Two Way Generalization of a Binary Burst Erasure Correcting Code

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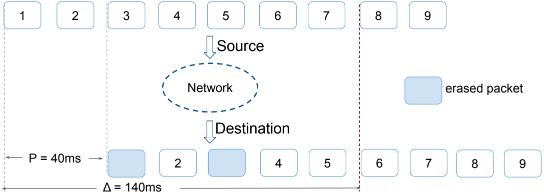
Joint Work With Vinayak Ramkumar (Postdoc, Tel Aviv), Nikhil Krishnan (Asst. Professor, IIT Palakkad), P. Vijay Kumar (Hon. Professor, IISc)

Motivation: Latency Sensitive Applications

- Several latency sensitive applications (real-time audio/video, AR/VR etc) that have end-to-end (E2E) latency requirements under packet erasures



Δ: E2E delay, $\tau = 4$: Delay in packet count

- E2E delay modeled through count of packets accessed in future.
- Goal: Design packet level FECs that can recover erasures within delay τ.

Delay Profile Of a Block Code

- An (n, k) linear block code can guarantee a worst case delay of $(n-1)$.
  - systematic case, the message symbols have delay profile $(n-1, n-2, n-3, \cdots, n-k)$.
- For a worst-case delay guarantee of τ, in presence of burst erasures of size b, any burst correcting code with parameters $n = \tau + 1, k = n - b$ will work.

  Can we do better (rate)?

- Yes. Can construct $(n = \tau + b, k = n - b)$ linear block code with delay profile $(\underbrace{\tau, \cdots, \tau}_{(b-1)}, \tau, \tau - 1, \cdots, b)$. (details to follow)
  - i-th message symbol with delay requirement $< n - i$ will be refered to as *urgent symbol*. Otherwise it is non-urgent symbol.

E. Martinian and M. Trott, "Delay-Optimal Burst Erasure Code Construction," ISIT 2007

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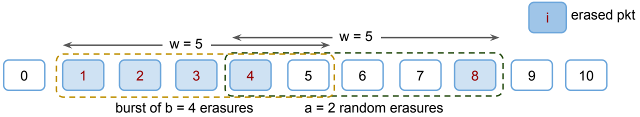
Erasure Model: Delay Constrained Sliding Window (DC-SW) Channel

(i) Admissible erasure patterns (AEP): within a sliding window of size w:

either $\leq a$ random erasures, or a burst of $\leq b$ erasures

(ii) Decoding-Delay Parameter: τ



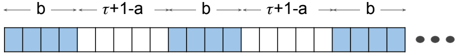
$$(a = 2, b = 4, w = 5, \tau = 4)$$

Badr et al., "Layered Constructions for Low-Delay Streaming Codes," *Trans. IT*, 2017.

Streaming Codes and Optimal Rate

- Streaming code is a packet-level FEC that can correct from all AEP of DCSW channel within a decoding delay constraint τ.
- It turns out that WOLOG, we can assume $w = \tau + 1$.
- The rate R of an $\{a, b, \tau\}$ streaming code has the upper bound:

$$R \leq \frac{(\tau + 1) - a}{(\tau + 1) + b - a} \triangleq R_{\text{opt}}.$$

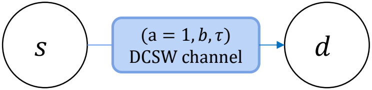


- Rate is 0 if $\tau < b$. Non-trivial only when $a \leq b \leq \tau$.

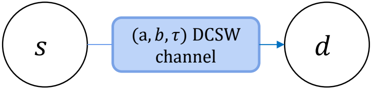
- Badr et al., "Layered Constructions for Low-Delay Streaming Codes," *Trans. IT*, 2017.
- M. Vajha, V. Ramkumar, M. Jhamtani and P. V. Kumar, "On the Performance Analysis of Streaming Codes over the Gilbert-Elliott Channel," ITW 2021

Settings Discussed in This Talk

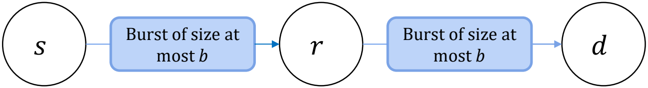
S1: Single link, burst erasures only



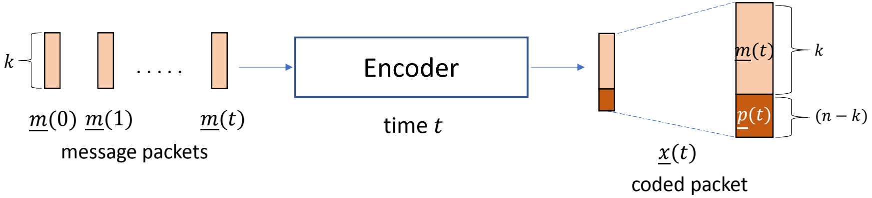
S2: Single link, burst or random erasures



