Streaming Codes for Multi-Hop Relay Networks With Burst Erasures

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***Abstract*—Streaming codes are packet-level codes designed to ensure the timely recovery of lost packets. This paper focuses on streaming codes for multi-hop relay networks that guarantee the recovery of message packets within a delay of τ time slots, even when a burst erasure affecting at most b packets occurs on each link. We first present a straightforward upper bound on the rate of burst-erasure-correcting streaming codes for multi-hop relay networks. The main contribution of this paper is a coding scheme that achieves rates arbitrarily close to this rate upper bound as the message size increases.**

I. Introduction

Driven by applications such as interactive video streaming, multiplayer gaming, augmented reality, and telemedicine, the demand for low-latency, reliable communication systems has grown significantly in recent years. The 5G cellular framework identifies Ultra-Reliable Low-Latency Communication (URLLC) as a key enabler for many such applications. Traditional retransmission-based error control strategies, such as ARQ, are often less preferred in such systems, as they incur large round-trip delays. Forward Error Correction (FEC) offers a more efficient alternative by adding redundancy to the transmitted data upfront, avoiding the need for feedback and retransmission. Streaming codes, introduced by Martinian and Sundberg [1], are a class of FEC schemes designed to provide reliable, low-latency communication by guaranteeing the recovery of transmitted packets within a fixed delay τ. The work of Martinian and Sundberg led to extensive follow-up work [2]–[19] on streaming codes for point-to-point networks under various channel models. Motivated by applications such as distributed sensor networks, multimedia streaming, and edge computing, where multiple intermediate relays enhance reliability and coverage, this paper focuses on a multi-node setting. Specifically, we study a scenario where a source communicates to a destination through $M \geq 1$ relays.

A. Notation

Let $\mathbb{Z}$ denote the set of all integers and $\mathbb{N}$ the set of natural numbers $\{1, 2, \ldots\}$. Let $a, b \in \mathbb{Z}$. The notation $[a : b]$ represents the set $\{i \mid a \leq i \leq b\} \cap \mathbb{Z}$ and we define $[a] \coloneqq [1 : a]$. We use $\mathbb{F}_q$ to denote the finite field with q elements. Column vectors are represented using underlined symbols, such as $\underline{x}$ and $\underline{y}$. Specifically, $\underline{0}$ denotes the all-zero column vector, with its size inferred from the context. For a positive integer u, $\mathbb{F}_q^u$ represents the set of all column vectors of length u over $\mathbb{F}_q$. The $u \times u$ identity matrix is denoted by I_u and its columns are the standard basis vectors $\underline{e}_0, \underline{e}_1, \ldots, \underline{e}_{u-1}$, i.e., $I_u = [\underline{e}_0 \ \underline{e}_1 \ \ldots \ \underline{e}_{u-1}]$. For any subset $S \subseteq \mathbb{F}_q^u$, $\langle S \rangle$ denotes the linear span of the elements in S.

B. Problem Setup

We consider a multi-hop relay network consisting of $M+2$ nodes, where $M \geq 1$. There is a source node (denoted by r_0), M relay nodes $(r_1, r_2, \ldots, r_M)$ and a destination node (denoted by r_{M+1}). At time slot $t \in \mathbb{N} \cup \{0\}$, the source node r_0 generates a *message packet* $\underline{m}(t) \in \mathbb{F}_q^k$ and transmits a *coded packet* $\underline{x}^{(r_0)}(t) \in \mathbb{F}_q^{n_{r_0}}$ to the relay node r_1. Subsequently, each relay node r_i, where $i \in [M]$, sequentially transmits its own coded packet $\underline{x}^{(r_i)}(t) \in \mathbb{F}_q^{n_{r_i}}$ to the next node r_{i+1}. Here, the parameter $k \in \mathbb{N}$ represents the length of the message packet and $\{n_{r_j}\}_{j \in [0:M]} \subseteq \mathbb{N}$ denote the lengths of the coded packets.

All $(M+1)$ communication links $\{(r_j, r_{j+1})\}_{j \in [0:M]}$ are modeled as packet erasure channels. At time t, the node r_l, where $l \in [M+1]$, receives $\underline{y}^{(r_l)}(t) \in \mathbb{F}_q^{n_{r_{l-1}}} \cup \{\star\}$, where $\underline{y}^{(r_l)}(t) = \underline{x}^{(r_{l-1})}(t)$ or $\star$ (indicating a packet erasure). For $j \in [0 : M]$, the erasure pattern on the link (r_j, r_{j+1}) is described by the binary sequence $e^{(r_j, r_{j+1})} = \{e_u^{(r_j, r_{j+1})}\}_{u=0}^{\infty}$. Here, $e_t^{(r_j, r_{j+1})} = 1$ indicates an erasure at time t and $e_t^{(r_j, r_{j+1})} = 0$ otherwise. In this paper, we assume a bursty erasure channel model, where each link (r_j, r_{j+1}) experiences a single burst erasure of length at most b. In other words, each sequence $\{e_u^{(r_j, r_{j+1})}\}_{u=0}^{\infty}$ contains at most b ones, which occur within b consecutive time slots. The positions of these ones may vary across links.

For $j \in [0 : M]$, the node r_j employs an encoding function $E_t^{(r_j)}$ at time t, which is defined as follows:

$$\begin{aligned} \underline{x}^{(r_0)}(t) &\coloneqq E_t^{(r_0)}(\underline{m}(0), \ldots, \underline{m}(t)), \\ \underline{x}^{(r_i)}(t) &\coloneqq E_t^{(r_i)}(\underline{y}^{(r_i)}(0), \ldots, \underline{y}^{(r_i)}(t)), \end{aligned}$$

where $t \in \mathbb{N} \cup \{0\}$, $i \in [M]$. The function $E_t^{(r_i)}$ may depend on the erasure pattern observed by the node r_i up to time t, i.e., $\{e_u^{(r_{i-1}, r_i)}\}_{u=0}^{t}$.

The destination node r_{M+1} aims to decode each message packet $\underline{m}(t)$ by time $(t+\tau)$, i.e., with a delay of at most τ time slots. At time $(t+\tau)$, the destination node uses the decoding function $D_{t+\tau}$ to estimate $\underline{m}(t)$:

$$\underline{\tilde{m}}(t) \coloneqq D_{t+\tau}(\underline{y}^{(r_{M+1})}(0), \underline{y}^{(r_{M+1})}(1), \ldots, \underline{y}^{(r_{M+1})}(t+\tau)).$$

A *multi-node streaming code* is characterized by the sequence of tuples $\{(\{E_t^{(r_j)}\}_{j\in[0:M]}, D_t)\}_{t=0}^{\infty}$ and the delay parameter τ. The rate of such a code is given by $\min(\{\frac{k}{n_{r_j}} \mid j \in [0:M]\})$. This code is defined to be a (b,τ,M) *multi-node burst-erasure-correcting streaming code (MBSC)* if it guarantees $\underline{\tilde{m}}(t) = \underline{m}(t)$, for all $t \in \mathbb{N}\cup\{0\}$, even in the presence of a single burst erasure of length at most b on each link. It is also desirable, as is the case with the code presented in this paper, for the encoding and decoding functions to have a finite and preferably small memory requirements.

Remark 1. A (b,τ,M) MBSC also ensures $\underline{\tilde{m}}(t) = \underline{m}(t)$ for all $t \in \mathbb{N}\cup\{0\}$ under a more general channel model, where each link experiences a single burst erasure of length at most b within any sliding window of $\tau+1$ consecutive time slots. This result follows from arguments similar to those in [20].

