On the Performance Analysis of Streaming Codes over the Gilbert-Elliott Channel

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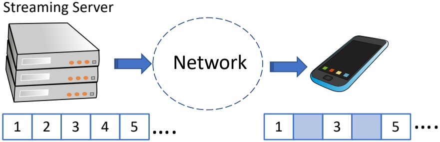
Setting Addressed by Streaming Codes

- Sequence of information-bearing packets $x(0), x(1), \ldots$, sent over an erasure channel
- Packet drops (erasures), occur due to
  - network congestion,
  - a degraded wireless link, or
  - a packet that arrives too late.



(blue rectangles denote erased packets)

Goal: use packet-level FEC to ensure best-possible tradeoff between rate and reliability

- under decoding-delay constraint of τ packets

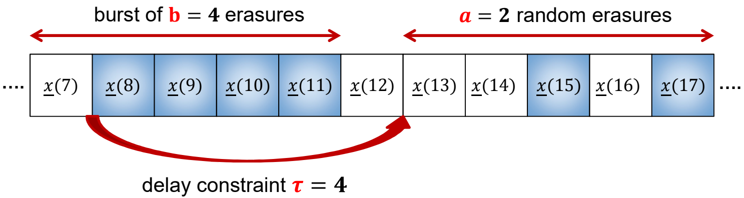
Tractable Packet Erasure Model

Delay Constrained Sliding-Window (DCSW) Channel Model

(a, b, w, τ)-DCSW Channel

- An *admissible erasure pattern (AEP)* is one in which, within each sliding window of w-packet duration, there are
  - either $\leq a$ random erasures,
  - or else, a burst of $\leq b$ erasures
- τ-packet decoding-delay constraint,

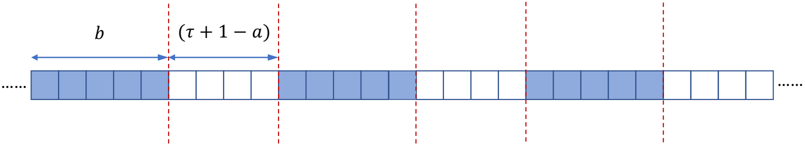
$$(a = 2, b = 4, w = 5, \tau = 4)$$



Badr et al., Layered Constructions for Low-Delay Streaming Codes," IEEE Trans. Inf. Theory, 2017.

Streaming Codes

- *Streaming codes* are packet-level FEC codes that can recover from all admissible erasure patterns seen in a DCSW channel.
- Can assume w.l.o.g that $w = \tau + 1$



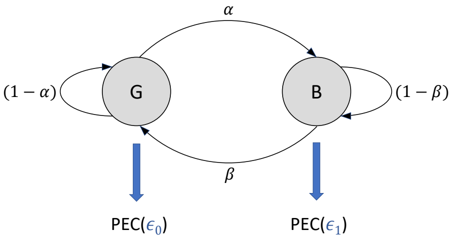
$$R_{opt}(a, b, \tau) \triangleq \frac{\tau + 1 - a}{\tau + 1 - a + b}$$

- Restrict to parameters $\{a, b, \tau\}$
  - $a \leq b \leq \tau$
  - $(a, b = a, \tau)$ case captures random erasures of size atmost a
  - $(a = 1, b, \tau)$ case captures burst erasures of size atmost b.

Badr et al., Layered Constructions for Low-Delay Streaming Codes," IEEE Trans. Inf. Theory, 2017.

How do Streaming Codes perform over Probabilistic Channels ?

The Gilbert-Elliott Channel



- Gilbert Elliott (GE) is a two-state channel model
  - **G** $\equiv$ Good State, **B** $\equiv$ Bad State
  - PEC is a packet-level erasure channel
    - ★ ϵ_0 is the probability of packet erasure in good state
    - ★ ϵ_1 is the probability of packet erasure in bad state
- capable of generating the random and burst erasures that one might encounter in practice

The Problem

Given

- a GE channel model,
- a delay constraint τ, and
- a desired reliability P_e
  - block erasure probability (BEP)
  - packet erasure probability (PEP)

our approach is to:

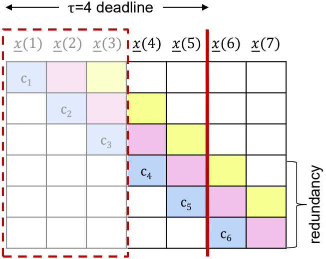
- select $(\hat{a}, \hat{b})$ parameters such that
  - $(\hat{a}, \hat{b}, \tau)$ streaming code achieves desired reliability P_e
  - results in highest rate among the set of (a, b) pairs that satisfy the reliability constraint
- can then employ a rate-optimal streaming code for $(\hat{a}, \hat{b}, \tau)$-DCSW channel.

