Adaptive Streaming Codes for Three-Node Relay Networks under Burst Erasure Channels

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***Abstract*—THIS PAPER IS ELIGIBLE FOR THE STUDENT PAPER AWARD. In this work, we develop packet-level streaming codes for a three-node relay network comprising a source, a relay, and a destination. We consider a burst erasure channel model wherein the source-relay link can introduce a single burst erasure of at most length b_1 and the relay-destination link can introduce a single burst erasure of at most length b_2 each within any sliding window of size $\tau+1$. The relay is assumed to have computational capability, and the encoding at the relay can adapt to the erasures on the source–relay link. We construct codes that recover all message packets with decoding delay τ over this channel. To the best of our knowledge, ours is the first construction to achieve a rate arbitrarily close to $\min(\frac{\tau-b_2}{\tau-b_2+b_1}, \frac{\tau-b_1}{\tau-b_1+b_2})$ for all feasible b_1 and b_2. This rate matches the theoretical upper bound for non-adaptive codes. In particular, when $b_1 \geq b_2$, this expression matches the rate upper bound for adaptive codes as well.**

I. Introduction

Reliable low-latency communication has become a critical enabler for a wide range of modern applications, including interactive multimedia, Internet of Things (IoT) connectivity, and advanced telecommunication services. In 5G cellular communication systems, Ultra-Reliable Low-Latency Communications (URLLC) is one of the key pillars [1], [2]. ARQ-based strategies for reliable transmission often incur large round-trip delays. In contrast, forward error correction (FEC) eliminates the need for retransmissions by introducing redundancy, allowing lost packets to be recovered from other received packets. Streaming codes are a class of FEC strategies that guarantee the recovery of all transmitted packets within a fixed delay. There has been extensive study of streaming codes for point-to-point communication [3]–[8]. However, since information in modern networks typically traverses multiple intermediate nodes such as routers or base stations, recent research has shifted toward streaming codes designed for multi-node networks [9]–[15]. Early work in this area by Fong et al. [9] developed streaming codes for three-node networks under random erasure constraints, with subsequent rate and construction refinements provided in [10]. Singhvi et al. [11] expanded this scope to channels subject to either random erasures or length-bounded burst erasures. More recently, studies such as [12]–[14] have focused on optimizing streaming codes for burst-erasure models in three-node relay architectures. In particular, Ramkumar et al. [12] construct streaming codes assuming the same maximum burst-erasure length b on both the source-relay and relay-destination links. In this paper, we extend their constructions to the more general case where the burst-erasure length observed on the source-relay link can differ from that on the relay-destination link.

A. Notation

Let $\mathbb{N}$ denote the set of all natural numbers $\{1, 2, 3, \dots\}$. For $a, b \in \mathbb{N}$, $[a : b]$ denotes the set $\{i \in \mathbb{N} : a \leq i \leq b\}$. $\mathbb{F}_q$ denotes a finite field with q elements. $\mathbb{F}_q^{a\times b}$ represents the set of all matrices with a rows and b columns, where each element is drawn from $\mathbb{F}_q$. For a matrix $A \in \mathbb{F}_q^{a\times b}$, the entry in the i^{th} row and j^{th} column is denoted by $A(i, j)$, where $i \in [1 : a]$ and $j \in [1 : b]$. The transpose of A is denoted by A^T. $\mathbb{F}_q^a$ denotes the set of length-a vectors over $\mathbb{F}_q$. Row vectors are denoted using an underline; for example, $\underline{x} \in \mathbb{F}_q^a$. Column vectors are denoted using bold lowercase letters; for example, $\mathbf{x} \in \mathbb{F}_q^a$. I_a denotes the $a \times a$ identity matrix. Let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_a$ denote the standard basis vectors of $\mathbb{F}_q^a$, i.e. $I_a := [\mathbf{e}_1 \ \ \mathbf{e}_2 \ \cdots \ \mathbf{e}_a]$.

B. Problem Setup

Consider a source node $\mathcal{S}$ that needs to transmit message packets to a destination node $\mathcal{D}$ through a relay node $\mathcal{R}$, as shown in Fig. 1. We assume zero time taken for transmission

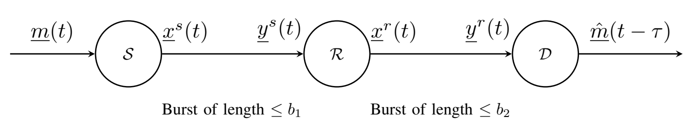

Fig. 1: A three-node network where the $\mathcal{S}$–$\mathcal{R}$ link introduces at most one burst of length $\leq b_1$ in any time window of size $\tau+1$, and the $\mathcal{R}$–$\mathcal{D}$ link introduces at most one burst of length $\leq b_2$ in any time window of size $\tau + 1$.

through each link, consistent with the existing literature on streaming codes. For each time slot $t \in \mathbb{N}$, a message packet $\underline{m}(t) := [m_1^s(t)\ m_2^s(t) \cdots m_k^s(t)] \in \mathbb{F}_q^k$ needs to be transmitted from the source. The source transmits coded packet $\underline{x}^s(t) := [x_1^s(t)\ x_2^s(t) \cdots\ x_{n_s}^s(t)] \in \mathbb{F}_q^{n_s}$ and the relay transmits coded packet $\underline{x}^r(t) := [x_1^r(t)\ x_2^r(t) \cdots x_{n_r}^r(t)] \in \mathbb{F}_q^{n_r}$ at time t. The rate of transmission in the $\mathcal{S}$-$\mathcal{R}$ link is defined as $R^s := \frac{k}{n_s}$, and the rate of transmission in the $\mathcal{R}$-$\mathcal{D}$ link is defined as $R^r := \frac{k}{n_r}$. The effective rate of transmission is given by $R := \min(R^s, R^r) = \frac{k}{\max(n_s, n_r)}$. Both the $\mathcal{S}$-$\mathcal{R}$ and $\mathcal{R}$-$\mathcal{D}$ links are modeled as an erasure channel with their erasure patterns represented by $\mathcal{E}_s = \{e_s(t)\}_{t=1}^{\infty}$ and $\mathcal{E}_r = \{e_r(t)\}_{t=1}^{\infty}$, respectively. Here, $e_s(t), e_r(t) \in \{0, 1\}$. The packet $\underline{x}^s(t)$ is erased if and only if $e_s(t) = 1$ and $\underline{x}^r(t)$ is erased if and only if $e_r(t) = 1$. The relay and destination observe $\underline{y}^s(t) \in \mathbb{F}_q^{n_s} \cup \{\star\}$ and $\underline{y}^r(t) \in \mathbb{F}_q^{n_r} \cup \{\star\}$, respectively, where

$$\underline{y}^s(t) = \begin{cases} \star & \text{if } e_s(t) = 1 \\ \underline{x}^s(t) & \text{otherwise,} \end{cases} \quad \underline{y}^r(t) = \begin{cases} \star & \text{if } e_r(t) = 1 \\ \underline{x}^r(t) & \text{otherwise.} \end{cases}$$

