BERMAN CODES

Achieving the Capacity of the Binary Erasure Channel

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BACKGROUND & MAIN RESULT

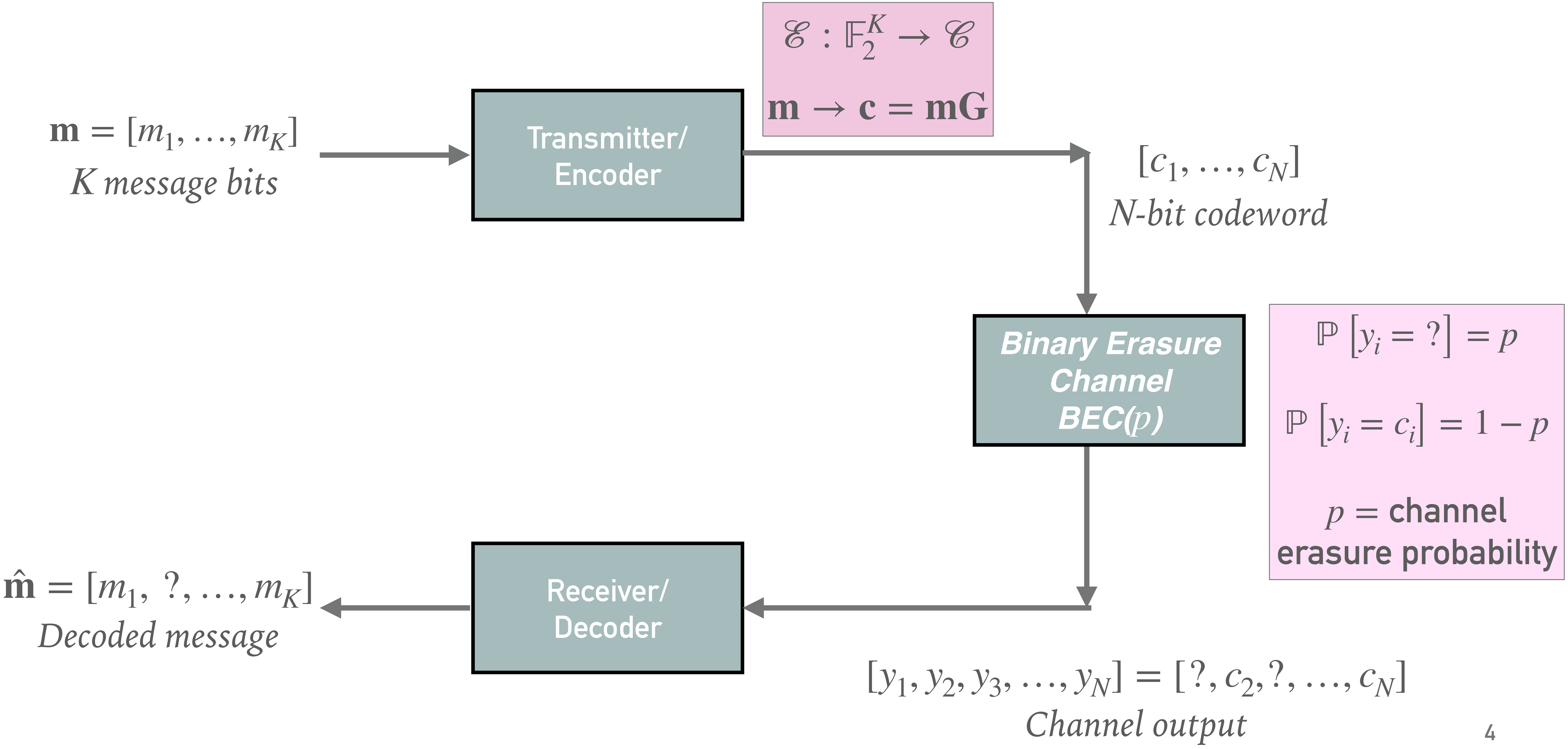
THE BINARY ERASURE CHANNEL (BEC)