C. Related Work

In [21], streaming codes for multi-hop relay networks are studied for cases where each (r_j, r_{j+1}) link, with $j \in [0:M]$, can have up to $N_{(r_j,r_{j+1})}$ arbitrary erasures. In contrast, the current paper focuses on links with burst erasures. If $N_{(r_j,r_{j+1})} = b$ for all $j \in [0:M]$, then the code in [21] is a (b,τ,M) MBSC with rate $\frac{\tau+1-(M+1)b}{\tau+1-Mb}$, where $\tau \geq (M+1)b$. It can be shown that this rate is sub-optimal when $b > 1$. Streaming codes for the case where there is a single relay between the source and destination (i.e., $M=1$) are studied in [20], [22]–[25]. In particular, the paper [25] presents a construction of $(b,\tau,M=1)$ MBSC for all valid $\{b,\tau\}$, achieving rates arbitrarily close to a rate upper bound with sufficiently large packet sizes. Our work non-trivially extends this construction to networks with multiple relays.

D. Our Contributions

We present a construction of a (b,τ,M) MBSC for all parameters $\{b,\tau,M\}$ such that $\tau \geq (M+1)b$. The rate of this code deviates from the upper bound in Lemma 1 due to the inclusion of a header. However, this discrepancy diminishes as the message packet length increases. The construction applies to any finite field; for simplicity, we use the binary field.

E. Organization of the Paper

In Section II, we present a rate upper bound for (b,τ,M) MBSCs and discuss a technique for constructing streaming codes from block codes. Section III provides an example of our MBSC construction. In Section IV, we introduce a new class of block codes, which are then used in Section V to construct a (b,τ,M) MBSC.

II. Preliminaries

This section derives a simple upper bound on the rate of MBSCs and summarizes the diagonal embedding technique used to obtain streaming codes from block codes.

A. Rate Upper Bound

In [1], streaming codes are studied for a setting without relay nodes between the source and destination, where the direct link can have a single burst erasure of length at most b. For this point-to-point (P2P) setting, the rate of streaming codes is upper bounded by $\frac{\tau}{\tau+b}$ if $\tau \geq b$ and 0 otherwise. Streaming codes achieving this rate upper bound are provided in [1], [2], [26]. Using the rate upper bound for the P2P setting, it is straightforward to obtain the following rate upper bound for (b,τ,M) MBSCs. We refer readers to the Appendix A of this paper for the proof.

Lemma 1. *The rate R of a (b,τ,M) MBSC is upper bounded as:*

$$R \leq \begin{cases} \frac{\tau-Mb}{\tau-(M-1)b} & \text{if } \tau \geq (M+1)b, \\ 0 & \text{otherwise}. \end{cases} \tag{1}$$

B. Diagonal Embedding

Diagonal embedding is a technique introduced in [2] to obtain packet-level codes from linear block codes. Consider the systematic generator matrix $G = [I_u \mid P]$ of an $[n,u]$ linear block code $\mathcal{C}$. Fix any positive integer L. We denote the message packet at time t as $\underline{m}(t) = [m_0(t)\ m_1(t) \ldots\ m_{uL-1}(t)]^T \in \mathbb{F}_q^{uL}$. For $t<0$, we assume that $\underline{m}(t) = \underline{0}$. A packet-level code of rate $\frac{u}{n}$ is then obtained by diagonal embedding of $\mathcal{C}$ in the following manner. For all $i \in [0:L-1]$ and $t \in \mathbb{Z}$,

$$\begin{aligned}&[x_{in}(t)\ x_{in+1}(t+1)\ \ldots\ x_{in+n-1}(t+n-1)] \\ &= [m_{iu}(t)\ m_{iu+1}(t+1)\ \ldots\ m_{iu+u-1}(t+u-1)]G,\end{aligned}$$

where $\underline{x}(t) = [x_0(t)\ x_1(t) \ldots\ x_{nL-1}(t)]^T \in \mathbb{F}_q^{nL}$ denotes the coded packet at time t.

III. An Example: $\{b=5, \tau=18, M=2\}$

In this section, we present an example MBSC construction for the parameters $\{b=5, \tau=18, M=2\}$. Since $M=2$, there are two relay nodes between the source and the destination. Let $\theta \coloneqq \tau - Mb = 18 - 5*2 = 8$. The source node ($r_0$) and relay nodes ($r_1$ and r_2) employ diagonal embedding of possibly distinct $[\theta+b=13, \theta=8]$ block codes to generate coded packets. In this example, for simplicity, we consider $L=1$ (see Sec. II-B) and ignore the header requirement. Thus, we have $k=8$, $n_{r_0} = n_{r_1} = n_{r_2} = 13$ and the rate is $\frac{8}{13}$, which matches the rate upper bound given by (1). In general, the header size can be kept constant with respect to L, and the rate can be made arbitrarily close to the optimal rate of $\frac{8}{13}$ by choosing a sufficiently large L.

Consider a time slot $t \in \mathbb{N}\cup\{0\}$. We define $m_i \coloneqq m_i(t+i) \in \mathbb{F}_2$, $i \in [0:7]$ and $p_j \coloneqq x_{8+j}^{(r_0)}(t+j+8) \in \mathbb{F}_2$, $j \in [0:4]$. Therefore, m_i and p_j are components of the coded packets

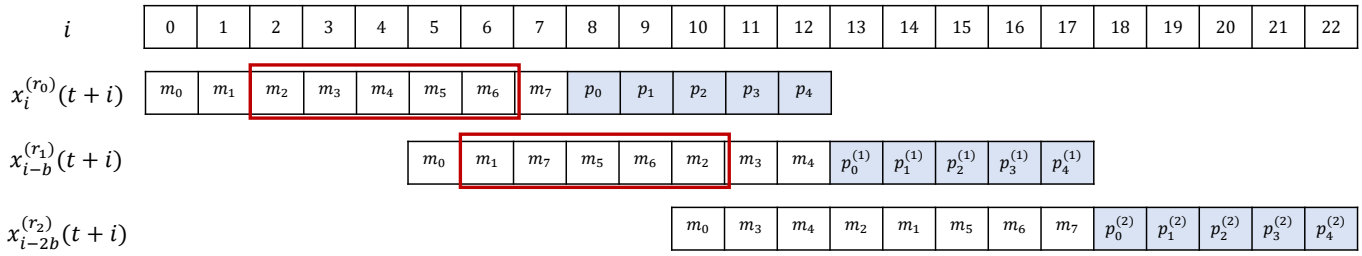

Fig. 1: The figure above illustrates an example scenario where a burst erasure of length $b = 5$ starts at time $t+2$ on the (r_0, r_1) link and at time $t+6$ on the (r_1, r_2) link, respectively. Symbols corresponding to a single diagonal in the packets transmitted from the source, relay r_1 and relay r_2 are shown.

$\underline{x}^{(r_0)}(t+i)$ and $\underline{x}^{(r_0)}(t+j+8)$, respectively, as depicted in Fig. 1. Our aim is to show that the destination can recover m_i by time $(t+i+18)$. Here, we show this only for a specific burst erasure case on the (r_0, r_1) and (r_1, r_2) links, where a burst erasure of length 5 starts at time $t+2$ on the (r_0, r_1) link and at time $t+6$ on the (r_1, r_2) link. We consider all burst erasure scenarios for the (r_2, r_3) link.

Source (r_0)*:* The source node r_0 uses the $[13, 8]$ binary linear code defined by the generator matrix G_{r_0} shown in (2) below, in conjunction with diagonal embedding to create the coded packets:

$$G_{r_0} = \left[\begin{array}{c|c} & I_5 \\ \hline I_8 & \begin{array}{c|c} I_3 & I_2 \\ \hline & \begin{array}{c|c} 1 & 1 \end{array} \end{array} \end{array}\right]. \tag{2}$$

We have $[m_0 \ \cdots \ m_7 \ p_0 \ \cdots \ p_4] = [m_0 \ \cdots \ m_7]G_{r_0}$. Therefore, $p_0 = m_0 + m_5$, $p_1 = m_1 + m_6$, $p_2 = m_2 + m_7$, $p_3 = m_3 + m_5 + m_7$, $p_4 = m_4 + m_6 + m_7$.