Let E_t^s and E_t^r be the causal encoding functions at the source and the relay at time t. i.e., we have:

$$\underline{x}^s(t) = E_t^s(\underline{m}(1), \ldots, \underline{m}(t)),\ \underline{x}^r(t) = E_t^r(\underline{y}^s(1), \ldots, \underline{y}^s(t)).$$

Let D_t be the decoding function used by the destination at time t. The decoding function attempts to output an estimate $\hat{\underline{m}}(t)$ of $\underline{m}(t)$ at time $t + \tau$, i.e., with a delay of τ time units. i.e., $\hat{\underline{m}}(t) = D_{t+\tau}(\underline{y}^r(1), \underline{y}^r(2), \ldots \underline{y}^r(t + \tau))$.

We consider a sliding-window-based burst-erasure channel model wherein $\mathcal{S}$–$\mathcal{R}$ and $\mathcal{R}$-$\mathcal{D}$ links can introduce bursts of lengths at most b_1 and b_2, respectively, within any window of $\tau+1$ time slots. In other words, if $e_s(t') = 1$ for some $t' \in \mathbb{N}$, then $e_s(t' + i) = 0$ for all $i \in [b_1 : \tau]$, and if $e_r(t') = 1$ for some $t' \in \mathbb{N}$, then $e_r(t'+i) = 0$ for all $i \in [b_2 : \tau]$. We refer to the tuple (E_t^s, E_t^r, D_t) as a (b_1, b_2, τ) *three-node burst erasure correcting streaming code (TBSC)* if D_t allows for the perfect recovery of message packets, i.e., $\hat{\underline{m}}(t) = \underline{m}(t)$ for all t.

In our proposed construction, we have a generalized architecture which allows for E_t^r to depend on the $\mathcal{S}$-$\mathcal{R}$ link's erasure history, $\{e_s(i)\}_{i=1}^{t}$. We refer to such a function as an *adaptive encoder* and the resulting code as an *adaptive streaming code*, while codes that do not depend on past erasures are called *non-adaptive codes*. Since the encoding operation at the relay may vary with the observed erasure history, the destination must be informed of the specific encoding used in order to enable decoding. To facilitate this, a header is appended to the relay-coded packets $\underline{x}^r(t)$, which conveys the required encoding information to the destination.

Construction	*Parameter Regime*	*Adaptive OR Non-Adaptive*
[11]	$u \mid \tau - u - v$	Non-Adaptive
[12]	$b_1 = b_2$	Adaptive
[13]	$\frac{\tau-u}{v} \geq \left\lfloor \frac{\tau-v}{u} \right\rfloor + 1$	Non-Adaptive
[14]	$\frac{\tau-u-v}{2u-v} \leq \left\lfloor \frac{\tau-u-v}{u} \right\rfloor$	Non-Adaptive
This paper	All possible parameters	Adaptive

TABLE I: Comparison of (b_1, b_2, τ) TBSC constructions and their achievable parameter regimes. Let $u := \max(b_1, b_2)$ and $v := \min(b_1, b_2)$. All codes listed in the table require the fundamental feasibility condition $\tau \geq b_1 + b_2$.

C. Our Contributions

We provide a family of (b_1, b_2, τ) adaptive TBSCs for all feasible parameters, i.e., $b_1, b_2, \tau \in \mathbb{N}$ such that $b_1 + b_2 \leq \tau$. To the best of our knowledge, ours is the first code family to achieve rates arbitrarily close to $\min(\frac{\tau-b_2}{\tau-b_2+b_1}, \frac{\tau-b_1}{\tau-b_1+b_2})$ for all feasible burst lengths. The marginal rate loss is due to the need to transmit a fixed-size header along with $\underline{x}^r(t)$ to convey erasure information on the $\mathcal{S}$–$\mathcal{R}$ link. However, this can be amortized by increasing the message packet size, for example by stacking multiple layers of coded packets. Also, for $b_1 \geq b_2$, we provide a straightforward matching converse bound, establishing optimality of the rate in the absence of a header. Table I summarizes and compares the parameter regimes under which TBSC constructions are developed in this work and in related literature.

II. Preliminaries

A. Rate Upper Bound

The paper [5] derived a rate upper bound for streaming codes with decoding delay τ over a point-to-point burst-erasure channel with burst length b. The optimal rate for this setting, denoted by $R^{\text{P2P}}(\tau, b)$, is given as:

$$R^{\text{P2P}}(\tau, b) = \begin{cases} \frac{\tau}{\tau+b} & \tau \geq b \\ 0 & \tau < b. \end{cases} \tag{1}$$

Theorem 1 ([12]). *Let R be the rate of transmission of a (b_1, b_2, τ) TBSC. Then the rate R must satisfy $R \leq R^{P2P}(\tau - b_2, b_1)$.*

The proof of Theorem 1 is provided in Appendix A.

Note that the upper bound on the rate of non-adaptive streaming codes in this setting, as derived by Singhvi et al. [11], is $R \leq \min\left(\frac{\tau-b_2}{\tau-b_2+b_1}, \frac{\tau-b_1}{\tau-b_1+b_2}\right)$. When $b_1 \geq b_2$, this bound coincides with the general rate upper bound stated in Theorem 1, and hence applies to all code constructions. Consequently, adaptive streaming codes can potentially exceed the non-adaptive bound in the regime $b_2 > b_1$.

