Streaming Codes for Multi-Hop Relay Networks With Burst Erasures

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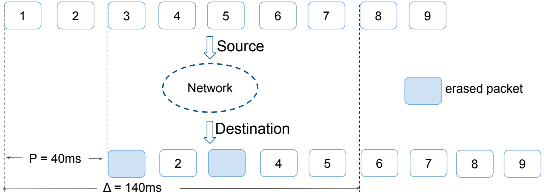
Motivation: Latency Sensitive Applications

- Several latency sensitive applications (real-time audio/video, AR/VR etc) that have end-to-end (E2E) latency requirements under packet erasures



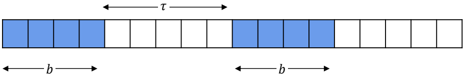
Δ: E2E delay, $\tau = 4$: Delay in packet count

- E2E delay modeled through count of packets accessed in future.
- Goal: Design packet level FECs that can recover erasures within delay τ.

Literature: Settings Considered

Point to Point (PP), Burst Erasures

- Martinian and Sundberg, TIT 2004
  - (b, τ)-streaming codes can recover from any burst of size b within delay τ



  - Rate upper bounded by:

$$R \leq \frac{\tau}{\tau + b} \triangleq R_{\text{opt}}^{(\text{PP})}(b, \tau)$$

  - $(n = \tau, k = \tau - b)$ wrap-around burst correcting code (WA-BCC) $\implies$ rate-optimal (b, τ) streaming code

- Hollmann and Tolhuizen (HT), TIT 2008
  - HT code shown to be binary WA-BCC and therefore results in a (b, τ) streaming code construction

Literature: Settings Considered

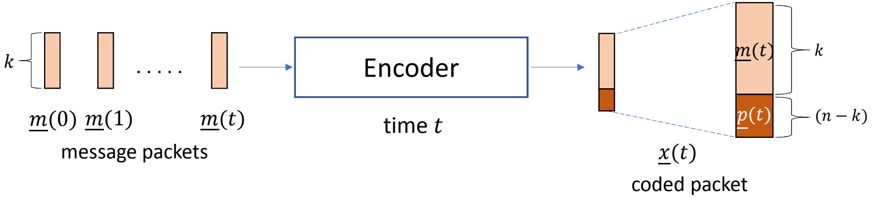
- P2P channels:
  - either burst b or random erasures a:
    - Badr et al. SPM 2017, Krishnan et al. ISIT 2018, Fong TIT 2019, Krishnan et al. TIT 2020, Vajha et. al TIT 2024 , Krishnan et al. Comm. Letters 2020, Domanovitz TIT 2022, Ramkumar et al. ISIT 2020
  - simultaneous bursts and random erasures (Shobhit et. al. ISIT 2024)

- 3-node relay setting
  - random erasures: Fong et al. TIT 2020, Facenda et al. TIT 23, Kaleem et al. ISIT 24
  - random and burst erasures: S. Singhvi et al. ISIT 2022.
  - burst erasures: both the links see atmost b size burst, Ramkumar et al., ISIT 2023, TIT 2024. (Generalizes HT code)

- Multiple node relay with random erasures: Domanovitz et al. TIT 2022

- Current work: Multiple-node relay network where each link sees a burst of size atmost b, delay constraint τ and M relays.
  - Rate upper bound
  - Further generalize HT code though Permuted Delay (PD) Code.
  - Constructions that achieve rate arbitrarily close to the upper bound for any b, τ and M.

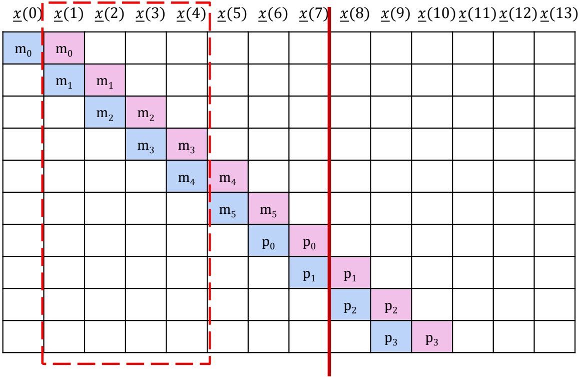
Redundancy through Packet Expansion Framework



- Can use scalar block codes to come up with streaming codes.

Diagonal Embedding (DE)

- Codewords of $[n, k]$ scalar block code are diagonally placed in the packet stream.