First Relay (r_1)*:* Since we consider a burst erasure of length 5 starting at time $t+2$ on the (r_0, r_1) link (see Fig. 1), the first relay receives symbols $m_0, m_1, m_7, p_0, p_1, p_2, p_3, p_4$ at times $t, t+1, t+7, t+8, \cdots, t+12$ respectively. Notice that from the definition of p_0, it follows that at time $t+8$, m_5 can be recovered, as the relay already has access to m_0, which it received at time t. Similarly, m_6, m_2, m_3, m_4 can be recovered at times $t+9, t+10, t+11, t+12$, respectively. Let

$$\underline{m}^{(1)} = [m_0^{(1)} \ \cdots \ m_7^{(1)}] := [m_0 \ m_1 \ m_7 \ m_5 \ m_6 \ m_2 \ m_3 \ m_4]. \tag{3}$$

The relay r_1 can transmit messages $m_i^{(1)}$ at time $t+5+i$ for $i \in [0:7]$, as these message are recovered and readily available. Therefore, let us define $x_i^{(r_1)}(t+5+i) := m_i^{(1)}$. Encoding at the relay is done using a $[13, 8]$ binary linear code defined by the systematic generator matrix G_{r_1} in (4). Let $[m_0^{(1)} \ \cdots \ m_7^{(1)} \ p_0^{(1)} \ \cdots \ p_4^{(1)}] = [m_0^{(1)} \ \cdots \ m_7^{(1)}]G_{r_1}$, and $x_{8+j}^{(r_1)}(t+13+j) = p_j^{(1)}$, where $j \in [0:4]$ and

$$G_{r_1} = \left[\begin{array}{c|c|c|c} & I_2 & & \\ \hline & & & 1 \\ \hline & & I_2 & \\ \hline I_8 & I_2 & I_2 & \\ \hline & & & 1 \end{array}\right]. \tag{4}$$

From (3) and (4), it follows that $p_0^{(1)} = m_0 + m_2$, $p_1^{(1)} = m_1 + m_3$, $p_2^{(1)} = m_2 + m_5$, $p_3^{(1)} = m_3 + m_6$ and $p_4^{(1)} = m_4 + m_7$.

Second Relay (r_2)*:* We consider a burst erasure of length 5 starting at time $(t+6)$ on the (r_1, r_2) link (see Fig. 1). The second relay receives symbols $m_0^{(1)} = m_0, m_6^{(1)} = m_3, m_7^{(1)} = m_4, p_0^{(1)}, \cdots, p_4^{(1)}$ at times $t+5, t+11, t+12, t+13, \cdots, t+17$, respectively. At time $t+13$, the relay can recover m_2 from $p_0^{(1)}$ and m_0, which it already received. Similarly, the symbols m_1, m_5, m_6, m_7 can be recovered at times $t+14, t+15, t+16, t+17$, respectively. Let

$$\underline{m}^{(2)} = [m_0^{(2)} \ \cdots \ m_7^{(2)}] := [m_0 \ m_3 \ m_4 \ m_2 \ m_1 \ m_5 \ m_6 \ m_7]. \tag{5}$$

The second relay can transmit messages $m_i^{(2)}$ at time $t+10+i$ for $i \in [0:7]$, as these messages are available. Therefore, we define $x_i^{(r_2)}(t+10+i) = m_i^{(2)}$. Encoding at the relay is done using a $[13, 8]$ binary linear code defined by the systematic generator matrix G_{r_2} in (6). Let $[m_0^{(2)} \ \cdots \ m_7^{(2)} \ p_0^{(2)} \ \cdots \ p_4^{(2)}] = [m_0^{(2)} \ \cdots \ m_7^{(2)}]G_{r_2}$, and $x_{8+j}^{(r_2)}(t+18+j) = p_j^{(2)}$, where $j \in [0:4]$ and

$$G_{r_2} = \left[\begin{array}{c|c|c|c} & 1 & & \\ \hline & & & I_2 \\ \hline & & 1 & \\ \hline & 1 & & \\ \hline I_8 & 1 & 1 & \\ \hline & & & I_2 \end{array}\right]. \tag{6}$$

From (5) and (6), it follows that $p_0^{(2)} = m_0 + m_5$, $p_1^{(2)} = m_1$, $p_2^{(2)} = m_2 + m_5$, $p_3^{(2)} = m_3 + m_6$, $p_4^{(2)} = m_4 + m_7$.

Destination (r_3)*:* It remains to show that each message symbol m_i can be recovered by time $(t+18+i)$ at the destination, irrespective of the position of a burst erasure of length 5 on the (r_2, r_3) link.

$\underline{m_0}$: m_0, m_5 and $p_0^{(2)}$ are transmitted at times $t+10$, $t+15$ and $t+18$, respectively. If m_0 is erased by the burst erasure, it implies that symbols m_5 and $p_0^{(2)}$ are available. Hence, m_0 can be recovered by time $t+18$.

$\underline{m_1}$: m_1 and $p_1^{(2)}$ are transmitted at times $t+14$ and $t+19$, respectively. If m_1 is erased, $p_1^{(2)}$ is available and m_1 can therefore be recovered by time $t+19$.

$\underline{m_2}$: $m_0, m_2, m_5, p_0^{(2)}$ and $p_2^{(2)}$ are transmitted at times $t+10, t+13, t+15, t+18$ and $t+20$, respectively. If m_2 is erased, then $p_0^{(2)}$ and $p_2^{(2)}$ are available. If m_5 is not erased, m_2 can be recovered from $p_2^{(2)}$. If m_5 is erased, then m_0 is available and $p_0^{(2)}$ can be used to recover m_5 first, which can then be used to recover m_2 by time $t+20$.

$\underline{m_3, m_6}$: m_3, m_6 and $p_3^{(2)}$ are transmitted at times $t+11, t+16$, and $t+21$, respectively. Only one of these three symbols can be erased by a burst erasure of length 5. Therefore, m_3 and m_6 can be recovered by time $t+21$.
$\underline{m_4, m_7}$: m_4, m_7 and $p_4^{(2)}$ are transmitted at times $t+12, t+17$ and $t+22$, respectively. As in the previous case, it follows that m_4 and m_7 can be recovered by time $t+22$.
$\underline{m_5}$: $m_0, m_2, m_5, p_0^{(2)}$ and $p_2^{(2)}$ are sent at $t+10, t+13, t+15, t+18$ and $t+20$, respectively. Assume that m_5 is erased. Thus, m_0 and $p_2^{(2)}$ are not erased. If m_2 is available, m_5 can be recovered from $p_2^{(2)}$. If m_2 is erased, then $p_0^{(2)}$ is available, as m_2 and $p_0^{(2)}$ are 5 time slots apart. Therefore, m_5 can be recovered from $p_0^{(2)}$ and m_0.

IV. Block Code Construction

We now present a class of systematic generator matrices that will be used in Section V to construct MBSCs. Let θ, b be positive integers with $\theta \geq b$. Let $\sigma : [0:\theta-1] \to [0:\theta-1]$ be a permutation with $\sigma(i) \leq \theta - b + i$ for all $i \in [0:b-2]$. Our aim is to construct a $\theta \times (\theta+b)$ systematic generator matrix $G_{\theta,b,\sigma} = [I_\theta \mid P_{\theta,b,\sigma}]$ for a $[\theta+b, \theta]$ binary linear block code $\mathcal{C}_{\theta,b,\sigma}$. Let $\underline{g}_u$ denote the $u-$th column of $G_{\theta,b,\sigma}$, for $u \in [0:\theta+b-1]$. We want $G_{\theta,b,\sigma}$ to satisfy the following two properties.