B. Diagonal Embedding

Diagonal embedding is a technique used to convert a block code into a packet-level streaming code. Let G be the generator matrix for an $[n, k]$ block code $\mathcal{C}$. Let $\underline{m}(t) = [m_1(t)\ m_2(t) \cdots m_k(t)]$ be the message packet and $\underline{x}(t) = [x_1(t)\ x_2(t) \cdots x_n(t)]$ be the coded packet at time t. Under the diagonal embedding framework, the underlying (coded) symbols of the coded packets are then obtained as follows:

$$[x_1(t)\ \cdots\ x_n(t + n - 1)] = [m_1(t) \cdots\ m_k(t + k - 1)]G.$$

Having explained diagonal embedding, we now return our focus to the underlying block code. Let $\mathcal{B} := \{i \in [1 : n] : x_i(t + i - 1) \text{ is erased}\}$ denote the erased symbol indices. We define the *decoding delay profile* of the code as $\mathcal{T} = [\tau_1\ \ \tau_2\ \ldots\ \tau_k]$, where τ_i denotes the *decoding delay* of $m_i(t + i - 1)$, provided the following condition holds:

$$\mathbf{e}_i \in span(\{\mathbf{g}_j : j \in [1 : k_i] \setminus \mathcal{B}\})\ \ \forall\ i \in [1 : n],$$

where $\mathbf{g}_i$ is the i^{th} column of G and $k_i := \min(i + \tau_i, n)$. Equivalently, each message symbol $m_i(t + i - 1)$ can be reconstructed from the non-erased symbols among the first k_i coded symbols. There are two properties of block code that allow us to construct the streaming codes of interest.

- Burst correction capability: A normal burst of size b where, $\max(\mathcal{B}) - \min(\mathcal{B}) + 1 \le b$.
- Wrap-around burst of size b where, there exists a cyclic interval of length b that contains all the erasures. In other words, $[1:n] \setminus \mathcal{B}$ contains a contiguous segment of length $n-b$.

C. Hollmann-Tolhuizen (HT) Code

Hollmann and Tolhuizen [3] have constructed a systematic, rate-optimal $[n,k]$ block code in $\mathbb{F}_2$ that can recover message symbols with a decoding delay $\tau = k$, in the presence of any burst erasures of length $b = n-k$. We will call this the HT code with generator matrix $G^{\mathrm{HT}}_{k,n} = \begin{bmatrix} I_k & P_{k,n-k} \end{bmatrix}$. For positive integers r and k, $P_{k,r}$ is recursively defined as

$$P_{k,r} = \begin{cases} \begin{bmatrix} I_r \\ P_{k-r,r} \end{bmatrix} & 1 \le r < k \\ I_k & r = k \\ \begin{bmatrix} I_k & P_{k,r-k} \end{bmatrix} & r > k. \end{cases}$$

For example:

$$G^{\mathrm{HT}}_{3,5} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}, G^{\mathrm{HT}}_{4,7} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \quad (2)$$

D. Generalized HT (GHT) Code

Ramkumar et al. [12] proposed a systematic $[n,k]$ block code over $\mathbb{F}_2$, parameterized by $\alpha \in [1:n-k]$, that enables recovery of all message symbols in the presence of a single burst erasure of length $n-k$. It is also assumed that $n-k \le k$. This code, which is referred to as the *Generalized HT code*, has the decoding delay profile $\mathcal{T} = [\tau_1 \ \ \tau_2 \ \ldots \ \tau_k]$ where the code reduced to $[n,k]$ HT code when $\alpha = n-k$. For $\alpha < n-k$ the delay profile is given by $\tau_i = k$ for all $i \in [1:\alpha-1]$, $\tau_i = b$ for $i \in [k-b+\alpha : k]$ and $\tau_i = n-i$ for the remaining message symbols. The generator matrix of $[n,k]$ GHT code with parameter α is denoted by $G^{(\alpha)}_{k,n}$ and its construction is given in Appendix B.

We present here some example GHT codes that will be used later in the example streaming code constructions in Sections III-A, III-B. It can be verified that the example codes described through generator matrices below have delay profiles [4 5 3 3], [3 3 3] respectively.

$$G^{(2)}_{4,7} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}, G^{(2)}_{3,5} = G^{HT}_{3,5}. \quad (3)$$

III. Code Construction

Our (b_1, b_2, τ) TBSC code construction follows a decode-and-forward relaying architecture, in which the relay decodes the received packets, generates new message packets, and re-encodes them for transmission. Both the $\mathcal{S}$-$\mathcal{R}$ and $\mathcal{R}$-$\mathcal{D}$ links employ systematic codes, with code packets as shown below:

$$\underline{x}^u(t) \coloneqq [m^u_1(t) \ \cdots \ m^u_k(t) \ \ p^u_1(t) \ \cdots \ p^u_{n-k}(t)], \quad u \in \{s,r\},$$

where $\{m^r_i\}_{i=1}^k$ denote the relay message symbols and $\{p^u_i\}_{i=1}^{n-k}$ denote parity symbols for $u \in \{s,r\}$.

The construction also appends a header, denoted by $\underline{h}(t)$, to each relay-coded packet $\underline{x}^r(t)$. The header is defined as

$$\underline{h}(t) = [e^s(t-n-b_1+1) \ \ e^s(t-n-b_1+2) \cdots e^s(t)].$$

For ease of exposition, we restrict attention in this section to single layer of message packets and omit the header in $\underline{x}^r(t)$. The general construction—including multiple message layers, the header, and the convergence of the rate—is presented in Appendix C.

The encoders operate on message diagonals using diagonal embedding, or, in certain cases, a combination of two diagonally embedded block codes, encoding one diagonal at a time:

$$\text{diag}^u(t) \coloneqq [m^u_1(t) \ \ m^u_2(t{+}1) \ \cdots \ m^u_k(t{+}k{-}1)], \quad u \in \{s,r\}.$$

To describe the code, we therefore focus on the block codes used to encode $\mathrm{diag}^s(t)$ and $\mathrm{diag}^r(t)$ and the resulting parity symbols. The source message diagonal at time t, $\mathrm{diag}^s(t)$, maps bijectively to the relay message diagonal at time $t+b_1$, $\mathrm{diag}^r(t+b_1)$. Accordingly, the recovery analysis focuses on ensuring a decoding delay of τ for all symbols in the source message diagonal $\mathrm{diag}^s(t)$ at the destination, which in turn guarantees the same decoding delay for all message packets using the TBSC.

Our code construction is divided into two cases, namely $b_1 \ge b_2$ and $b_2 > b_1$. We first start by presenting examples for both these cases in Sections III-A and III-B, followed by a description of the aspects of the construction that are common to both cases in Section III-C. The detailed constructions for the two cases, which specify the mapping from $\mathrm{diag}^s(t)$ to $\mathrm{diag}^r(t{+}b_1)$ and the corresponding block codes, are presented in Sections III-D and III-E.

In both cases of the construction $k_1 \coloneqq \tau - b_2$ message symbols from $\mathrm{diag}^s(t)$ are encoded using a $[n_1, k_1]$ HT code, where $n_1 \coloneqq \tau - b_2 + b_1$.