S3: 3-node relay, burst erasures only, delay constraint τ



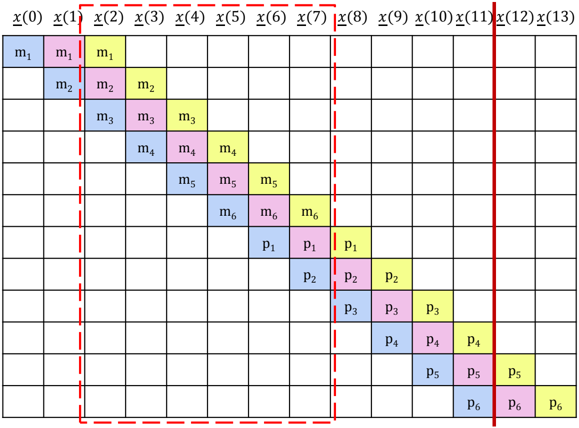
Redundancy through Packet Expansion Framework



- Can use scalar block codes to come up with streaming codes.

Diagonal Embedding (DE)

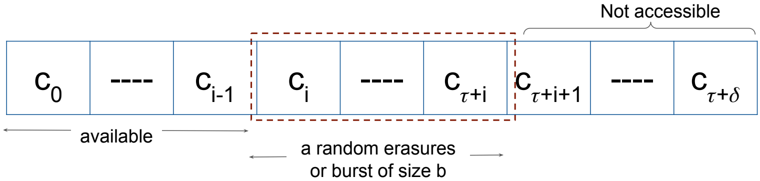
- Codewords of $[n, k]$ scalar block code are diagonally placed in the packet stream.
- This approach needs $n - k \geq b$



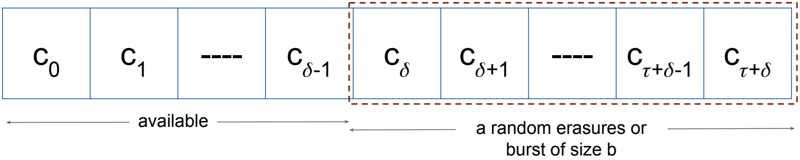
DE of $[12, 6]$ scalar code where $a = 4$, $b = 6$, $\tau = 9$

Scalar Code Properties

- For a given $\{a, b, \tau\}$ let
$$n = \tau + 1 + \delta, k = n - b \quad \text{where} \quad \delta = b - a$$
- For $i \in [0 : \delta - 1]$, to recover c_i:



- Let $E \subset [\delta : \tau + \delta]$ be either the set of a random erasures or a set of consecutive b erasures. To recover $\{c_j \mid j \in E\}$:



- The delay profile requirement here is $(\underbrace{\tau, \cdots, \tau}_{\delta \text{ urgent symbols}}, \underbrace{\tau, \tau - 1, \cdots, b}_{(k-\delta) \text{ non-urgent symbols}})$
- $\delta = b - 1$ when $a = 1$ for the burst only case.

S1: Single Link, Burst Erasures

- $R_{\text{opt}} = \frac{\tau}{\tau+b}$ (by setting $a = 1$ in R_{opt})
- Use $(\tau, \tau - b)$ wrap-around burst correcting code (WA-BCC) to construct $(a = 1, b, \tau)$ streaming code with $(n = \tau + b, k = n - b)$.

$$H_{WB} = \left[\begin{array}{cc} \underbrace{P}_{(b\times\tau-b)} & I_b \end{array} \right]$$

- $(n = \tau + b, k = n - b)$ code with pc matrix H is an $(a = 1, b, \tau)$ streaming code

$$H_{WBS} = \left[\begin{array}{ccc} I_b & P & I_b \end{array} \right]$$

  - Clearly burst erasure correcting code
  - What about worst case delay ?
  - $i \in [0 : b - 1]$, m_i requires delay τ, (until i-th parity available)
  - m_0 recovery: use 0-th pc
  - m_i recovery $i \in [1 : b - 1]$: need a pc equation with 0's in locations $[i + 1 : i + b - 1]$, $[\tau + i + 1 : n - 1]$.