DE of $[10, 6]$ scalar code where $b = 4, \tau = 6$

In-Time Recovery and Urgent Symbols

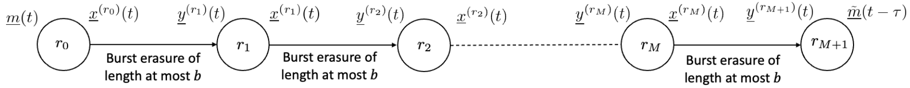
- Let $(m_0, \cdots, m_{k-1}, p_0, \cdots, p_{b-1})$ be a codeword of $[n = \tau + b, k = \tau]$ streaming code. m_i to be recovered by p_i for $i \in [0 : b-2]$.
- Urgent-Symbols: Message symbols whose recovery can't use all the b parity symbols
  - Then $m_0, m_1, \cdots, m_{b-2}$ are urgent symbols.
- HT code with generator matrix $G = [I \ P_{\tau,b}]$ where $P_{\tau,b}$ is defined recursively below:

$$P_{u,v} = \begin{cases} [I_u \ P_{u,v-u}] & v > u \\ \begin{bmatrix} I_v \\ P_{u-v,v} \end{bmatrix} & \text{otherwise} \end{cases}$$

- E. Martinian and M. Trott, "Delay-Optimal Burst Erasure Code Construction," ISIT 2007
- H. D. L. Hollmann and L. M. G. M. Tolhuizen, "Optimal Codes for Correcting a Single (Wrap-Around) Burst of Erasures," in TIT 2008

Multiple Node Relay, Burst Erasures

Rate Bound: M relay



- An example, permissible erasure pattern in the $(M+1)$-th link

$x^{r_M}(t)$	$\dots$	$x^{r_M}(t+\tau-b)$	$x^{r_M}(t+\tau-b+1)$	$\dots$	$x^{r_M}(t+\tau)$

- $m(t)$ should be recovered at relay M by $t+\tau-b$

- Rate in the M relay network is upper bounded by rate of $M-1$ relay network

$$\begin{aligned} R_M(b,\tau) &\leq R_{M-1}(b,\tau-b) \\ &\leq R_{opt}^{\text{PP}}(b,\tau-Mb) = \begin{cases} \frac{\tau-Mb}{\tau-(M-1)b} & \tau \geq (M+1)b \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

- We construct (b,τ,M) multihop burst-correcting streaming codes (MBSCs) that achieve rates arbitrarily close to the upper bound.

Delayed Transmission at Each Relay

- m_0 transmitted by source at time 0.
  - first relay is guaranteed to receive a symbol only at time b.
  - i-th relay transmits at time ib.
- 3-node relay example: M=1, $\tau = 13, b = 5$. Using $(b = 5, \tau - b = 8)$ streaming code with $[n = \tau = 13, \tau - b = 8]$ code.

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
x^{r_0}	m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7	p_0	p_1	p_2	p_3	p_4										
x^{r_1}						m_5	m_6	m_7	m_0	m_1	m_2	m_3	m_4	$p_0^{(1)}$	$p_1^{(1)}$	$p_2^{(1)}$	$p_3^{(1)}$	$p_4^{(1)}$					
x^{r_0}	m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7	p_0	p_1	p_2	p_3	p_4										
x^{r_1}						m_0	m_1	m_7	m_5	m_6	m_2	m_3	m_4	$p_0^{(1)}$	$p_1^{(1)}$	$p_2^{(1)}$	$p_3^{(1)}$	$p_4^{(1)}$					

  - Depending on the burst start index, message symbols are transmitted in different positions due to their availability at relay.
  - Urgent symbols: m_0, m_1, m_2, m_3

Generalized HT Code

- Doesn't assume that the $b-1$ urgent symbols are at the beginning of the codeword.
- Let β be the index of the start of the burst
  - $\beta \geq b$

$$(m_0, \cdots, m_{b-1}, \text{✗}, \text{✗}, \cdots, \text{✗})$$

  - $\beta < b$

$$(m_0, \cdots, m_{\beta-1}, \text{✗}, \cdots, \text{✗}, m_\beta, \cdots, m_{b-1})$$

- GHT codes parameterized by $(n = k + b, k, \alpha)$ where $\alpha = \min(\beta, b-1)$
  - can recover urgent-symbols in timely manner under burst erasures
  - can recover remaining message-symbols under burst erasures
  - strong recovery property
    - ★ allows recovery of message symbols that are part of parity equation without using later parity symbols.
    - ★ Also satisfied by HT code.
    - ★ Allows for transmission of message symbols in systematic form.

Ramkumar, Vajha, Krishnan: "Streaming Codes for Three-Node Relay Networks With Burst Erasures", TIT 2024

Extending to $M \geq 2$

- GHT code not enough to extend the MBSC construction for $M \geq 2$.
- Example: $(b = 5, \tau = 18, M = 2)$:
  - HT and GHT codes used at source and relay 1.
  - Burst seen at 2 in first link and at 6 in second link.

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
$x_i^{(r_0)}(t+i)$	m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7	$m_0 + m_5$	$m_6 + m_1$	$m_7 + m_2$	$m_7 + m_5 + m_3$	$m_7 + m_6 + m_4$										
$x_{i-b}^{(r_1)}(t+i)$						m_0	m_1	m_7	m_5	m_6	m_2	m_3	m_4	$m_0 + m_2$	$m_3 + m_1$	$m_2 + m_5$	$m_3 + m_6$	$m_4 + m_7$					
$x_{i-2b}^{(r_2)}(t+i)$											m_0	m_3	m_4	m_2	m_1	m_5	m_6	m_7	$p_0^{(2)}$	$p_1^{(2)}$	$p_2^{(2)}$	$p_3^{(2)}$	$p_4^{(2)}$

$(m_0, m_3, m_4, m_2, m_1, m_5, m_6, m_7)$ is not a permutation supported by GHT code.