A. Example: $\tau = 6$, $b_1 = 2$ and $b_2 = 3$

To illustrate the operation of the code, we demonstrate the recovery of $\mathrm{diag}^s(1)$ for an example with $\tau = 6$, $b_1 = 2$ and $b_2 = 3$, in which $k = \tau - b_1 = 4$ and $n = \tau - b_1 + b_2 = 7$. For this example, we consider a specific erasure pattern in the $\mathcal{S}$-$\mathcal{R}$ link from time 1 to 2, as depicted in Fig. 2 and show the messages can be recovered in time for a $b_2 = 3$ burst erasure pattern in the $\mathcal{R}$-$\mathcal{D}$ link.

Let $\mathrm{diag}^s(1) = [m_1 \ \ m_2 \ \ m_3 \ \ m_4]$. The first $b_2 - b_1$ symbols, which in this case reduces to m_1, are transmitted at time stamps 1 and 3 using a repetition code. While $[m_2 \ m_3 \ m_4]$ is encoded using $G^{\mathrm{HT}}_{3,5}$ (see equation (2)), i.e., $[m_2 \ \ m_3 \ \ m_4 \ \ p_1 \ \ p_2] = [m_2 \ \ m_3 \ \ m_4] G^{\mathrm{HT}}_{3,5}$. It follows that $p_1 = m_2 + m_4$ and $p_2 = m_3 + m_4$.

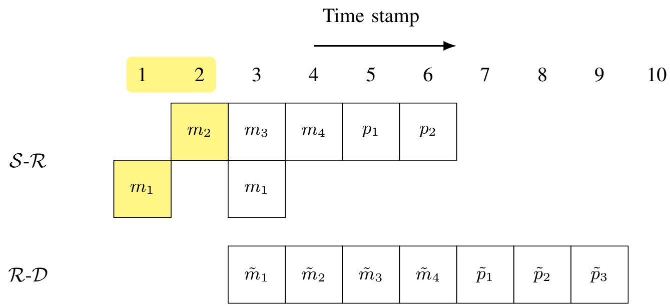

Fig. 2: The temporal distribution of coded symbols corresponding to $\text{diag}^s(1)$. The highlight represents the erasures in the $\mathcal{S}$-$\mathcal{R}$ link.

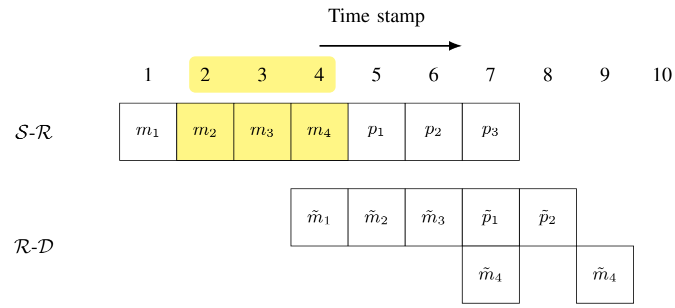

Fig. 3: The temporal distribution of coded symbols corresponding to $\text{diag}^s(1)$. The highlight represents the erasures in the $\mathcal{S}$-$\mathcal{R}$ link.

Let $\beta = 2$ denote the index of the first erasure among the HT code symbols in $\text{diag}^s(1)$. The message symbols in $\text{diag}^r(3)$ are denoted by $\tilde{m}_1, \cdots, \tilde{m}_4$ and they can be determined as follows. Clearly, $\tilde{m}_1, \tilde{m}_2, \tilde{m}_4$ can be chosen to be m_1, m_4, m_3 respectively as these symbols are available at the desired time as shown in Fig. 2. We can set $\tilde{m}_3$ to m_2 as it can be determined using p_1 along with m_4.

The relay message diagonal $\text{diag}^r(3)$ is encoded using a $[7,4]$ GHT code with parameter $\alpha = \min(b_2, \beta) = 2$ and generator matrix $G_{4,7}^{(2)}$ (see equation (3)), as follows:

$$[\tilde{m}_1\ \tilde{m}_2\ \tilde{m}_3\ \tilde{m}_4\ \tilde{p}_1\ \tilde{p}_2\ \tilde{p}_3] = [\tilde{m}_1\ \tilde{m}_2\ \tilde{m}_3\ \tilde{m}_4]G_{4,7}^{(2)},$$

Note that $\tilde{p}_1 = \tilde{m}_1 + \tilde{m}_4 = m_1 + m_3$, $\tilde{p}_2 = \tilde{m}_3 = m_2$ and $\tilde{p}_3 = \tilde{m}_2 + \tilde{m}_4$. Now we verify if message m_i can be recovered by time $6+i$. From the decoding delay properties of the GHT code, $\tilde{m}_1 = m_1$ can be recovered by time 7, $\tilde{m}_3 = m_2, \tilde{m}_4 = m_3$ can be recovered by times $8, 9$ respectively. $\tilde{m}_2 = m_4$ can be recovered by time 9 ensuring timely recovery for all the message symbols.

B. Example: $\tau = 6$, $b_1 = 3$, and $b_2 = 2$

To illustrate the operation of the code, we demonstrate the recovery of $\text{diag}^s(1)$ for the example with $\tau = 6$, $b_1 = 3$, and $b_2 = 2$, in which we set $k = \tau - b_2 = 4$ and $n = \tau - b_2 + b_1 = 7$. For this example, we consider a specific erasure pattern, depicted in Fig. 3, consisting of a burst erasure on the $\mathcal{S}$-$\mathcal{R}$ link from time 2 to 4.

Let $\text{diag}^s(1) = [m_1\ m_2\ m_3\ m_4]$, which gets encoded using $G_{4,7}^{\text{HT}}$ (see equation (2)), i.e., $[m_1\ m_2\ m_3\ m_4\ p_1\ p_2\ p_3] = [m_1\ m_2\ m_3\ m_4]G_{4,7}^{\text{HT}}$. It follows that $p_1 = m_1 + m_4$, $p_2 = m_2 + m_4$, and $p_3 = m_3 + m_4$.

Let $\beta = 2$ denote the index of the first erasure among the HT code symbols in $\text{diag}^s(1)$. The message symbols in $\text{diag}^r(4)$ are denoted as $\tilde{m}_1, \cdots, \tilde{m}_4$ and clearly from Fig. 3 the message $\tilde{m}_1$ has to chosen as m_1. We can determine m_4 from p_1 and m_1 and therefore set $\tilde{m}_2 = m_4$. From the decoding delay properties of the HT code by using parities p_2 and p_3 we can set $\tilde{m}_3 = m_2$ and $\tilde{m}_4 = m_3$.