How to Estimate Reliability of Streaming Codes?

Two Approaches to Constructing Streaming Codes

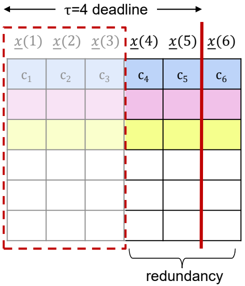
$$(a = 2, b = 3, \tau = 4)$$

Diagonal Embedding (DE)



DE of $(n = 6, k = 3)$ Scalar Code

Horizontal Embedding (HE)



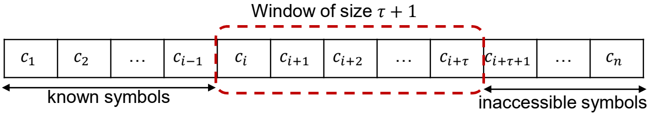
HE of $(n = 6, k = 3)$ Scalar Code

Parity-Check of Scalar Code:

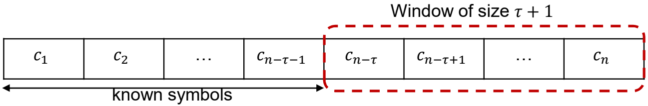
$$\begin{bmatrix} 1 & 0 & 0 & 1 & \alpha & 0 \\ 1 & 0 & c_{11} & c_{12} & c_{13} & 1 \\ 0 & 1 & c_{21} & c_{22} & c_{23} & 0 \end{bmatrix}$$

Properties Required from Embedded Scalar Code (ESC)

- i-th window for $i \in [n-\tau-1]$: Should be able to recover c_i from available symbols



- $n-\tau$-th window: Should be able to recover all the erased code symbols in this window



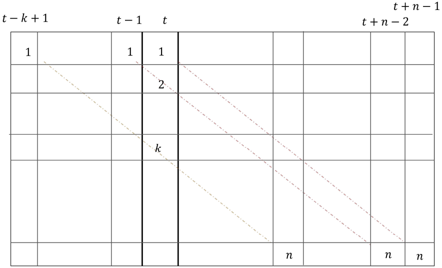
- There are rate-optimal ESC code constructions for any $\{a, b, \tau\}$ with field size $O(\tau^2)$ with $(n = \tau + 1 - a + b, k = n - b)$.

Two Approaches to Constructing Streaming Codes

- HE of ESC results in parity packet insertion approach. Can compute the block erasure probability (BEP) as follows:

$$\text{BEP}(n, a, b, \tau) = P(E_1^n \notin \text{AEP}) \triangleq \Delta$$

- DE of ESC results in packet expansion approach. Can use Δ to determine upper bounds on packet erasure probability (PEP).

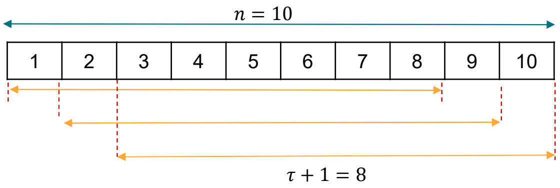


Trivial bound on PEP

$$\begin{aligned} PEP &= P(E_t = 1 \cap \cup_{i=1}^{k} D_i(E_{t-i+1}^{t-i+n})) \\ &\leq P\left((E_t = 1) \cap \left(\cup_{i=1}^{k}(E_{t-i+1}^{t-i+n} \notin AEP)\right)\right) \\ &\leq kP(E_1^n \notin AEP) \end{aligned}$$

Goal: To compute $P(E_1^n \notin \text{AEP})$.