E. Martinian and M. Trott, "Delay-Optimal Burst Erasure Code Construction," ISIT 2007

S1: Single Link, Burst Erasures

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  - Clearly burst erasure correcting code
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S1: Single Link, Burst Erasures

m_i recovery

$$H_{WB} = \begin{bmatrix} P & I_b \end{bmatrix}, \quad H_{WBS} = \begin{bmatrix} I_b & P & I_b \end{bmatrix}$$

$$H_{WBS}([0:i],:) \quad = \quad [I_{i+1} \quad \underbrace{0}_{(i+1)\times(b-i-1)} \quad P([0:i],:) \; I_{i+1} \quad \underbrace{0}_{(i+1)\times(b-i-1)} \quad]$$

- "*top-left*" $(i \times i)$ sub-matrix of P (say P_i) is invertible for any $i \in [\min\{b, n-b\} - 1]$
  - $\exists v \in \mathbb{F}_2^i$ such that $[v^T \;\; 1] P([0:i],[0:i-1]) = 0$.

$$\begin{bmatrix} v^T & 1 \end{bmatrix} H_{WBS}([0:i],:) \quad = \quad [v^T \;\; 1 \;\; \underbrace{0}_{b-1} \;\; X \;\; v^T \;\; 1 \;\; \underbrace{0}_{b-i-1}]$$

- if $n-b < b$, any $n-b$ "*consecutive-rows*" of P are l.i. Can show m_i recoverable in delay for $i \in [n-b : b-1]$

- Hollmann and Toulhuizen (HT) code is a binary WA-BCC.

E. Martinian and M. Trott, "Delay-Optimal Burst Erasure Code Construction," ISIT 2007

WA-BCC: Hollmann and Toulhuizen (HT) Code

- Recursive construction of $(u \times v)$ matrix $P_{u,v}$ that appends identity matrices column-wise or row-wise.

$$P_{u,v} = \begin{cases} [I_u \;\; P_{u,v-u}] & v > u \\ \begin{bmatrix} I_v \\ P_{u-v,v} \end{bmatrix} & \text{otherwise} \end{cases}$$

- Let $H_{WB} = [P \;\; I_b]$ be the pc matrix of $(n, n-b)$ WA-BCC. P is an $(b \times n-b)$ matrix.
  - clear that $H_{WB}^{\text{col}} = [I_b \;\; P \;\; I_b]$ is pc matrix of $(n+b, n)$ WA-BCC.
  - Can show that row extension also retains the WA-BCC (using the fact that dual of WA-BCC is also WA-BCC)

$$H_{WB}^{\text{row}} = \left[\begin{array}{c|c} I_n & \begin{array}{c} I_{n-b} \\ \hline P \end{array} \end{array} \right]$$

H. D. L. Hollmann and L. M. G. M. Tolhuizen, "Optimal Codes for Correcting a Single (Wrap-Around) Burst of Erasures," in TIT 2008

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S2: Single Link, Random and Burst Erasures

JigSaw code

- $(a = 3, b = 6, \tau = 8)$ streaming code defined by (6×12) parity check matrix

	0	1	2	3	4	5	6	7	8	9	10	11
0	α	0	0	0	0	0	$v_{0,6}$	$v_{0,7}$	$v_{0,8}$	0	0	0
1	0	α	0	0	0	0	0	$v_{1,7}$	$v_{1,8}$	$v_{1,9}$	0	0
2	0	0	α	0	0	0	0	0	$v_{0,8}$	$v_{1,9}$	$v_{2,10}$	0
3	1	0	0	*	*	*	*	*	*	0	0	1
4	0	1	0	*	*	*	*	*	*	0	0	0
5	0	0	1	*	*	*	*	*	*	0	0	0

- JigSaw is rate-optimal streaming code for any (a, b, τ) with field size of q^2 where $q \geq \tau$.
- $\alpha \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$

M. Nikhil Krishnan, Deeptanshu Shukla and P. Vijay Kumar, "Rate-Optimal Streaming Codes for Channels With Burst and Random Erasures", IEEE Trans. Info. Theory, 2020

S2: Explicit JigSaw Code

- $(a = 3, b = 6, \tau = 8)$

	0	1	2	3	4	5	6	7	8	9	10	11
0	α	0	0	0	0	0			α	0	0	0
1	0	α	0	0	0	0	P		0	1	0	0
2	0	0	α	0	0	0			0	0	1	0
3	1	0	0	*	*	*	*	*	*	0	0	1
4	0	1	0	*	*	*	*	*	*	0	0	0
5	0	0	1	*	*	*	*	*	*	0	0	0

- Last $a = 3$ rows of the pc matrix unchanged. Random erasure recovery follows from the structure of JigSaw.
- Support of the first $\delta = b - a = 3$ rows changed construction. Matrix P has 0, 1 elements.