We set $\alpha = \min(b_2, \beta) = 2$ and use $[5,3]$ GHT code with parameter $\alpha = 2$ to encode the first 3 message symbols of $\text{diag}^r(4)$, $\tilde{m}_1, \tilde{m}_2, \tilde{m}_3$. The generator matrix of $G_{3,5}^{(2)}$ used to encode is as shown in equations (3), (2). Note that from this definition the two parity symbols can be obtained as $\tilde{p}_1 = \tilde{m}_1 + \tilde{m}_3$ and $\tilde{p}_2 = \tilde{m}_2 + \tilde{m}_3$. In conjunction with the GHT code, we also use repetition code to encode $\tilde{m}_4$ with repeated symbols sent at times $7, 9$ respectively. It can be verified that each message symbol m_i can be recovered by time $6+i$ under any $b_2 = 2$ burst scenario in $\mathcal{R}$-$\mathcal{D}$ link.

C. Symbol Availability at Relay

We define the following matrix to characterize the symbols obtained at the relay from the HT code.

Definition 1. *For a given $\beta \in [1 : k_1 + 1]$, we define $V(\beta) \in \mathbb{F}_2^{k_1 \times k_1}$ as $V(\beta) := [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_{k_1}]$, where*

$$\mathbf{v}_i := \begin{cases} \mathbf{e}_i, & i \in [1, \beta - 1], \\ \mathbf{g}_{i+b_1}^{\text{HT}}, & i \in [\beta : \min(k_1, k_1 - b_1 + \beta - 1)], \\ \mathbf{e}_{i-k_1+b_1}, & i \in [k_1 - b_1 + \beta : k_1], \end{cases}$$

and $\mathbf{g}_j^{\text{HT}}$ denotes the j^{th} column of the generator matrix $G_{k_1,k_1+b_1}^{\text{HT}}$.

We now formalize the properties of the symbols obtained at the relay from the HT code under a burst erasure.

Theorem III.1. *Consider the message symbols $\{m_i\}_{i=1}^{k_1}$, which are encoded at the source using the generator matrix G_{k_1,n_1}^{HT} to produce*

$$[x_1\ \ x_2\ \ \cdots\ x_{n_1}] = [m_1\ \ m_2\ \ \cdots\ m_{k_1}]G_{k_1,n_1}^{\text{HT}}.$$

Let $\{x_i\}_{i=1}^{n_1}$ denote the resulting coded symbols, with x_i transmitted at time $t' + i - 1$. The set of indices of erased symbols is denoted by $\mathcal{B}_{\text{HT}} := \{i : x_i \text{ is erased}\}$, with $\beta := \min\big(k_1 + 1, \min(\mathcal{B}_{\text{HT}})\big)$ denoting the position of the first erased symbol counted from the beginning of the codeword. Suppose that $\mathcal{B}_{\text{HT}}$ corresponds to a single burst erasure of length b_1 (including wrap-around).

For $[\tilde{m}_1\ \cdots\ \tilde{m}_{k_1}] = [m_1\ \cdots\ m_{k_1}]V(\beta)$, the following statements hold:

1) *For each $i \in [1 : k_1]$, the symbol $\tilde{m}_i$ is available at the relay at time $t' + b_1 + i - 1$.*
2) *The matrix $V(\beta)$ is invertible. This ensures that the message symbols $\{m_i\}_{i=1}^{k_1}$ can be recovered from $\{\tilde{m}_i\}_{i=1}^{k_1}$.*

The proof of Theorem III.1 is provided in Appendix D.

For notational simplicity, we introduce shorthand notation for a subset of the symbols associated with $\text{diag}^s(t)$. In particular, we define $m_i := m_i^s(t+i-1)$ and $\tilde{m}_i := m_i^r(t+b_1+i-1)$ for $i \in [1:k]$, $p_\ell := p_\ell^s(t+k+\ell-1)$ for $\ell \in [1:b_1]$, and $\tilde{p}_j := p_j^r(t+\tau+j-1)$ for $j \in [1:b_2]$.

D. Code Construction for $b_1 \geq b_2$

Our construction has a transmission rate that is $R = \frac{\tau-b_2}{\tau-b_2+b_1}$, with $k = \tau-b_2$ and $n = \tau-b_2+b_1$, while excluding the header in the relay coded packets. Notice that in this case $k = k_1$ and $n = n_1$.

1) $\mathcal{S}$-$\mathcal{R}$ Transmission: $\text{diag}^s(t)$ is encoded using a $[n_1, k_1]$ HT code as follows:

$$[m_1 \ \cdots \ m_{k_1}]G^{\text{HT}}_{k_1,n_1} = [m_1 \cdots m_{k_1} \ \ p_1 \cdots p_{b_1}].$$

During the $\mathcal{S}$–$\mathcal{R}$ transmission, a subset of the source coded symbols associated with $\text{diag}^s(t)$ is erased, and these are denoted by $\mathcal{B}_{\text{HT}} := \{i \in [1:n_1] : x_i^s(t+i-1) \text{ is erased}\}$, with $\beta := \min\big(k+1, \min(\mathcal{B}_{\text{HT}})\big)$ being the position of the first erasure. Under the $\mathcal{S}$-$\mathcal{R}$ link model, $\mathcal{B}_{\text{HT}}$ corresponds to a burst of length at most b_1, including wrap-around.

2) Relay Message Symbols: The relay message diagonal $\text{diag}^r(t+b_1)$ is formed as

$$[\tilde{m}_1 \ \cdots \ \tilde{m}_k] = [m_1 \ \cdots \ m_k]V(\beta).$$

This transmission is feasible by Theorem III.1.

3) $\mathcal{R}$-$\mathcal{D}$ Transmission: $\text{diag}^r(t+b)$ is encoded using a hybrid coding scheme combining GHT and repetition codes.

- The first $\tau-b_1$ symbols $\{m_i^r(t+b_1+i-1)\}_{i=1}^{\tau-b_1}$ (or equivalently $\{\tilde{m}_i\}_{i=1}^{\tau-b_1}$) are encoded using a $[\tau-b_1+b_2, \tau-b_1]$ GHT code with $\alpha = \min(b_2,\beta)$ as follows:

$$[\tilde{m}_1 \cdots \tilde{m}_{\tau-b_1}]G^{(\alpha)}_{\tau-b_1,\tau-b_1+b_2} = [\tilde{m}_1 \cdots \tilde{m}_{\tau-b_1} \ \tilde{p}_1 \cdots \tilde{p}_{b_2}].$$

- A repetition code of the following construction is used to encode the last b_1-b_2 message symbols. The symbols $m_i^r(t+b_1+i-1)$, equivalently denoted by $\tilde{m}_i$, is transmitted at time $t+b_1+i-1$ and $t+b_1+b_2+i-1$ for all $i \in [\tau-b_1+1 : \tau-b_2]$. The symbol repetitions are spaced by b_2 time gap so that the code can endure a burst of length b_2.