Admissible Erasure Patterns



$$\text{AEP} = \cap_{i=1}^{n-\tau} (A_i \cup B_i)$$

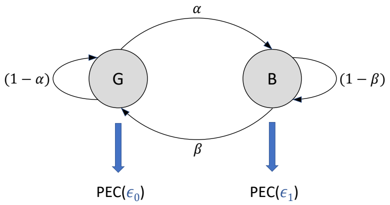
- A_i is the set of erasure patterns that have weight $\leq a$ in window $[i : i + \tau]$
- B_i is the set of erasure patterns that have span $\leq b$ in window $[i : i + \tau]$

Goal: To get a handle on the $P(\text{AEP})$

What is Known for GE Channels ?

- Closed form expression for $P(A_i)$ and $P(B_i)$ known.
- We provide an expression for $P(A_i \cup B_i)$
- We come up with bounds for $P(\text{AEP})$.

Computing Probability of an Erasure Pattern



Can show that

$$P(E_1^n = e_1^n) = 1^T \Psi(e_n) \cdots \Psi(e_1)\underline{\pi}$$

- $\underline{\pi} = [\frac{\beta}{\alpha+\beta} \quad \frac{\alpha}{\alpha+\beta}]$ is the stationary probability vector
- Ψ is defined as below:

$$\Psi(e) = \begin{cases} \Gamma S & e = 1 \\ (I - \Gamma)S & e = 0 \end{cases}$$

- $S = \underbrace{\begin{bmatrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{bmatrix}}_{\text{transitional probability matrix}}$ and $\Gamma = \begin{bmatrix} \epsilon_0 & \\ & \epsilon_1 \end{bmatrix}$.
- Notice that $\Psi(0) + \Psi(1) = S$

Computing Random Erasure Probabilities

- Let A be the set of erasures whose weight is atmost a in window of length n.

$$P(A) = \sum_{i=0}^{a} \underbrace{P(w(E_1^n) = i)}_{\text{closed form expression known}}$$

- BEP of an $[n, k = n - a]$ MDS code when used over GE channel is given by $1 - P(A)$.

C. Pimentel and I. F. Blake, "Enumeration of markov chains and burst error statistics for finite state channel models," IEEE Transactions on Vehicular Technology, 1999.

Computing Burst Erasure Probabilities

- Let B be the set of erasures whose span is atmost b in window of length n.
- Let b_i be the probability of erasures where the first erasure appears at index i and the span $\leq b$.

1	...	$i-1$	i	$i+1$	...	$i+b'-1$	$i+b'$	...	n
0	...	0	1	X	...	X	0	...	0

don't care

$$P(B) = P(E_1^n = \underline{0}) + \sum_{i=1}^{n} b_i$$

$$b_i = 1^T \Psi(0)^{n-i-b'+1} S^{b'-1} \Psi(1) \Psi(0)^{i-1} \underline{\pi}$$

where $b' = \min\{b, n-i+1\}$.

- Any cyclic code with parameters $[n, k = n-b]$ has BEP upper bounded by $1 - P(B)$.

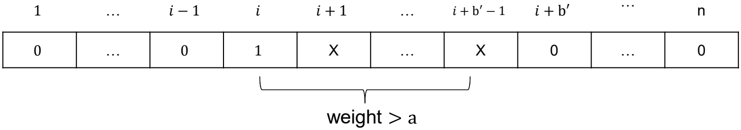
G. Haßlinger and O. Hohlfeld, "Analysis of random and burst error codes in 2-state markov channels," in 34th International Conference on Telecommunications and Signal Processing (TSP 2011).

Computing $P(A \cup B)$

- $A \cup B$ is the set of erasure patterns either have weight atmost a or span atmost b in window of length n.

$$P(A \cup B) \quad = \quad P(A) + P(B \setminus A) \triangleq P_{ws}(n, a, b)$$

- Let a_i be the probability of erasures where the first erasure appears at index i and the span $\leq b$ and weight $> a$.



