\ x^{\epsilon} & x^{-\epsilon} \\ x & x \end{array} \begin{array}{l} \rightarrow T_2 \\ \rightarrow T_3 \\ \rightarrow T_j \end{array} \quad d_{TV}(T_2 \pm E_2, T_3) \leq d_{TV}(T_2, T_3) + \epsilon \leq \delta \quad [\text{can choose } \epsilon \text{ to be small}]$$

$$d_{TV}(T_2 \pm E_2, T_j) \leq \delta \quad (\text{always}).$$

$$T = \begin{bmatrix} x & & & \\ x & x & & \\ x & & x & \\ x & & & x \\ x & & & & x \end{bmatrix}$$

Claim: All non-localized entries have to be δ .

Otherwise $E_2 = (\epsilon \ -\epsilon \ 0 \dots 0)$ results in $T \pm E \in P_{\delta}$.

Existence of a Tight Points

Elangoan & Jog conjecture :

$\exists$ a tight point in every extreme configuration

i.e., $\exists$ a row i st $d_{TV}(T_i, T_{i'}) = \delta \quad \forall i' \in [k] \setminus \{i\}$.

Row i is tight w.r.t i' if $d_{TV}(T_i, T_{i'}) = \delta$.

We prove the conjecture for $k=3$ and also characterize the extreme configurations with 2, 3 rows.

Existence of a Tight Pair

Given $T \in E(P_s)$ for every row $i \in [k]$
 $\exists i' \in [k] \setminus \{i\}$ s.t. $d_{TV}(T_i, T_{i'}) = \delta$.

Proof:

Suppose there is a singleton row in T
Then we know that T is of form

$$\begin{bmatrix} 1 & & & \\ 1-\delta & \delta & & \\ 1-\delta & & \delta & \\ \vdots & & & \\ 1-\delta & & & \delta \end{bmatrix}$$

(upto permutation)

and every row is
tight w.r.t every
other row.

Suppose there is no singleton row & for $i \in [k]$
 $d_{TV}(T_i, T_{i'}) < \delta \quad \forall i' \neq i$

Existence of a Tight Pair

W.l.o.g assume $T_{i1}, T_{i2} > 0$.

$$\exists \text{ st } \epsilon_{i1} = \epsilon, \quad \epsilon_{i2} = -\epsilon.$$

$$d_{TV}(T_i + \epsilon_i, T_{i'}) \leq d_{TV}(T_i, T_{i'}) + \epsilon.$$

by choosing $\epsilon \leq \delta$.

$$\epsilon \leq \delta - \max_{i' \neq i} d_{TV}(T_i, T_{i'}).$$

Therefore for T to be extreme $\exists$ a tight pair
for every row.

Existence of Tight Point for $k \leq 3$.

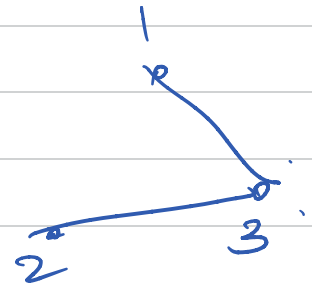
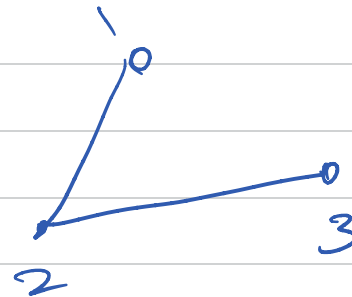
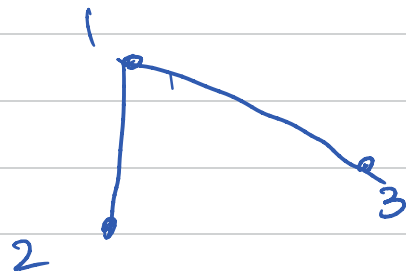
Consider a graph with k vertices.

$\exists$ an edge between node i, i' if they are tight w.r.t each other.

For $k=2$



For $k=3$



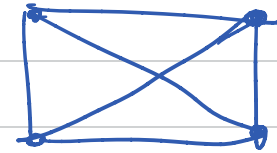
The only possible options are shown above.

In every case $\exists$ a vertex with degree 2

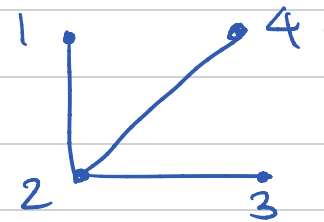
$\Rightarrow$ existence of tight point.

What can we say about $k=4$

T has singleton row



Without Singleton rows, it needs to have at least one sub-graph structure of form.



For existence of tight point need a vertex with degree 3.

Existence of tight point for $k \geq 4$ open!

Extreme Matrices for $k=2$

Extreme Configurations with one row are of the form $[1 \ 0 \ 0 \ \dots \ 0]$.

Extreme configurations with 2 rows are of the form
$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 1-s & s & & \end{bmatrix}, \begin{bmatrix} 1-s & s & 0 & 0 & \dots & 0 \\ 1-s & 0 & s & 0 & \dots & 0 \end{bmatrix}.$$

upto row / column permutations.

↓
extreme matrix with
Singleton row.