We define the relay coded packet as

$$\underline{x}^r(t) = [m_1^r(t) \cdots m_k^r(t) \ \ p_1^r(t) \ \ p_2^r(t) \cdots \cdots p_{b_2}^r(t) \ \ m^r_{\tau-b_1+1}(t-b_2) \cdots m_k^r(t-b_2)],$$

where the last b_1-b_2 symbols $\{m_i^r(t-b_2)\}_{i=\tau-b_1+1}^{k}$ represent the parity symbols from the repetition code.

The timely recovery of the message symbols $\{m_i\}_{i=1}^k$ at the destination is established in Appendix E.

E. Code Construction for $b_2 > b_1$

Our construction has a transmission rate that is $R = \frac{\tau-b_1}{\tau-b_1+b_2}$, with $k = \tau-b_1$ and $n = \tau-b_1+b_2$, while excluding the header in the relay coded packets.

1) $\mathcal{S}$-$\mathcal{R}$ Transmission: $\text{diag}^s(t)$ is encoded using a hybrid coding scheme combining HT and repetition codes.

- A Repetition code of the following construction is used to encode the first b_2-b_1 message symbols. The symbol $m_i^s(t+i-1)$, equivalently denoted by m_i, is transmitted at times $t+i-1$ and $t+i+b_1-1$ for all $i \in [1:b_2-b_1]$. The symbol repetitions are spaced by a b_1 time gap so that the code can endure a burst of length b_1.
- The last $\tau-b_2$ message symbols $\{m_i^s(t+i-1)\}_{i=b_2-b_1+1}^{\tau-b_1}$ (or equivalently $\{m_i\}_{i=b_2-b_1+1}^{\tau-b_1}$) are encoded using a $[n_1 = \tau-b_2+b_1, k_1 = \tau-b_2]$ HT code as follows:

$$[m_{b_2-b_1+1} \cdots m_{\tau-b_1}]G^{\text{HT}}_{k_1,n_1} = [m_{b_2-b_1+1} \cdots m_{\tau-b_1} \ \ p_1 \cdots p_{b_1}]$$

The source coded packet $\underline{x}^s(t)$ is given by

$$\underline{x}^s(t) = [m_1^s(t) \cdots m_k^s(t) \ \ p_1^s(t) \ \ p_2^s(t) \cdots \cdots p_{b_1}^s(t) \ \ m_1^s(t-b_1) \cdots m^s_{b_2-b_1}(t-b_1)].$$

2) Relay Message Symbols: The first b_2-b_1 symbols of $\text{diag}^r(t+b_1)$ are obtained as follows:

$$\tilde{m}_i = m_i^r(t+b_1+i-1) = m_i \ \forall \ i \in [1:b_2-b_1].$$

This is feasible as $m_i \ \forall \ i \in [1:b_2-b_1]$ are encoded with repetition code that has a decoding delay of b_1 and are available at the relay at time $t+b_1+i-1$.

The remaining symbols of $\text{diag}^r(t+b_1)$ are obtained from the $[n_1,k_1]$ HT codeword as follows. Let $\mathcal{B}_{\text{HT}} := \{i \in [b_2-b_1+1:\tau] : x_i^s(t+i-1) \text{ is erased}\}$ denote the erasure indices corresponding to the HT-coded symbols. Under the $\mathcal{S}$-$\mathcal{R}$ link model, $\mathcal{B}_{\text{HT}}$ corresponds to a burst of length at most b_1. Define $\beta := \min\big(k+1, \min(\mathcal{B}_{\text{HT}})\big)$ as the starting index of the burst with respect to the original HT code indexing. Since the HT-coded symbols in our construction are indexed from b_2-b_1+1 instead of 1, we define $\tilde{\beta} := \max(1, \beta-b_2+b_1)$ to represent the position of the first erasure within the HT code segment. The relay symbols are then given by

$$[\tilde{m}_{b_2-b_1+1} \ \cdots \ \tilde{m}_{\tau-b_1}] = [m_{b_2-b_1+1} \ \cdots \ m_{\tau-b_1}]V(\tilde{\beta}),$$

3) $\mathcal{R}$-$\mathcal{D}$ Transmission: The relay encodes $\text{diag}^r(t+b_1)$ using a GHT code with $\alpha = \min(b_2,\beta)$ as follows:

$$[\tilde{m}_1 \cdots \tilde{m}_{\tau-b_1}]G^{(\alpha)}_{\tau-b_1,\tau-b_1+b_2} = [\tilde{m}_1 \cdots \tilde{m}_{\tau-b_1} \ \tilde{p}_1 \cdots \tilde{p}_{b_2}].$$

The timely recovery of the message symbols $\{m_i\}_{i=1}^k$ at the destination is established in Appendix F.

IV. Conclusion

Prior works such as [11], [13], [14] constructed (b_1,b_2,τ) TBSCs that achieve the rate $R = \min(\frac{\tau-b_2}{\tau-b_2+b_1}, \frac{\tau-b_1}{\tau-b_1+b_2})$ only for restricted parameter regimes. In contrast, this paper provides a construction for all feasible parameters satisfying $\tau \geq b_1+b_2$. We also prove that the construction is asymptotically rate-optimal when $b_1 \geq b_2$. Whether the proposed construction is optimal when $b_2 > b_1$ remains to be established.

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APPENDIX A
PROOF OF THEOREM 1

The proof follows directly from an extension of the argument in [12, Thm. 1], which treats the special case $b_1 = b_2 := b$. The proof outline is as follows. Consider the scenario where $\mathcal{R}$-$\mathcal{D}$ link introduces erasures in the interval $[t+\tau-b_2+1 : t+\tau]$. Since $\underline{m}(t)$ has to be recovered in time $t+\tau$, and no new packets are available at the destination in the interval $[t+\tau-b_2+1 : t+\tau]$, $\underline{m}(t)$ has to be recovered by time $t+\tau-b_2$. This is regardless of whether the relay encoder is adaptive or non-adaptive. Hence, the scenario at the $\mathcal{S}-\mathcal{R}$ link reduces to that of a P2P link with delay constraint $\tau - b_2$ (with burst length b_1).