M. Vajha, V. Ramkumar, M. N. Krishnan and P. Vijay Kumar, "Explicit Rate-Optimal Streaming Codes With Smaller Field Size," in IEEE ISIT 2021, Trans. Info. Theory, 2024

S2: Explicit JigSaw Code

- The definition of $(u \times v)$ matrix $P^a_{u,v}$ is recursive.

$$\mathbf{P}^a_{u,v} = \begin{cases} \begin{bmatrix} I_u & \underbrace{\mathbf{0}}_{(u\times a)} & \mathbf{P}^a_{u,v-u-a} \end{bmatrix} & u+a<v \\ \begin{bmatrix} I_u & \underbrace{\mathbf{0}}_{(u\times(v-u))} \end{bmatrix} & u \le v \le u+a \\ \begin{bmatrix} I_v \\ \mathbf{P}^a_{u-v,v} \end{bmatrix} & v<u \end{cases}$$

- This structure of the P matrix results in two properties on $\hat{P} = [\underbrace{0}_{\delta\times a} \; P]$
  - *consecutive-columns:* any b consecutive columns of $\hat{P}$ have a-zero columns and δ linearly independent columns.
  - *bottom-right:* bottom right sub-matrix of $\hat{P}$ of size $\theta \times (\theta + a)$, there are a-zero columns and θ linearly independent columns.
  - top-left, consecutive-rows properties hold for P.

$$P^3_{3,2} = \begin{bmatrix} I_2 \\ P^3_{1,2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

S2: Explicit JigSaw Code

- ($a = 3, b = 6, \tau = 8$). Support of the first $\delta = b - a = 3$ rows changed.

	0	1	2	3	4	5	6	7	8	9	10	11
0	α	0	0	0	0	0	1	0	α	0	0	0
1	0	α	0	0	0	0	0	1	0	1	0	0
2	0	0	α	0	0	0	1	0	0	0	1	0
3	1	0	0	*	*	*	*	*	*	0	0	1
4	0	1	0	*	*	*	*	*	*	0	0	0
5	0	0	1	*	*	*	*	*	*	0	0	0

- top-left, consecutive rows properties can be used to show urgent symbol recovery from burst.

S2: Explicit JigSaw Code

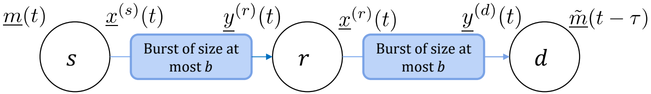
- $(a = 3, b = 6, \tau = 8)$.Burst erasure starting at index 4.

	0	1	2	3	4	5	6	7	8	9	10	11
0	α	0	0	0	0	0	1	0	α	0	0	0
1	0	α	0	0	0	0	0	1	0	1	0	0
2	0	0	α	0	0	0	1	0	0	0	1	0
3	1	0	0	*	*	*	*	*	*	0	0	1
4	0	1	0	*	*	*	*	*	*	0	0	0
5	0	0	1	*	*	*	*	*	*	0	0	0

- bottom-right, consecutive columns properties used to show burst recovery of non-urgent symbols

S3: 3 Node Relay, Burst Erasures

Rate Bound



- An example, permissible erasure pattern in the R-D link

$x^r(t)$	$\dots$	$x^r(t+\tau-b)$	$x^r(t+\tau-b+1)$	$\dots$	$x^r(t+\tau)$

- $m(t)$ should be recovered at relay by $t+\tau-b$
- Rate in the S-R link upper bounded by rate of $(a=1, b, \tau-b)$ DCSW channel.
- (b,τ) streaming code for S3 satisfies

$$R \leq \begin{cases} \frac{\tau-b}{\tau} & \tau \geq 2b \\ 0 & \text{otherwise} \end{cases}$$

- Generalized HT based streaming code achieves rate arbitrarily close to the upper bound.