T needs to have
Support structure of
form.

$$\begin{bmatrix} a_1 & a_2 & 0 & \dots \\ b_1 & & b_2 & 0 & \dots & 0 \end{bmatrix}$$

Extreme Matrices for $k=3$.

* $T \in E(P_8)$ needs to have following support structure

$$\left[\begin{array}{cccc|ccc} a_1 & a_2 & a_3 & a_4 & & & & 0 & \dots & 0 \\ b_1 & b_2 & & & b_5 & b_6 & & & & \\ c_1 & & b_3 & & c_5 & & c_7 & 0 & \dots & 0 \end{array} \right]$$

upto row/column permutations.

* Atleast four of the following conditions need to hold for T to be extreme

$$\left\{ \begin{array}{l} a_2 = b_2 = 0, \quad a_3 = b_3 = 0, \quad b_5 = c_5 = 0, \quad b_6 = 0, \quad a_4 = 0, \\ b_5 = c_5, \quad c_7 = 0, \quad a_2 = b_2, \quad a_3 = b_3 \end{array} \right\}.$$

Extreme Matrices for $k=3$.

* Only possible extreme configurations with three rows

$$\begin{bmatrix} 1 & 0 & 0 & - & - & - \\ 1-\delta & \delta & 0 & - & - & - \\ 1-\delta & 0 & \delta & - & - & - \end{bmatrix}, \begin{bmatrix} 1-\delta & \delta & - & - & - & - \\ 1-\delta & - & \delta & - & - & - \\ 1-\delta & - & - & \delta & - & - \end{bmatrix},$$

$$\begin{bmatrix} 1-2\delta & \delta & \delta & 0 & - & - \\ 1-\delta & \delta & - & - & - & - \\ 1-\delta & - & \delta & 0 & - & - \end{bmatrix}, \begin{bmatrix} 1-2\delta & \delta & \delta & - & - & - \\ 1-2\delta & - & \delta & - & - & - \\ 1-2\delta & - & - & \delta & \delta & - \end{bmatrix}$$

upto row/column permutation.

Extreme Points for any k .

* Not yet known for $k \geq 4$, $m \geq 4$.

* Support structure needs to be of the form.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & & a_{1k} & 0 & \cdots & 0 & & 0 \\ a_{21} & a_{22} & \vdots & & & a_{2k+1} & & a_{2,2k-1} & & \vdots \\ \vdots & \vdots & \vdots & & & \vdots & & \vdots & & \vdots \\ \vdots & \vdots & & a_{k-2,3} & & \vdots & & \vdots & & \vdots \\ & a_{k-1,2} & & & & \vdots & & \vdots & & \vdots \\ a_{k1} & 0 & & & & a_{k,k+1} & & & & a_{k,2k} \end{bmatrix}$$

upto row / column permutations.

Example Extreme Configurations (for general k).

Star Configuration (Elangovan & Jog).

Let $\underline{a} = (a_1, \dots, a_{m-1})$ st $\sum_{i=1}^{m-1} a_i = 2$ and
 $0 \leq a_i \leq 1$

Rows of matrix T given by:

tight point $\swarrow$ $T_0 = (a_1\delta, a_2\delta, \dots, a_{m-1}\delta, 1-2\delta).$

$$T_{\underline{b}} = (a_1b_1\delta, a_2b_2\delta, \dots, a_{m-1}b_{m-1}\delta, 1-\delta).$$

for every $\underline{b} \in \{0, 1\}^{m-1}$ st $\sum_{i=1}^{m-1} b_i a_i = 1$

$$d_{TV}(T_0, T_{\underline{b}}) = \delta, \quad d_{TV}(T_{\underline{b}}, T_{\underline{b}'}) \leq \delta.$$

Example Extreme Configurations (for general k).

* Family 1 : (Elangovan & Jog).

$$\underline{a} = \left(\frac{u}{v} \quad \frac{1}{v} \quad \dots \quad \frac{1}{v} \right), \quad u < v$$

* Family 2 : $\underline{a} = \left(\frac{u_1}{v_1} \quad \underbrace{\frac{1}{v_1} \dots \frac{1}{v_1}}_{v_1 - u_1 \text{ times}} \quad \underbrace{\frac{1}{v_2} \dots \frac{1}{v_2}}_{v_2 \text{ times}} \right)$

st $u_1 \leq v_1 - \frac{v_1}{g}$ where $g = \gcd(v_1, v_2)$.

Family 1 $\subseteq$ Family 2.

* Family 3 : $\underline{a} = \left(\underbrace{\frac{1}{v_1} \dots \frac{1}{v_1}}_{x_1} \quad \underbrace{\frac{1}{v_2} \dots \frac{1}{v_2}}_{x_2} \quad \underbrace{\frac{1}{v_3} \dots \frac{1}{v_3}}_{v_3 \text{ times}} \right)$

st $\frac{x_1}{v_1} + \frac{x_2}{v_2} = 1$ $\gcd(v_3, v_2) > 1$, $\gcd(v_3, v_1) > 1$

One
Work.

Thank You!

Ref: <https://people.iith.ac.in/mynav/pdfs/ExtremePointsDeltaLDP.pdf>