APPENDIX B
CONSTRUCTION OF $G_{k,n}^{(\alpha)}$

Let $G_{k,n}^{\mathrm{HT}} = [I_k \ P_{k,n-k}]$ denote the generator matrix of an $[n,k]$ HT code. As shown in [12], there exists a permutation $\sigma_\alpha : [1 : n-k] \to [1 : n-k]$ such that

- $\sigma_\alpha(i) = i \ \forall \ i \in [1 : \alpha - 1]$,
- $P_{k,n-k}(2k-n+i, \sigma_\alpha(i)) = 1 \ \forall \ i \in [\alpha, n-k]$,
- $P_{k,n-k}(2k-n+i, j) = 0 \ \forall \ i \in [\alpha, n-k] \ \& \ j \in [\alpha, \sigma_\alpha(i) - 1]$.

Let the columns of $P_{k,n-k}$ be denoted by $P_{k,n-k} = [\mathbf{p}_1 \ \ \mathbf{p}_2 \ \cdots \ \mathbf{p}_{n-k}]$. Permuting these columns according to σ_α yields $M_{k,n-k} := [\mathbf{p}_{\sigma_\alpha(1)} \ \cdots \ \mathbf{p}_{\sigma_\alpha(n-k)}]$.

The generator matrix of the GHT code is defined as

$$G_{k,n}^{(\alpha)} := [I_k \ P_{k,n-k}^{(\alpha)}],$$

where $P_{k,n-k}^{(\alpha)}$ is given entrywise by

$$P_{k,n-k}^{(\alpha)}(i,j) = \begin{cases} 0, & j \geq \alpha, \ i > j + 2k - n, \\ M_{k,n-k}(i,j), & \text{otherwise.} \end{cases}$$

APPENDIX C
MULTILAYERED CONSTRUCTION WITH HEADER

As noted earlier in Section I-C, the proposed construction incorporates multiple message layers and a header in each relay-coded packet. We describe this general construction here.

Consider a setting with L layers of message packets, each of size $\hat{k}$. The ℓ^{th} message layer at time t is given by

$$\underline{m}_\ell^s(t) = [m_{(\ell-1)\hat{k}+1}^s(t) \ m_{(\ell-1)\hat{k}+2}^s(t) \cdots m_{\ell\hat{k}}^s(t)] \ \forall \ \ell \in [1 : L]$$

were $\underline{m}_\ell^s(t)$ denotes the ℓ^{th} layer of message symbols. The source coded packet is given by $\underline{x}^s(t) = [x_1^s(t) \ \ x_2^s(t) \cdots x_{L\hat{n}}^s]$ where $\underline{x}_\ell^s = [x_{(\ell-1)\hat{n}+1}^s(t) \quad x_{(\ell-1)\hat{n}+2}^s(t) \cdots x_{\ell\hat{n}}^s(t)]$ denotes the ℓ^{th} layer of the $\underline{x}^s(t)$ and is obtained by encoding $\{\underline{m}_\ell^s(i)\}_{i=1}^t$. The relay message symbols for a specific layer are derived from the corresponding layer in the source coded packets; i.e., $\underline{m}_\ell^r(t)$ is obtained from the set $\{\underline{x}_\ell^s(i)\}_{i=1}^t$. Each relay coded packet includes a header that conveys the erasure history of the $\mathcal{S}$-$\mathcal{R}$ link. Using this notation, the relay-coded packet transmitted at time t is written as

$$\underline{x}^r(t) = [x_1^s(t) \ x_2^s(t) \cdots x_{L\hat{n}}^r(t) \ \underline{h}(t)]$$

where the first entries are the relay coded symbols and $\underline{h}(t)$ is a vector-valued header, defined as:

$$\underline{h}(t) = [e^s(t - \hat{n} - b_1 + 1) \ \ e^s(t - \hat{n} - b_1 + 2) \cdots e^s(t)].$$

Furthermore, $\underline{x}_\ell^r = [x_{(\ell-1)\hat{n}+1}^r(t) \quad x_{(\ell-1)\hat{n}+2}^r(t) \cdots x_{\ell\hat{n}}^r(t)]$ denotes the ℓ^{th} layer of the $\underline{x}^r(t)$ and is obtained by encoding $\{\underline{m}_\ell^r(i)\}_{i=1}^t$.

The encoding and relay message mapping for each layer follow the single-layer construction of Section III, taking the parameters $\hat{k} = \tau - \min(b_1, b_2)$ and $\hat{n} = \tau - \min(b_1, b_2) + \max(b_1, b_2)$.

The resulting rate of the TBSC construction is therefore $\frac{L\hat{k}}{L\hat{n}+(\hat{n}+b_1)}$. As L increases, this rate approaches the asymptotic value, since $\lim_{L\to\infty} \frac{L\hat{k}}{L\hat{n}+\hat{n}+b_1} = \frac{\hat{k}}{\hat{n}} = \min\left(\frac{\tau-b_1}{\tau-b_1+b_2}, \frac{\tau-b_2}{\tau-b_2+b_1}\right)$.

Appendix D
Proof of Theorem III.1

The second statement of Theorem III.1 follows directly from [12, Lemma 2] under the parameter substitutions $b = b_1$ and $\tau' = \tau - b_2 + b_1$.

While the first statement can be established by adapting the proof of [12, Lemma 7], that work does not explicitly consider wrap-around bursts. Thus, although the core logical arguments remain the same, we restate the proof of the first statement for completeness as follows:

- Given that $[x_1 \cdots x_{n_1}] = [m_1 \cdots m_{k_1}]G^{\text{HT}}_{k_1,n_1}$ and $[\tilde{m}_1 \cdots \tilde{m}_{k_1}] = [m_1 \cdots m_{k_1}]V(\beta)$, it follows that $\tilde{m}_i = m_i = x_i \ \forall \ i \in [1 : \beta - 1]$. Since these symbols are not erased, they are available at time $t' + i - 1$, and consequently remain available at time $t' + b_1 + i - 1$.
- Similarly, we observe that $\tilde{m}_i = x_{i+b_1} \ \forall \ i \in [\beta : \min(k_1, k_1 - b_1 + \beta - 1)]$. Moreover, since any two bursts are separated by τ, the symbols $x_j \ \forall \ j \in [\beta + b_1 : k_1 + \beta - 1]$ are guaranteed not to be erased. Consequently, the symbols $\tilde{m}_i = x_{i+b_1}$ are available at time $t' + b_1 + i - 1$ for all $i \in [\beta : \min(k_1, k_1 - b_1 + \beta - 1)]$.
- Given the channel model, all message symbols $\{m_j\}_{j=1}^{k_1}$ can be recovered with a decoding delay of k_1. Hence, m_ℓ is available at the relay at time $\ell + k_1$ for all $\ell \in [\beta : b_1]$. Consequently, $\tilde{m}_i = m_{i-k_1+b_1}$ is available at the relay at time $t + b_1 + i - 1 \ \forall \ i \in [k_1 - b_1 + \beta : k_1]$.