V. Ramkumar, M. Vajha, M. N. Krishnan, "Streaming Codes for Three-Node Relay Networks With Burst Erasures", *to appear in ISIT 2024*

S3: 3 Node Relay, Burst Erasures

S-R link

- $(n = \tau, k = \tau - b)$ HT code is used at the source.
- Example: $\tau = 10, b = 4$, (4×10) pc matrix

$$H_{HT} = \left[\begin{array}{c|c|c} I_4 & P_{4,2} & I_4 \end{array} \right] = \left[\begin{array}{c|c} P_{4,6} & I_4 \end{array} \right] = \left[\begin{array}{c|c|c} I_4 & \begin{array}{c} I_2 \\ \hline I_2 \end{array} & I_4 \end{array} \right]$$

  - $p_0 = m_0 + m_4, p_1 = m_1 + m_5, p_2 = m_2 + m_4, p_3 = m_3 + m_5$

- Transmit at Source

t	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$x^s(t)$	m_0	m_1	m_2	m_3	m_4	m_5	p_0	p_1	p_2	p_3				

- Receive at Relay: Burst starting at time 0/1 in S-R link

$y^r(t)$					m_4	m_5	p_0	p_1	p_2	p_3				
$x^r(t)$					m_4	m_5	m_0	m_1	m_2	m_3	$\hat{p}_0$	$\hat{p}_1$	$\hat{p}_2$	$\hat{p}_3$
$y^r(t)$	m_0					m_5	p_0	p_1	p_2	p_3				
$x^r(t)$					m_0	m_5	p_0	m_1	m_2	m_3	$\hat{p}_0$	$\hat{p}_1$	$\hat{p}_2$	$\hat{p}_3$

S3: 3 Node Relay, Burst Erasures

R-D Link

t	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$x^s(t)$	m_0	m_1	m_2	m_3	m_4	m_5	p_0	p_1	p_2	p_3				
$y^r(t)$	m_0					m_5	p_0	p_1	p_2	p_3				
$x^r(t)$					m_0	m_5	p_0	m_1	m_2	m_3	$\hat{p}_0$	$\hat{p}_1$	$\hat{p}_2$	$\hat{p}_3$

Burst at 1 in S-R link

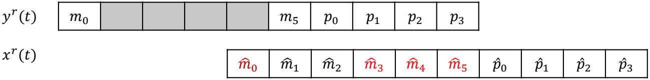
- Information needs to be preserved.
  - Can resolve m_4 from $p_0 = m_0 + m_4$, and m_0
  - $\alpha = \min\{\beta, b-1\}$ where β is burst start and set $(\hat{m}_0, \cdots, \hat{m}_{k-1})$ as

$$\Big(\underbrace{m_0, \cdots, m_{\alpha-1}}_{\alpha \text{ urgent symbols}}, \underbrace{x^s(\alpha), \cdots, x^s(k-b+\alpha-1)}_{(k-\alpha) \text{ non-urgent symbols}}, \underbrace{m_\alpha, \cdots, m_{b-1}}_{(b-\alpha) \text{ urgent symbols}}\Big)$$

  - HT code guarantees the information preservation property
  - GHT code parameterized by α. Slight overhead to communicate α to destination
- Urgent messages:
  - For $i \in [0 : \alpha - 1]$ $\hat{m}_i$ requires delay $\leq \tau - b$ (until $\hat{p}_i$ available)
  - For $i \in [0 : b - \alpha - 1]$ $\hat{m}_{k-1-i}$ requires delay $\leq b$ (until $\hat{p}_{b-1-i}$ is available)

S3: 3 Node Relay, Burst Erasures

GHT construction



Burst at 1 in S-R link

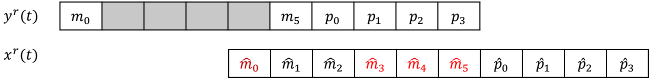
	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	1	0	1			
1	0	1	0	0	0	1		1		
2	0	0	1	0	1	0			1	
3	0	0	0	1	0	1				1

pc matrix of $(b = 4, \tau - b = 6)$ HT code

- Retain row 0, ensures in time recovery of $\hat{m}_0$

S3: 3 Node Relay, Burst Erasures

GHT Construction



Burst at 1 in S-R link

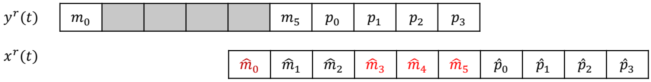
$$
\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\
\left[\begin{array}{ccc|ccc|cccc}
1 & 0 & 0 & 0 & 1 & 0 & 1 & & & \\
* & * & * & 1 & * & * & & 1 & & \\
* & * & * & * & 1 & * & & & 1 & \\
* & * & * & * & * & 1 & & & & 1
\end{array}\right]
\end{array}
$$

pc matrix of $(b = 4, \tau - b = 6)$ HT code

- Structured Bottom-Right Property: Permute rows of parity check matrix to get 1 in the diagonal of bottom-right $(b - \alpha) \times (b - \alpha)$ sub-matrix