- *Strong-Recovery*: Fix any $i \in [0:\theta-1]$ and $j \in [0:b-1]$. If $P_{\theta,b,\sigma}(i,j) = 1$, then
$$\underline{e}_i \in \langle \{\underline{g}_u \mid u \in [0:\theta+j] \setminus \mathcal{B}\}\rangle,$$
for all $\mathcal{B} \subseteq [0:\theta+j-1]$ with $\max(\mathcal{B}) - \min(\mathcal{B}) + 1 \leq b$.
- *Timely-Recovery*: For all $i \in [0:\theta-1]$,
$$\underline{e}_{\sigma(i)} \in \langle \{\underline{g}_u \mid u \in [0:\min(\{\theta+i, \theta+b-1\})] \setminus \mathcal{B}\}\rangle,$$
for all $\mathcal{B} \subseteq [0:\theta+b-1]$ with $\max(\mathcal{B}) - \min(\mathcal{B}) + 1 \leq b$.

We construct the generator matrix $G_{\theta,b,\sigma}$ in $b-1$ steps. We start with a $[\theta+b, \theta]$ matrix $G^{(-1)} = [I_\theta \mid P^{(-1)}]$. In each step $\ell \in [0:b-2]$, we construct a $[\theta+b, \theta]$ matrix $G^{(\ell)} = [I_\theta \mid P^{(\ell)}]$. After the $b-1$ steps, we define $G_{\theta,b,\sigma} \coloneqq G^{(b-2)}$.

To describe $G^{(\ell)}$, it is sufficient to specify the support sets of the columns in P^ℓ, i.e., $\mathcal{P}_j^{(\ell)} \coloneqq \{i \in [0:\theta-1] \mid P^{(\ell)}(i,j) = 1\}$, where $j \in [0:b-1]$. We begin the construction by initializing,
$$\mathcal{P}_j^{(-1)} = \{\theta - ib + j \mid i \in \mathbb{N}, \theta - ib + j \geq 0\}, \tag{7}$$
for all $j \in [0:b-1]$. In the ℓ-th step, $\ell \in [0:b-2]$, we obtain $\{\mathcal{P}_j^{(\ell)} \mid j \in [0:b-1]\}$ from $\{\mathcal{P}_j^{(\ell-1)} \mid j \in [0:b-1]\}$ as follows. We first determine $\beta_\ell = \max\{j \in [0:b-1] \mid \sigma(\ell) \in \mathcal{P}_j^{(\ell-1)}\}$. If $\beta_\ell \leq \ell$, then $\mathcal{P}_j^{(\ell)} = \mathcal{P}_j^{(\ell-1)}$ for all $j \in [0:b-1]$. On the other hand, if $\beta_\ell > \ell$, then
$$\mathcal{P}_j^{(\ell)} = \begin{cases} \mathcal{P}_{\beta_\ell}^{(\ell-1)} & \text{if } j = \ell, \\ \mathcal{P}_\ell^{(\ell-1)} \cup \{\theta - b + \beta_\ell\} & \text{if } j = \beta_\ell \\ \mathcal{P}_j^{(\ell-1)} & \text{else.} \end{cases} \tag{8}$$
We note that $\cup_{j=0}^{b-1} \mathcal{P}_j^{(\ell)} = [0:\theta-1]$ for every $\ell \in [-1:b-2]$ and therefore β_ℓ is always well defined. It can also be seen that for all $j > \ell$, $\max(\mathcal{P}_j^{(\ell)}) = \theta - b + j$.

We now illustrate how the above construction method yields the generator matrices used in the MBSC example in Section III.

Example: Let $\theta = 8, b = 5, \sigma = (0,4,3,1,2,5,6,7)$, i.e., $\sigma(0) = 0$, $\sigma(1) = 4$ and so on. Notice that σ satisfies the requirement $\sigma(i) \leq \theta + i - b = 3 + i$ for $i \in [0:3]$. We will now construct $G_{\theta,b,\sigma}$ recursively by first defining the supports of the parity portion of $G^{(-1)}$ as shown below:
$$\mathcal{P}_0^{(-1)} = \{3\}, \mathcal{P}_1^{(-1)} = \{4\}, \mathcal{P}_2^{(-1)} = \{0,5\}, \mathcal{P}_3^{(-1)} = \{1,6\}, \mathcal{P}_4^{(-1)} = \{2,7\}.$$
Since $\sigma(0) = 0 \in \mathcal{P}_2^{(-1)}$, we have $\beta_0 = 2$ and from (8), we have
$$\mathcal{P}_0^{(0)} = \{0,5\}, \mathcal{P}_1^{(0)} = \{4\}, \mathcal{P}_2^{(0)} = \{3,5\}, \mathcal{P}_3^{(0)} = \{1,6\}, \mathcal{P}_4^{(0)} = \{2,7\}. \tag{9}$$
Now, it can be seen that $\sigma(1) = 4 \in \mathcal{P}_1^{(0)}$ i.e, $\beta_1 = 1$. Therefore, no update is required, i.e., $\mathcal{P}_i^{(1)} = \mathcal{P}_i^{(0)}$ for all $i \in [0:4]$. Similarly it can be checked that for the steps $\ell = 2, 3$ as well there will be no update as $\sigma(2) = 3 \in \mathcal{P}_2^{(1)}$ and $\sigma(3) = 1 \in \mathcal{P}_3^{(1)}$. Therefore, the resultant generator matrix $G_{8,5,\sigma} = G^{(0)}$ and has parity support as shown in (9). It can be verified that $G_{8,5,\sigma} = G_{r_2}$, which is used by the second relay (see (6)) in the example. It can also be seen that the permutation σ is such that $\sigma(i)$ identifies the position of m_i in the vector $\underline{m}^{(2)}$ in (5).

It can be checked that for the identity permutation $\mathbb{I} = (0,1,\cdots,7)$, the resultant generator matrix $G_{8,5,\mathbb{I}}$ is equal to G_{r_0} in (2), which is used in the example to encode at the source. Similarly, for the permutation $\hat{\sigma} = (0,1,5,6,7,3,4,2)$, which identifies the positions of m_i's in the vector $\underline{m}^{(1)}$ (see (3)), $G_{8,5,\hat{\sigma}} = G_{r_1}$ as defined in (4).

Remark 2. When σ is the identity permutation, our block code construction reduces to the code presented in [26]. If σ satisfies the condition that there exists a $j \in [0:b-2]$ such that $\sigma(i) = i$ for $i \in [0:j]$ and $\sigma(i) = \theta - b + i$ for $i \in [j+1:b-2]$, our construction reduces to the one in [25]. For $M > 1$, the MBSC construction presented in Section V requires permutations that do not satisfy this condition. In the above example, $\sigma = (0,4,3,1,2,5,6,7)$ does not satisfy the condition and hence $G_{8,5,\sigma}$ used by the second relay cannot be obtained from the construction in [25].

We now focus on proving that $G_{\theta,b,\sigma}$ possesses the two recovery properties stated previously. To prove the strong-recovery property, we use the following lemma.

Lemma 2. *For any $\ell \in [-1:b-2]$, $G^{(\ell)}$ satisfies the following property. Let $\underline{g}_u^{(\ell)}$ denote the $u-$th column of $G^{(\ell)}$, for $u \in [0:\theta+b-1]$. If $i \in \mathcal{P}_j^{(\ell)}$, $j \in [0:b-1]$, then*
$$\underline{e}_i \in \left\langle \left\{\underline{g}_u^{(\ell)} \mid u \in \left(\{\theta+j\} \cup [0:\theta+\min\{\ell, j-1\}]\right) \setminus \mathcal{B}\right\}\right\rangle,$$
for all $\mathcal{B} \subseteq [0:\theta+j-1]$ with $\max(\mathcal{B}) - \min(\mathcal{B}) + 1 \leq b$.