Appendix E
Timely Recovery of $\{m_i\}_{i=1}^{k}$ for $b_1 \geq b_2$

The recovery of $\{\tilde{m}_i\}_{i=1}^{\tau-b_2}$ implies recovery of $\{m_i\}_{i=1}^{\tau-b_2}$. On the $\mathcal{R}$–$\mathcal{D}$ link, we assume at most one burst of size b_2 in any sliding window of length $\tau + 1$. Because the temporal span of both the GHT and repetition codes is less than $\tau + 1$, each code encounters at most one burst, which both codes can correct. The block codeword terminates at time $t + \tau + b_1 - 1$, ensuring recovery of $\{m_i\}_{i=1}^{\tau-b_2}$ by this time. Consequently, the symbols $\{m_i\}_{i=b_1+1}^{\tau-b_2}$ are decoded within delay τ, and it remains to verify that $\{m_i\}_{i=1}^{b_1}$ also achieve delay τ for different values of β, as discussed below.

1) $\beta \leq b_2$: We first determine the decoding delay of $\{m_i\}_{i=1}^{\beta-1}$, followed by that of $\{m_i\}_{i=\beta}^{b_1}$, as detailed below.

- The first $\beta - 1$ symbols satisfy $\tilde{m}_i = m_i$ for $i \in [1 : \beta - 1]$ and are encoded by the GHT code. Since $\alpha = \beta$ in this case, these symbols incur a decoding delay of $\tau - b_1$ on the $\mathcal{R}$-$\mathcal{D}$ link. As these same symbols $\{m_i\}_{i=1}^{\beta-1}$ are transmitted from the relay after b_1 time slots, their total decoding delay is τ.
- The last $b_2 - \beta + 1$ message symbols encoded by the GHT code, $\{\tilde{m}_i\}_{i=\tau-b_1-b_2+\beta}^{\tau-b_1}$ (equivalently, $\{m_i\}_{i=\beta}^{b_2}$), incur a decoding delay of b_2 on the $\mathcal{R}$-$\mathcal{D}$ link. Additionally, the last $b_1 - b_2$ message symbols $\{\tilde{m}_i\}_{i=\tau-b_1+1}^{\tau-b_2}$ (i.e., $\{m_i\}_{i=b_2+1}^{b_1}$) are transmitted via the repetition code, which also has a decoding delay of b_2. Thus the symbols $\{m_i\}_{i=\beta}^{b_1}$ are transmitted from the relay after $\tau - b_2$ time slots and incur a decoding delay of b_2 on the $\mathcal{R}$-$\mathcal{D}$ link, yielding an overall decoding delay of τ.

2) Case: $b_2 < \beta$: In this case, $\alpha = b_2$, and the GHT code simplifies to an HT code with $G^{\text{HT}}_{\tau-b_1,\tau-b_1+b_2}$ as the generator matrix. The message symbols are recovered with a decoding delay of $\tau - b_1$ or b_2, depending on whether they are encoded using the GHT code or the repetition code, respectively.

- The first $\beta - 1$ symbols ($\{\tilde{m}_i\}_{i=1}^{\beta-1}$ or equivalently $\{m_i\}_{i=1}^{\beta-1}$) have a decoding delay of either $\tau - b_1$ or b_2, depending on whether they belong to the HT-code or repetition code, respectively. Therefore, their effective decoding delay is either $(\tau - b_1) + b_1 = \tau$ or $b_2 + b_1 \leq \tau$.
- Since $\beta > b_2$, $\tau - b_1 - b_2 + \beta > \tau - b_1 \implies \{\tilde{m}_i\}_{i=\tau-b_1-b_2+\beta}^{\tau-b_2} \subseteq \{\tilde{m}_i\}_{i=\tau-b_1+1}^{\tau-b_2}$. Thus, the last $b_1 - \beta + 1$ symbols, represented by $\{\tilde{m}_i\}_{i=\tau-b_1-b_2+\beta}^{\tau-b_2}$ (or equivalently $\{m_i\}_{i=\beta}^{b_1}$) belong entirely to the set of symbols transmitted under the repetition code. These symbols are transmitted from the relay with a delay of $\tau - b_2$, and since the repetition code ensures a decoding delay of b_2 in $\mathcal{R}$-$\mathcal{D}$ transmission, their effective decoding delay becomes $b_2 + \tau - b_2 = \tau$.

Appendix F
Timely Recovery of $\{m_i\}_{i=1}^{k}$ for $b_2 > b_2$

Considering the $\mathcal{R}$-$\mathcal{D}$ link model and the GHT code length, the span of the GHT code could encounter 1 or 2 burst erasures.

1) Single Burst Case: For $i \in [1 : \beta - 1]$, we have $\tilde{m}_i = m_i$, where $i \in [1 : b_2 - b_1]$ correspond to repetition-coded symbols and $i \in [b_2 - b_1 + 1 : \beta - 1]$ to HT coded symbols on the $\mathcal{S}$-$\mathcal{R}$ link; these symbols are transmitted at time $t + b_1 + i - 1$. The symbols $\{m_i\}_{i=\beta}^{b_2}$ are transmitted from the relay after $\tau - b_2$ time slots. Comparing with delay recovery analysis for the $b_1 \geq b_2$ construction in Appendix E, we observe that, for a given value of β, the delay requirements for the message symbols encoded using the GHT code are identical in both cases. Since this delay analysis was established for the earlier case, it follows that the message symbols in the present case also achieve an effective decoding delay of τ.

2) Two Burst Case: Now consider the case where two distinct bursts affect a single GHT codeword. Suppose the first burst ends with the erasure of $\tilde{m}_\ell$, i.e., at time $t + b_1 + \ell - 1$, and the second burst begins after time $t + b_1 + \ell + \tau$. From the $\mathcal{R}$-$\mathcal{D}$ link model and the presence of two burst erasures within the span of the GHT codeword, it follows that $\ell \leq b_2 - b_1 - 1$. Since $\alpha > b_2 - b_1 > \ell$, $\{\tilde{m}_i\}_{i=1}^{\ell}$ are decoded at the destination by time $t + b_1 + \ell + \tau - b_1 - 1$, which is feasible because only a single burst has occurred by then. From time $t + b_1 + \ell + \tau - b_1 - 1$ onward, the erasure scenario reduces to the single-burst case and is therefore resolved as discussed before.