S3: 3 Node Relay, Burst Erasures

GHT Construction



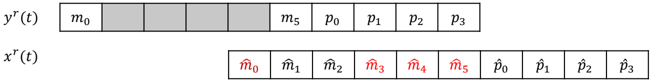
Burst at 1 in S-R link

	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	1	0	1			
1	0	0	0	1	0	1		1		
2	0	0	1	0	1	0			1	
3	0	1	0	0	0	1				1

- Set the upper triangular elements to 0

S3: 3 Node Relay, Burst Erasures

GHT Construction



Burst at 1 in S-R link

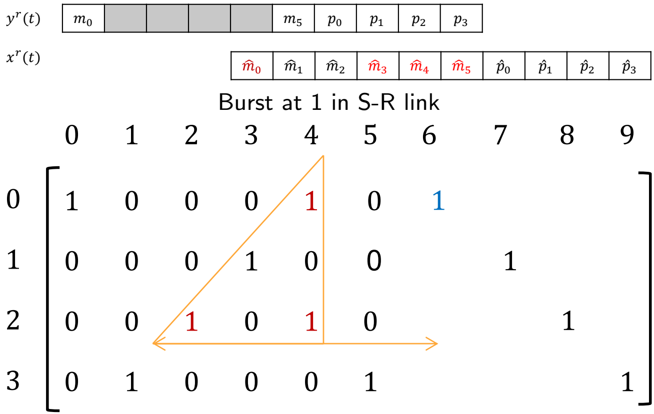
$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \left[\begin{array}{cccccccccc} 1 & 0 & 0 & 0 & 1 & 0 & 1 & & & \\ 0 & 0 & 0 & 1 & 0 & 0 & & 1 & & \\ 0 & 0 & 1 & 0 & 1 & 0 & & & 1 & \\ 0 & 1 & 0 & 0 & 0 & 1 & & & & 1 \end{array}\right] \end{array}$$

pc matrix of $(b = 4, \tau - b = 6)$ GHT code for $\alpha = 1$.

- $\hat{m}_3, \hat{m}_5$ can be recovered in delay
- $\hat{m}_1$ can be recovered from any burst

S3: 3 Node Relay, Burst Erasures

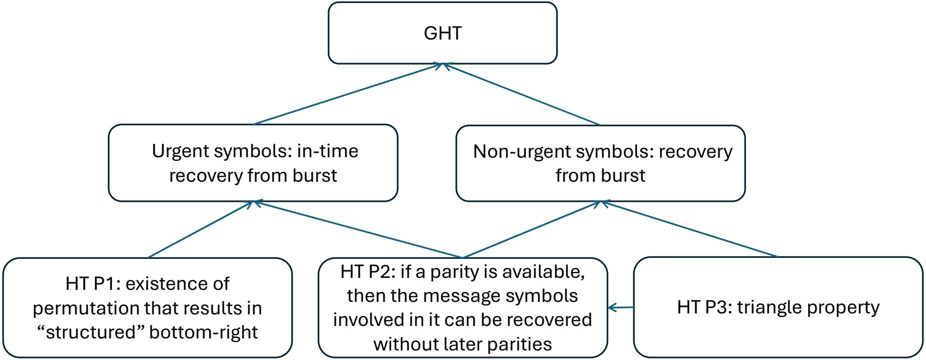
GHT Construction



pc matrix of $(b = 4, \tau - b = 6)$ GHT code for $\alpha = 1$.

- $\hat{m}_4$ recovery: if $\hat{m}_2$ is available can recover it using $\hat{p}_2$ otherwise from $\hat{p}_0$.
- $\hat{p}_0$ and $\hat{m}_2$ are spaced by b by triangle property
- $\hat{m}_2$ recovery follows.
- Can recover m_i within delay.

Key Ideas in General Proof



Future Directions

- Construction of codes for any ordering of the urgent symbols $(m_0, \cdots, m_{b-1})$ that are "allowed".

$$(m_0, *, *, m_1, m_2, m_3, p_0, p_1, p_2, p_3) \quad \checkmark$$
$$(m_1, *, *, m_0, m_2, m_3, p_0, p_1, p_2, p_3) \quad ✗$$

  - leads to streaming codes for m-node relay settings

- 3-node relay settings with different size bursts b_1, b_2.

- Characterize the delay-profiles for which constructions are possible

Thank You!