Proof: See the Appendix B. □

In particular, for $\ell = b-2$, Lemma 2 implies the strong-recovery property for $G_{\theta,b,\sigma} = G^{(b-2)}$. We will now show the timely-recovery property for $G_{\theta,b,\sigma}$ by using the strong-recovery and the following lemma.

Lemma 3. *For any $i \in [0:\theta-1]$, there exists a $j \in [0:b-1]$ such that $P_{\theta,b,\sigma}(i,j) = 1$ and $\theta + j \geq i + b$. Furthermore, for $i \in [0:b-2]$, there exists a $j \in [0:i]$ such that $P_{\theta,b,\sigma}(\sigma(i),j) = 1$ and $\theta + j \geq \sigma(i) + b$.*

Proof: See the Appendix C. □

Fix any $\mathcal{B} \subseteq [0:\theta+b-1]$ with $\max(\mathcal{B}) - \min(\mathcal{B}) + 1 \leq b$. We first argue that for all $i \in [0:\theta-1]$,

$$\underline{e}_{\sigma(i)} \in \langle \{\underline{g}_u \mid u \in [0:\theta+b-1] \setminus \mathcal{B}\} \rangle. \tag{10}$$

If $\sigma(i) \notin \mathcal{B}$, this is trivially true. If $\sigma(i) \in \mathcal{B}$, then by the first part of Lemma 3, there exists a $j \in [0:b-1]$ such that $\max(\mathcal{B}) \leq \theta + j - 1$ and $P_{\theta,b,\sigma}(\sigma(i),j) = 1$. It follows from the strong-recovery property that (10) holds. To complete the proof of timely-recovery, it remains to show that for every $i \in [0:b-2]$,

$$\underline{e}_{\sigma(i)} \in \langle \{\underline{g}_u \mid u \in [0:\theta+i] \setminus \mathcal{B}\} \rangle. \tag{11}$$

Again, if $\sigma(i) \notin \mathcal{B}$, this is trivially true. If $\sigma(i) \in \mathcal{B}$, then by the second part of Lemma 3, there exists a $j \in [0:i]$ such that $\max(\mathcal{B}) \leq \theta + j - 1$ and $P_{\theta,b,\sigma}(\sigma(i),j) = 1$. It follows from the strong-recovery property that (11) holds.

Using the two recovery properties, we obtain the following lemma which is crucial for our streaming code construction.

Lemma 4. *Let θ, b be positive integers with $\theta \geq b$ and $\sigma : [0:\theta-1] \to [0:\theta-1]$ be a permutation with $\sigma(i) \leq \theta - b + i$ for all $i \in [0:b-2]$. Let $[c_0\ c_1\ \dots\ c_{\theta+b-1}] = [m_{\sigma^{-1}(0)}\ m_{\sigma^{-1}(1)}\ \dots\ m_{\sigma^{-1}(\theta-1)}]G_{\theta,b,\sigma}$. For each $j \in [0:\theta-1]$, there is a unique permutation $\sigma^{(j)} : [0:\theta-1] \to [0:\theta-1]$ such that*

- *$\sigma^{(j)}(i) = \sigma(i)$ for all $i \in [0:j-1]$,*
- *$\sigma^{(j)}(i) \leq \theta - b + i$ for all $i \in [0:b-2]$ and*
- *$\{m_{(\sigma^{(j)})^{-1}(u)} \mid u \in [0:i]\}$ is recoverable from $\{c_u \mid u \in [0:i+b] \setminus [j:j+b-1]\}$, for all $i \in [0:\theta-1]$.*

We will refer to $\sigma^{(j)}$ as the $j-$variant of σ.

Proof: See the Appendix D. □

It can be verified that $\sigma = (0,4,3,1,2,5,6,7)$ is the 1-variant of $\hat{\sigma} = (0,1,5,6,7,3,4,2)$ and $\hat{\sigma}$ is 2-variant of identity permutation $\mathbb{I}$. The permutations $\mathbb{I}, \sigma, \hat{\sigma}$ appear at nodes r_0, r_1, r_2 respectively.

V. Streaming Code Construction

In this section, we present our MBSC construction, which works for all $\{b, \tau, M\}$ with $\tau \geq (M+1)b$. Recall from Lemma 1 that non-zero rates are possible only if $\tau \geq (M+1)b$. Let $\theta = \tau - Mb$ and $0 < b \leq \theta$. For a positive integer L, we set $k = \theta L$, $n_s = (\theta+b)L$ and $n_i = (\theta+b)L + \delta$ for all $i \in [1:M]$. We will add a header $\underline{h}^{(r_i)}(t) \in \mathbb{F}_2^{\delta}$ to the coded packet transmitted at time t by relay r_i. Let $\underline{m}(t) = [m_0(t)\ m_1(t) \dots\ m_{\theta L-1}(t)]^T \in \mathbb{F}_2^{\theta L}$, $\underline{x}^{(r_0)}(t) = [x_0^{(r_0)}(t)\ x_1^{(r_0)}(t) \dots\ x_{(\theta+b)L-1}^{(r_0)}(t)]^T \in \mathbb{F}_2^{(\theta+b)L}$ and $\underline{x}^{(r_i)}(t) = [x_0^{(r_i)}(t)\ x_1^{(r_i)}(t) \dots\ x_{(\theta+b)L-1}^{(r_i)}(t)\ \underline{h}(t)^T]^T \in \mathbb{F}_2^{(\theta+b)L+\delta}$, for all $i \in [1:M]$. For $t < 0$, we set $\underline{m}(t) = \underline{0}$.

A. Encoding at the Source

The encoding at the source for any $t \in \mathbb{Z}$ and $\ell \in [0:L-1]$ can be described as follows:

$$\begin{aligned}&[x_{\ell(\theta+b)}^{(r_0)}(t)\ x_{\ell(\theta+b)+1}^{(r_0)}(t+1)\ \dots\ x_{(\ell+1)(\theta+b)-1}^{(r_0)}(t+\theta+b-1)]\\ &\quad = [m_{\ell\theta}(t)\ m_{\ell\theta+1}(t+1)\ \dots\ m_{\ell\theta+\theta-1}(t+\theta-1)]G_{\theta,b,\mathbb{I}}.\end{aligned}$$

Here $\mathbb{I}$ is the the identity permutation, i.e., $\mathbb{I}(i) = i$. Since the delay requirement is τ, we want $m_{\ell\theta+j}(t+j)$ to be recoverable at the destination by time $t + j + \tau$.

B. Encoding at the Relay r_i

The encoding at the relay r_i is given by

$$\begin{aligned}&[x_{\ell(\theta+b)}^{(r_i)}(t+ib)\ x_{\ell(\theta+b)+1}^{(r_i)}(t+ib+1)\ \dots\ x_{(\ell+1)(\theta+b)-1}^{(r_i)}(t+ib+\theta+b-1)]\\ &\quad = [\hat{m}_0\ \hat{m}_1\ \cdots\ \hat{m}_{\theta-1}]G_{\theta,b,\pi_{t,i}} \text{ where } \hat{m}_j = m_{\ell\theta+\pi_{t,i}^{-1}(j)}(t+\pi_{t,i}^{-1}(j)).\end{aligned}$$

The permutation $\pi_{t,i}$ is chosen in the following fashion. Let $\pi_{t,0} = \mathbb{I}$. If $e_{t+(i-1)b+u}^{(r_{i-1},r_i)} = 0$ for all $u \in [0:\theta-1]$, then $\pi_{t,i} = \pi_{t,i-1}$. Else, define $\alpha_{t,i} = \min\{u \in [0:\theta-1] \mid \min e_{t+(i-1)b+u}^{(r_{i-1},r_i)} = 1\}$. Then, $\pi_{t,i} = \pi_{t,i-1}^{(\alpha_{t,i})}$, where $\pi_{t,i-1}^{(\alpha)}$ is the $\alpha_{t,i}-$variant of $\pi_{t,i}$ given by Lemma 4.