$$\begin{aligned} P(B \setminus A) &= \sum_{i=1}^{n-a} a_i \\ a_i &= 1^T \Psi(0)^{n-i-b'+1} Q(b'-1, a-1) \Psi(1) \Psi(0)^{i-1} \underline{\pi} \end{aligned}$$

where $b' = \min\{b, n-i+1\}$

Bounding $P(\text{AEP})$

$$P(\text{AEP}) = P(\cap_{i=1}^{n-\tau}(A_i \cup B_i))$$

- $A \cup B$ is the set of erasure patterns that either have weight atmost a or span atmost b in a window $[1 : n]$.
- $A_i \cup B_i$ is the set of erasure patterns that either have weight atmost a or span atmost b in a window $[i : \tau + i]$.

$$\begin{array}{rcl} (A \cup B) \subseteq & \text{AEP} & \subseteq (A_1 \cup B_1) \\ P_{ws}(n, a, b) \leq & P(\text{AEP}) & \leq P_{ws}(\tau + 1, a, b) \end{array}$$

Bounds on BEP of streaming code

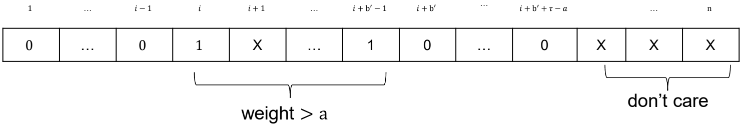
- Improved the bounds by coming up with tractable sets L, U such that:

$$\begin{aligned} (A \cup B) \subseteq L \subseteq \quad & \text{AEP} \quad \subseteq U \subseteq (A_1 \cup B_1) \\ 1 - P(U) \leq \quad & \text{BEP} \quad \leq 1 - P(L) \end{aligned}$$

$$L \quad \triangleq \quad L_A \cup L_B$$

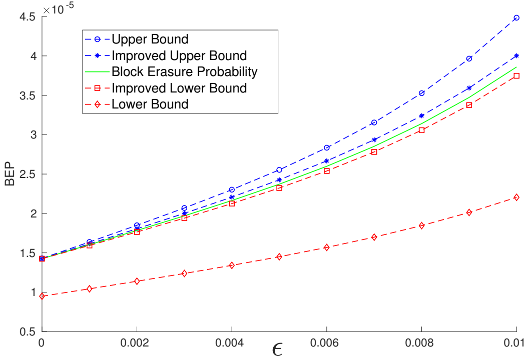
- L_A and L_B defined such that $A \subseteq L_A$ and $B \setminus A \subseteq L_B$

$$\begin{aligned} L_B &= \cup_{i=1}^{n-a} \cup_{b'=a+1}^{\min\{n-i+1,b\}} L_{B,i,b'} \\ L_{B,i,b'} &= \{e_1^n \mid w(e_1^{i-1}) = 0, e_i = e_{i+b'-1} = 1, \\ & \quad w(e_{i+b'}^{\min\{i+b'+\tau-a,n\}}) = 0, w(e_i^{i+b'-1}) > a\} \end{aligned}$$



Bounds on BEP of streaming code

$(a = 3, b = 6, \tau = 10)$ streaming code

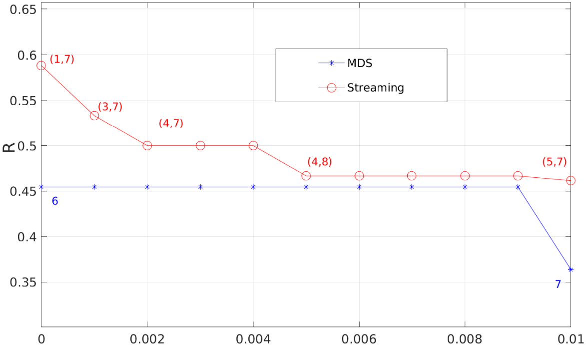


$\text{GE}(\alpha = 10^{-4}, \beta = 0.5, \epsilon_0 = \epsilon, \epsilon_1 = 1)$

Choosing a, b Using BEP Upper Bound

- (a, b) is picked to give best rate while meeting BEP$\leq P_e$ requirement for $(n = \tau + 1 + b - a, k = n - b)$ streaming code.

- For $[\tau + 1, \tau + 1 - a]$ MDS codes minimal value of a is picked to satisfy BEP requirement.



$$\text{GE}(\alpha = 10^{-4}, \beta = 0.5, \epsilon_0 = \epsilon, \epsilon_1 = 1),\ \tau = 10 \text{ and } P_e = 10^{-5}$$

Thanks!