Permuted Delay (PD) Code

- Supported permutations: Urgent symbols position can't be too far in the codeword.
  - $\sigma(i)$ be position of i-th symbol, then $\sigma(i) + b \leq k + i$ for $i \in [0 : b-2]$

$$(m_0, *, *, m_1, m_2, m_3, p_0, p_1, p_2, p_3) \quad \checkmark$$
$$(m_1, *, *, m_0, m_2, m_3, p_0, p_1, p_2, p_3) \quad ✗$$

- Construction of PD code through b intermediate codes, $\mathcal{C}^{-1}, \mathcal{C}^{0}, \cdots, \mathcal{C}^{b-2}$ described by parities $(p_0^\ell, \cdots, p_{b-1}^\ell)$ for $\ell = -1, 0, 1, \cdots, b-2$.

Permuted Delay (PD) Code: Through an Example

- $(m_0, m_3, m_4, m_2, m_1, m_5, m_6, m_7)$ is not a permutation supported by GHT code.
- Initial step:
  - leaders: Allocate last b message symbols to the b parities
  - group message symbols separated by b within one-parity results in burst erasure correcting code

$$p_0^{-1} = m_2,\ p_1^{-1} = m_1, p_2^{-1} = m_5 + m_0, p_3^{-1} = m_6 + m_3, p_4^{-1} = m_7 + m_4,$$

- ℓ-th step: Check if ℓ-th urgent symbol is supported by $\leq \ell$-th parity. If not swap with $\hat{\ell}$ to which m_ℓ belongs. Retain the leader of $\hat{\ell}$

$$p_0^0 = m_5 + m_0,\ p_1^0 = m_1, p_2^0 = m_5 + m_2, p_3^0 = m_6 + m_3, p_4^0 = m_7 + m_4,$$

- No changes done after 0-th step in this example

$$p_0 = m_5 + m_0, p_1 = m_1, p_2 = m_5 + m_2, p_3 = m_6 + m_3, p_4 = m_7 + m_4$$

Permuted Delay (PD) Code: Through an Example

- Codeword as shown below:

$$(m_0, m_3, m_4, m_2, m_1, m_5, m_6, m_7,$$
$$p_0 = m_5 + m_0, p_1 = m_1, p_2 = m_5 + m_2, p_3 = m_6 + m_3, p_4 = m_7 + m_4)$$

- Urgent symbols: In time recovery of m_0, m_1, m_2, m_3
  - m_2 recovery: if m_5 available, use p_2. Otherwise, m_0 will be available, use p_o to recover m_5 and then recover m_2 from p_2
- Remaining symbols: Recoverable under burst of size 5.

Permuted Delay (PD) Code: Properties

- Very strong recovery property: In the intermediate code $\mathcal{C}_\ell$ say $p_j^\ell = m_i + \text{✗}$, m_i can be recovered from p_j^ℓ by not accessing parities beyond $\min(\ell, j-1)$-th parity.

$\Downarrow$

- Strong recovery property: Say $p_j = m + i + \text{✗}$, m_i can be recovered from p_j without accessing parities beyond j

$\Downarrow$

- Timely recovery property: urgent symbols recovered in time

- Permuted Delay condition: ensured to recover urgent symbols early enough such that the permutation requirement of PD code holds at each realy.

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
$x_i^{(r_0)}(t+i)$	m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7	$m_0 + m_5$	$m_6 + m_1$	$m_7 + m_2$	$m_7 + m_5 + m_3$	$m_7 + m_6 + m_4$										
$x_{i-b}^{(r_1)}(t+i)$						m_0	m_1	m_7	m_5	m_6	m_2	m_3	m_4	$m_0 + m_2$	$m_3 + m_1$	$m_2 + m_5$	$m_3 + m_6$	$m_4 + m_7$					
$x_{i-2b}^{(r_2)}(t+i)$											m_0	m_3	m_4	m_2	m_1	m_5	m_6	m_7	$p_0^{(2)}$	$p_1^{(2)}$	$p_2^{(2)}$	$p_3^{(2)}$	$p_4^{(2)}$

  - m_1 available in the worst case by time 9 at relay 1 and by time 14 at relay 2

Conclusions

- Can use $[n = k + b, k = \tau - Mb, \sigma]$ PD code at source and the relays where σ is the order in which message symbols can be transmitted
- message order/permutation information communicated through header.
- Cost of header can be made negligible in comparison to the message packet size by picking larger packet sizes

- Open question: Bounds and constructions for multihop relay network with unequal burst sizes across the links

Thank You!