By definition, $G_{\theta,b,\pi_{t,i}}$ is a systematic generator matrix. The following lemma ensures that the relay r_i can send the systematic symbols on time.

Lemma 5. *The relay r_i is capable of recovering $m_{\ell\theta+\pi_{t,i}^{-1}(j)}(t+\pi_{t,i}^{-1}(j))$ by time $t+ib+j$.*

Proof: See the Appendix E □

The purpose of the header is to convey $\pi_{t,i-1}$ to relay r_i. A permutation can be represented using $\theta\lceil\log_2(\theta)\rceil$ bits. If $\pi_{t,i-1}$ is included in $\underline{h}^{(r_{i-1})}(t+(i-1)b+u)$, for all $u \in [\theta-1:\theta+b-1]$, then the permutation $\pi_{t,i-1}$ will be available at relay r_i by time $t+ib+\theta-1$ even in the presence of a burst erasure in the (r_{i-1}, r_i) link. The relay r_i can also compute $\alpha_{t,i}$ by time $t+ib+\theta-1$. Thus, the relay r_i can determine the generator matrix $G_{\theta,b,\pi_{t,i}}$ by time $t+ib+\theta-1$ and can generate the required parity symbols on time for transmission. This scheme requires a header of size $\delta = (b+1)\theta\lceil\log_2(\theta)\rceil$ to be added to each coded packet transmitted by relays.

C. Recovery at the Destination

By treating the destination as r_{M+1}, one can define permutation $\pi_{t,M+1}$ in the same manner as above. It follows from Lemma 5 that the destination can recover $m_{\ell\theta+\pi_{t,M+1}^{-1}(u)}(t+\pi_{t,M+1}^{-1}(u))$ by time $t+(M+1)b+u = t+\tau-\theta+b+u$. Therefore, every $m_{\ell\theta+j}(t+j)$, $j \in [0:\theta-1]$, is recoverable at the destination by time $t+\tau-\theta+b+\pi_{t,M+1}(j) \leq t+\tau-\theta+b+\theta-1 = t+\tau+b-1$. It remains to argue that the delay constraint is satisfied for $m_{\ell\theta+j}(t+j)$ with $j \in [0:b-2]$. From Lemma 4, we have $\pi_{t,M+1}(j) \leq \theta-b+j$ for all $j \in [0:b-2]$. It follows that, for all

$j \in [0 : b-2]$, $m_{\ell\theta+j}(t+j)$ is recoverable at the destination by time $t+\tau-\theta+b+\theta-b+j = t+\tau+j$.

The rate of construction is $R = \frac{\theta L}{(\theta+b)L+(b+1)\theta\lceil\log_2(\theta)\rceil}$. It is easy to see that as message packet size increases i.e., $L \to \infty$, the rate R approaches the rate upper bound (see Lemma 1).

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APPENDIX

A. Proof of Lemma 1

Let us define:

$$R_{\max}(b,\tau,M) = \begin{cases} \frac{\tau-Mb}{\tau-(M-1)b} & \tau \geq (M+1)b \\ 0 & \text{otherwise} \end{cases}$$

We would like to show that rate of any (b,τ,M) MBSC is upper bounded by $R_{\max}(b,\tau,M)$. We will prove this bound by induction on the number of relays M. The upper bound holds true for the $M=1$ case from [25]. Let us assume that rate of a $(b,\tau,M-1)$ MBSC is upper bound by: $R_{\max}(b,\tau,M-1)$.

Suppose there exists a (b,τ,M) MBSC with message packet length k and coded packet lengths n_{r_i} for $i \in [0:M]$. For some, $t \in \mathbb{N}$, consider the decoding of $\underline{m}(t)$ at the destination node $d = r_{M+1}$ under the following erasure pattern in the (r_M, r_{M+1}) link:

$$e_j^{(r_M,r_{M+1})} = \begin{cases} 1 & j \in [t+\tau-b+1 : t+\tau] \\ 0 & \text{otherwise} \end{cases}$$

For the destination node to decode $m(t)$ as time $t+\tau$, under this erasure pattern the relay r_M must be able to recover $m(t)$ by time $t+\tau-b$. This is equivalent to having a delay gaurantee of $\tau-b$ over multi-hop relay network with $M-1$ relays. This results in the upper bound:

$$R \leq R_{\max}(b,\tau-b,M-1) = \begin{cases} \frac{\tau-Mb}{\tau-(M-1)b} & \tau \geq (M+1)b \\ 0 & \text{otherwise.} \end{cases}$$

B. Proof of Lemma 2

We prove the lemma by induction. It is easy to see that if $i, i' \in \mathcal{P}_j^{(-1)}$, then $|i-i'| \geq b$. Therefore, the statement of the lemma is true for $G^{(-1)}$. This will serve as the base case for induction. As induction assumption, we will assume that the statement is true for $G^{(\ell-1)}$. To prove the lemma, one has to show that $G^{(\ell)}$ satisfies the lemma statement. Suppose $\beta_\ell \leq \ell$, then $G^{(\ell)} = G^{(\ell-1)}$ and the lemma statement holds. Therefore, we need to only consider $\beta_\ell > \ell$. We divide proof into three cases.

a) $j \in [0 : b-1] \setminus \{\ell, \beta_\ell\}$*:* In this case, for all $u \in \{\theta + j\} \cup [0 : \theta - 1] \cup [\theta : \theta + \min\{\ell - 1, j - 1\}]$, we have $\underline{g}_u^{(\ell)} = \underline{g}_u^{(\ell-1)}$. Therefore for $j \in [0 : b-1] \setminus \{\ell, \beta_\ell\}$, if the lemma statement is true for $G^{(\ell-1)}$, then it is true for $G^{(\ell)}$ as well.

b) $j = \ell$*:* Consider any $\mathcal{B} \subseteq [0 : \theta + \ell - 1]$ with $\max(\mathcal{B}) - \min(\mathcal{B}) + 1 \leq b$. Let $i \in \mathcal{P}_\ell^{(\ell)}$. Then $i \in \mathcal{P}_{\beta_\ell}^{(\ell-1)}$. Recall that $\beta_\ell > \ell$. By induction assumption,

$$\underline{e}_i \in \left\langle \left\{ \underline{g}_u^{(\ell-1)} \mid u \in \left(\{\theta + \beta_\ell\} \cup [0 : \theta - 1] \cup [\theta : \theta + \ell - 1] \right) \setminus \mathcal{B} \right\} \right\rangle.$$

For all $u \in [0 : \theta - 1] \cup [\theta : \theta + \ell - 1]$, we have $\underline{g}_u^{(\ell)} = \underline{g}_u^{(\ell-1)}$. We also have $\underline{g}_{\theta+\ell}^{(\ell)} = \underline{g}_{\theta+\beta_\ell}^{(\ell-1)}$. Therefore,

$$\underline{e}_i \in \left\langle \left\{ \underline{g}_u^{(\ell)} \mid u \in \left(\{\theta + \ell\} \cup [0 : \theta - 1] \cup [\theta : \theta + \ell - 1] \right) \setminus \mathcal{B} \right\} \right\rangle.$$

Thus we have argued the $j = \ell$ case.

c) $j = \beta_\ell$*:* Consider any $\mathcal{B} \subseteq [0 : \theta + \beta_\ell - 1]$ with $\max(\mathcal{B}) - \min(\mathcal{B}) + 1 \leq b$. We look at two sub-cases.

$\underline{\theta - b + \beta_\ell \notin \mathcal{B}}$ **:** Let $i \in \mathcal{P}_{\beta_\ell}^{(\ell)}$. If $i = \theta - b + \beta_\ell$, then the statement is trivially true. Suppose $i \in \mathcal{P}_{\beta_\ell}^{(\ell)} \setminus \{\theta - b + \beta_\ell\}$. Then, $i \in \mathcal{P}_\ell^{(\ell-1)}$. Let $\mathcal{B}' = \mathcal{B} \cap [0 : \theta + \ell - 1]$. By induction assumption,

$$\underline{e}_i \in \left\langle \left\{ \underline{g}_u^{(\ell-1)} \mid u \in \left(\{\theta + \ell\} \cup [0 : \theta - 1] \cup [\theta : \theta + \ell - 1] \right) \setminus \mathcal{B}' \right\} \right\rangle.$$

For all $u \in [0 : \theta - 1] \cup [\theta : \theta + \ell - 1]$, we have $\underline{g}_u^{(\ell)} = \underline{g}_u^{(\ell-1)}$. Moreover, $\underline{g}_{\theta+\beta_\ell}^{(\ell)} = \underline{g}_{\theta+\ell}^{(\ell-1)} + \underline{e}_{\theta-b+\beta_\ell}$. Since $\theta - b + \beta_\ell \notin \mathcal{B}$ and $\beta_\ell > \ell$, we get

$$\underline{e}_i \in \left\langle \left\{ \underline{g}_u^{(\ell)} \mid u \in \left(\{\theta + \beta_\ell\} \cup [0 : \theta - 1] \cup [\theta : \theta + \ell - 1] \right) \setminus \mathcal{B} \right\} \right\rangle.$$

and the lemma statement follows.

$\underline{\theta - b + \beta_\ell \in \mathcal{B}}$**:** First we consider the case where $\theta + \ell \in \mathcal{B}$. It results in $\min(\mathcal{B}) > \theta - b + \ell$. Recall that $\max(\mathcal{P}_\ell^{(\ell-1)}) = \theta - b + \ell$ and $\mathcal{P}_{\beta_\ell}^{(\ell)} = \mathcal{P}_\ell^{(\ell-1)} \cup \{\theta - b + \beta_\ell\}$. Hence, we have $\max(\mathcal{P}_{\beta_\ell}^{(\ell)} \setminus \{\theta - b + \beta_\ell\}) = \theta - b + \ell$. Therefore, if $i \in \mathcal{P}_{\beta_\ell}^{(\ell)} \setminus \{\theta - b + \beta_\ell\}$, then $i \notin \mathcal{B}$. We also get,

$$\underline{e}_{\theta-b+\beta_\ell} \in \left\langle \left\{ \underline{g}_u^{(\ell)} \mid u \in \left(\{\theta + \beta_\ell\} \cup [0 : \theta - 1] \right) \setminus \mathcal{B} \right\} \right\rangle.$$

It follows that lemma statement is true for this case.

We now consider the case where $\theta + \ell \notin \mathcal{B}$. We have $\max(\mathcal{P}_{\beta_\ell}^{(\ell-1)}) = \theta - b + \beta_\ell$. We first look at $\theta - b + \beta_\ell \in \mathcal{P}_{\beta_\ell}^{(\ell)}$. Since $\theta - b + \beta_\ell \in \mathcal{P}_{\beta_\ell}^{(\ell-1)}$, the induction assumption gives, $\underline{e}_{\theta-b+\beta_\ell} \in \langle \{ \underline{g}_u^{(\ell-1)} \mid u \in (\{\theta + \beta_\ell\} \cup [0 : \theta - 1] \cup [\theta : \theta + \ell - 1]) \setminus \mathcal{B} \} \rangle$. For all $u \in [0 : \theta - 1] \cup [\theta : \theta + \ell - 1]$, we have $\underline{g}_u^{(\ell)} = \underline{g}_u^{(\ell-1)}$. We also have $\underline{g}_{\theta+\ell}^{(\ell)} = \underline{g}_{\theta+\beta_\ell}^{(\ell-1)}$. Since $\theta + \ell \notin \mathcal{B}$, this results in

$$\underline{e}_{\theta-b+\beta_\ell} \in \left\langle \left\{ \underline{g}_u^{(\ell)} \mid u \in \left([0 : \theta - 1] \cup [\theta : \theta + \ell] \right) \setminus \mathcal{B} \right\} \right\rangle, \quad (12)$$

which satisfies the requirement of the lemma. Now we focus on $i \in \mathcal{P}_{\beta_\ell}^{(\ell)} \setminus \{\theta - b + \beta_\ell\}$. It follows that $i \in \mathcal{P}_\ell^{(\ell-1)}$. From induction assumption and $\theta + \ell \notin \mathcal{B}$ we get,

$$\underline{e}_i \in \left\langle \left\{ \underline{g}_u^{(\ell-1)} \mid u \in \left(\{\theta + \ell\} \cup [0 : \theta - 1] \cup [\theta : \theta + \ell - 1] \right) \setminus \mathcal{B} \right\} \right\rangle.$$

For all $u \in [0 : \theta - 1] \cup [\theta : \theta + \ell - 1]$, we have $\underline{g}_u^{(\ell)} = \underline{g}_u^{(\ell-1)}$. We also have $\underline{g}_{\theta+\beta_\ell}^{(\ell)} = \underline{g}_{\theta+\ell}^{(\ell-1)} + \underline{e}_{\theta-b+\beta_\ell}$. Therefore, $\underline{e}_i \in \langle \{ \underline{g}_u^{(\ell)} \mid u \in (\{\theta + \beta_\ell\} \cup [0 : \theta - 1] \cup [\theta : \theta + \ell - 1]) \setminus \mathcal{B} \} \cup \{ \underline{e}_{\theta-b+\beta_\ell} \} \rangle$. Combining the above equation with (12), we get

$$\underline{e}_i \in \left\langle \left\{ \underline{g}_u^{(\ell)} \mid u \in \left(\{\theta + \beta_\ell\} \cup [0 : \theta - 1] \cup [\theta : \theta + \ell] \right) \setminus \mathcal{B} \right\} \right\rangle,$$

as required. Thus, we have argued for all $j \in [0 : b-1]$ that $G^{(\ell)}$ satisfies the lemma statement.

C. Proof of Lemma 3

By definition, $\cup_{j=0}^{b-1} \mathcal{P}_j^{(b-2)} = [0 : \theta - 1]$. It follows that the first part of the lemma is true for $i \in [0 : \theta - b]$, as $i + b \leq \theta$. Now consider $i \in [\theta - b + 1 : \theta - 1]$. From (7), we have $i \in \mathcal{P}_{i+b-\theta}^{(-1)}$. Suppose $i \in \mathcal{P}_v^{(\ell)}$. Then, (8) ensures that either $i \in \mathcal{P}_v^{(\ell+1)}$ or $i \in \mathcal{P}_u^{(\ell+1)}$ for some $u > v$. Therefore, $i \in \mathcal{P}_j^{(b-2)}$ for some $j \geq i + b - \theta$. This completes the proof of the first part of the lemma.

Now we focus on the second part of the lemma. Fix any $i \in [0 : b-2]$. The construction guarantees that there exists a $j \in [0 : i]$ such that $\sigma(i) \in \mathcal{P}_j^{(b-2)}$. For $\sigma(i) \in [0 : \theta - b]$, we have $\theta \geq \sigma(i) + b$ and the second part holds. Now suppose $\sigma(i) \in [\theta - b + 1 : \theta - 1]$. It can be seen that $\beta_i \geq \sigma(i) + b - \theta$. We recall that σ satisfies $i \geq \sigma(i) + b - \theta$. It follows from (8) that there exists a $j \in [\sigma(i) + b - \theta : i]$ such that $\sigma(i) \in \mathcal{P}_j^{(i)}$. For $j \leq i$, we have $\mathcal{P}_j^{(i)} = \mathcal{P}_j^{(b-2)}$. Hence, we have $\sigma(i) \in \mathcal{P}_j^{(b-2)}$ for some $j \in [\sigma(i) + b - \theta : i]$, thereby completing the proof.

D. Proof of Lemma 4

Given a permutation σ and $j \in [0 : \theta - 1]$, we will first show that there exists a unique permutation $\sigma^{(j)}$ such that $\sigma^{(j)}(i) = i$ for all $i < j$ and $\{m_{(\sigma^{(j)})^{-1}(u)} \mid u \in [0 : i]\}$ is recoverable from $\{c_u \mid u \in [0 : i + b] \setminus [j : j + b - 1]\}$, for all $i \in [0 : \theta - 1]$.

(a) $i \in [0 : j - 1]$: The assignment of $\sigma^{(j)-1}(i)$ is unique for $i \in [0 : j - 1]$ and is given by $\sigma^{(j)-1}(i) = i$ for $i \in [0 : j - 1]$.

(b) $i \in [j : \theta - 1]$: We will now assume that there is a unique assignment of $\sigma^{(j)-1}(u)$ for all $u \in [0 : i - 1]$ and then show that there exists a unique assignment for $\sigma^{(j)-1}(u)$ such that $\{m_{\sigma^{(j)-1}(u)} \mid u \in [0 : i]\}$ is recoverable from $\{c_u \mid u \in [0 : i + b] \setminus [j : j + b - 1]\}$.

(b.1) If $i + b < \theta$, then $c_{i+b} = m_{\sigma^{-1}(i+b)}$. Therefore the only assignment possible for $\sigma^{(j)-1}(i) = \sigma^{-1}(i + b)$ as only this can ensure the recover-ability condition.

(b.2) If $i + b \geq \theta$ then $c_{i+b} = [m_{\sigma^{-1}(0)} \ \cdots \ m_{\sigma^{-1}(\theta-1)}] \underline{g}_{i+b}$ and from the strong recovery property, if $P(u, i + b - \theta) = 1$ for $u \in [0 : \theta - 1]$, then $m_{\sigma^{-1}(u)}$ can be recovered from $\{c_v \mid v \in [0 : j - 1] \cup [j + b : i + b]\}$.

Claim: There exists a unique u such that $P(u, i+b-\theta) = 1$ and $\sigma^{-1}(u) \notin \{\sigma^{(j)-1}(v) \mid v \in [0 : i - 1]\}$. If this claim is true then we can set that $\sigma^{(j)-1}(i) = \sigma^{-1}(u)$.

Proof of Claim: Since $G_{\theta,b,\sigma}$ is the generator matrix of a burst correcting code the following statements follow:

$$\dim(< \{\underline{g}_v \mid v \in [0 : i + b] \setminus [j : j + b - 1]\} >) = i + 1$$
$$\dim(< \{\underline{g}_v \mid v \in [0 : i + b - 1] \setminus [j : j + b - 1]\} >) = i$$

From the first equation it follows that:

$$\begin{aligned} \underline{g}_{i+b} &\notin < \{g_v \mid v \in [0 : i+b-1] \setminus [j : j+b-1]\} > \\ &= < \{e_{\sigma(\sigma^{(j)-1}(v))} \mid v \in [0 : i-1]\} > \end{aligned}$$

and the last equality above follows from recover-ability condition. It therefore follows that there exists a u such that $P(u, i+b-\theta) = 1$ and $\sigma^{-1}(u) \notin \{\sigma^{(j)-1}(v) \mid v \in [0 : i-1]\}$. To show uniqueness, suppose there exist u, u' that satisfy the above condition, then we have from strong recovery that:

$$e_u, e_{u'} \in < \{\underline{g}_v \mid v \in [0 : i+b] \setminus [j : j+b-1]\} >$$

But we know that:

$$\begin{aligned} i+1 &= \dim(< \{\underline{g}_v \mid v \in [0 : i+b] \setminus [j : j+b-1]\} >) \\ &\geq \dim(< \{e_u, e_{u'}, e_{\sigma(\sigma^{(j)-1}(v))} \mid v \in [0 : i-1] >) \\ &= i+2 \text{ (a contradiction.)} \end{aligned}$$

Therefore there exists an unique u and the claim follows.

We will now proceed to show that the permutation $\sigma^{(j)}$ satisfies the condition that $\sigma^{(j)}(i) \leq \theta - b + i$ for all $i \in [0 : b-2]$. We know that the permutation σ satisfies $\sigma(i) \leq \theta - b + i$ for all $i \in [0 : b-2]$.

For $i \in [0 : j-1]$, $\sigma^{(j)}(i) = \sigma(i) \leq \theta - b + i$. For $i \in [j : b-2]$ from the timely recovery property we have:

$$e_{\sigma(i)} \in < \underline{g}_u \mid u \in [0 : \theta + i] \setminus [j : j+b-1] >$$

i.e., m_i is recoverable from $\{c_u \mid u \in [0 : \theta+i] \setminus [j : j+b-1]\}$. We know that $m_{\sigma^{(j)-1}(\hat{i})}$ is recoverable from $\{c_u \mid u \in [0 : \hat{i}+b] \setminus [j : j+b-1]\}$. Therefore i should be assigned to $\sigma^{(j)-1}(u)$ such that $u \in [0 : \theta + i - b]$ i.e., $\sigma(i) \leq \theta + i - b$.

E. Proof of Lemma 5

Let $c_u = x^{(r_{i-1})}_{\ell\theta+b}(t + (i-1)b + u)$ for $u \in [0 : \theta + b - 1]$, $\sigma = \pi_{t,i-1}$ and $\hat{m}_u = m_{\ell\theta+u}(t+u)$. Then from the encoding definition at relay r_{i-1} it follows that:

$$[c_0 \ c_1 \ \cdots, c_{\theta+b-1}] = [\hat{m}_{\sigma^{-1}(0)}, \cdots, \hat{m}_{\sigma^{-1}(b-1)}] G_{\theta,b,\sigma}.$$

If $e^{(r_{i-1},r_i)}_{t+(i-1)b+u} = 0$ for all $u \in [0 : \theta - 1]$ then $\pi_{t,i} = \pi_{t,i-1} = \sigma$ then: $c_j = x^{(r_{i-1})}_{\ell(\theta+b)+u}(t + (i-1)b + j)$ for all $j \in [0 : \theta - 1]$ and it is available at relay r_i by time $t+(i-1)b+j$. Notice that $c_j = \hat{m}_{\sigma^{-1}(u)} = m_{\ell\theta+\sigma^{-1}(j)}(t + \sigma^{-1}(j)) = m_{\ell\theta+\pi^{-1}_{t,i}(j)}(t + \pi^{-1}_{t,i}(j))$. Therefore $m_{\ell\theta+\pi^{-1}_{t,i}(j)}(t + \pi^{-1}_{t,i}(j))$ is available by $(t + (i-1)b + j)$ and hence also by $t + ib + j$ at the relay r_i.

Now let us consider the scenario where there is an erasure in interval $[t+(i-1)b : t+(i-1)b+\theta-1]$ and let $\alpha_{t,i} = \min\{u \in [0 : \theta-1] \mid e^{r_{i-1},r_i}_{t+(i-1)+u}\}$. Since $\pi_{t,i}$ is defined to be $\alpha_{t,i}$ variant of $\pi_{t,i-1}$, we are guaranteed recovery of message symbols $\{\hat{m}_{\pi^{-1}_{t,i}(u)} \mid u \in [0 : j]\}$ from $\{c_u \mid u \in [0 : j+b] \setminus [\alpha_{t,i} : \alpha_{t,i} + b - 1]\}$. Therefore $\hat{m}_{\pi^{-1}_{t,i}(j)} = m_{\ell\theta+\pi^{-1}_{i,t}(j)}(t + \pi^{-1}_{i,t}(j))$ is recoverable by accessing until symbol c_{j+b} that is sent at time $t + ib + j$. This completes the proof.