- BEC(p) Channel:
  - communication medium with binary input $\{0,1\}$, ternary output $\{0,1,?\}$
  - each transmitted bit is erased with probability p (where $0 < p < 1$)
  - each use of the channel is independent of the other uses

$$\underset{\textit{Transmitted}}{(0,1,0,1,1,0)} \xrightarrow{BEC} \underset{\textit{Received}}{(0,1,?,1,?,0)}$$

$$P\left[(0,1,?,1,?,0) \text{ received} \mid (0,1,0,1,1,0) \text{ transmitted}\right] = p^2(1-p)^4$$

CODES IN DIGITAL COMMUNICATION



CODES / ERASURE CORRECTING CODES / ERROR CORRECTING CODES

$$(m_1, m_2) \xrightarrow{\textit{encoding}} (m_1, m_2, m_1 \oplus m_2) \xrightarrow{\textit{BEC}} (m_1, ?, m_1 \oplus m_2) \xrightarrow{\textit{decoding}} (m_1, m_2)$$

Message — *Transmitted Codeword* — *Received Vector* — *Message*

- **Code $\mathscr{C}$:** set of all possible transmitted sequences

  Single Parity Check Code (of length 3) $\mathscr{C} = \{(0,0,0), (0,1,1), (1,0,1), (1,1,0)\}$

- **Codewords:** sequences/vectors in $\mathscr{C}$

- **Parameters of Interest:** **Rate** and **Probability of Decoding Failure**

CODES / ERASURE CORRECTING CODES / ERROR CORRECTING CODES

$$(m_1, m_2) \xrightarrow{\textit{encoding}} (m_1, m_2, m_1 \oplus m_2) \xrightarrow{\textit{BEC}} (m_1, ?, m_1 \oplus m_2) \xrightarrow{\textit{decoding}} (m_1, m_2)$$

Message | *Transmitted Codeword* | *Received Vector* | *Message*

- **Dimension of a (linear) code, K:** number of message bits
  Length of a code, N: number of channel uses (number of bits in the transmitted sequence)

- Robustness is increased at the cost of reduction in communication rate

$$\textbf{Rate } R = \frac{K}{N} = \frac{\text{number of message bits}}{\text{number of channel uses}}$$

BIT ERASURE RATE (BER) & CODEWORD ERASURE RATE (CER)

$$(m_1, m_2) \xrightarrow{\textit{encoding}} (m_1, m_2, m_1 \oplus m_2) \xrightarrow{\textit{BEC}} (m_1, ?, ?) \xrightarrow{\textit{decoding}} (m_1, ?)$$

Message — *Transmitted Codeword* — *Received Vector* — *Message*

- Decoding is not always successful

$$\text{BER} = \frac{1}{\text{num of message bits}} \sum_i P[m_i \text{ is not decodable}]$$

$$\text{CER} = P[\text{ entire message can not be decoded }] \qquad (\text{Note that BER} \le \text{CER})$$

Performance Metrics

$$(m_1, m_2) \xrightarrow{\textit{encoding}} (m_1, m_2, m_1 \oplus m_2) \xrightarrow{\textit{BEC}} (m_1, ?, m_1 \oplus m_2) \xrightarrow{\textit{decoding}} (m_1, m_2)$$

Message — *Transmitted Codeword* — *Received Vector* — *Message*

- ***Rate of the Code***

$$R = \frac{\text{number of messages bits}}{\text{number of code bits}} = \frac{2}{3}$$

- Higher rate ⇒ Faster Communication

- ***Bit Erasure Rate (BER)***

$$\frac{P[\hat{m}_1 = ?] + P[\hat{m}_2 = ?]}{2} = 2p^2 - p^3$$

- Smaller BER ⇒ Better Quality of Communication

A Main Problem in Coding Theory:

For a given $p \in (0,1)$, Design Codes with High Rate and BER, CER ≈ 0

CAPACITY OF THE ERASURE CHANNEL

[Shannon'48] ***Capacity Theorem for the Erasure Channel***

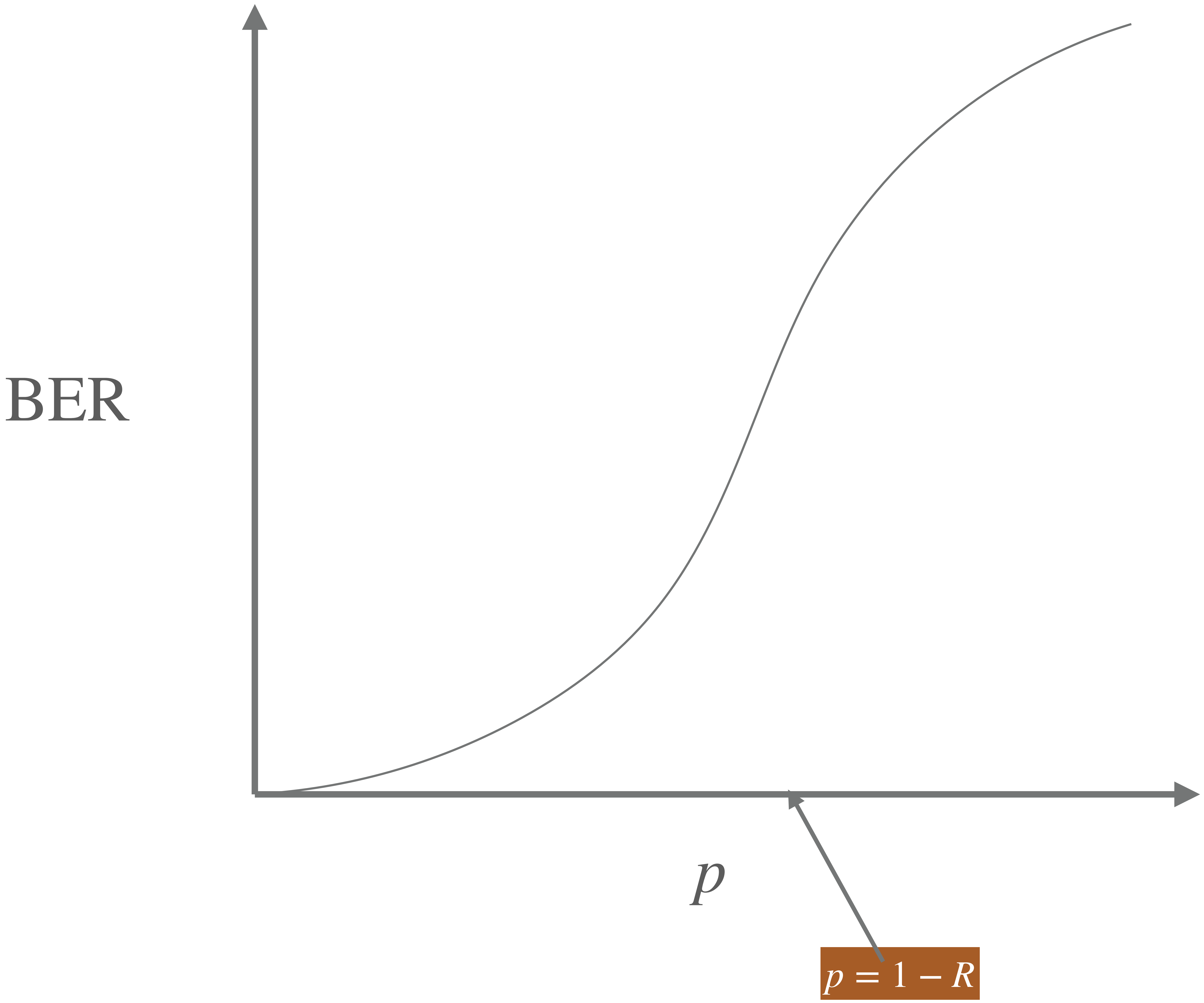
Codes can be designed with $\mathrm{BER}, \mathrm{CER} \to 0$ *if and only if* $R < 1 - p$ and $N \to \infty$

- ***Capacity*** of the binary erasure channel $\mathrm{BEC}(p)$ is $C \triangleq 1 - p$

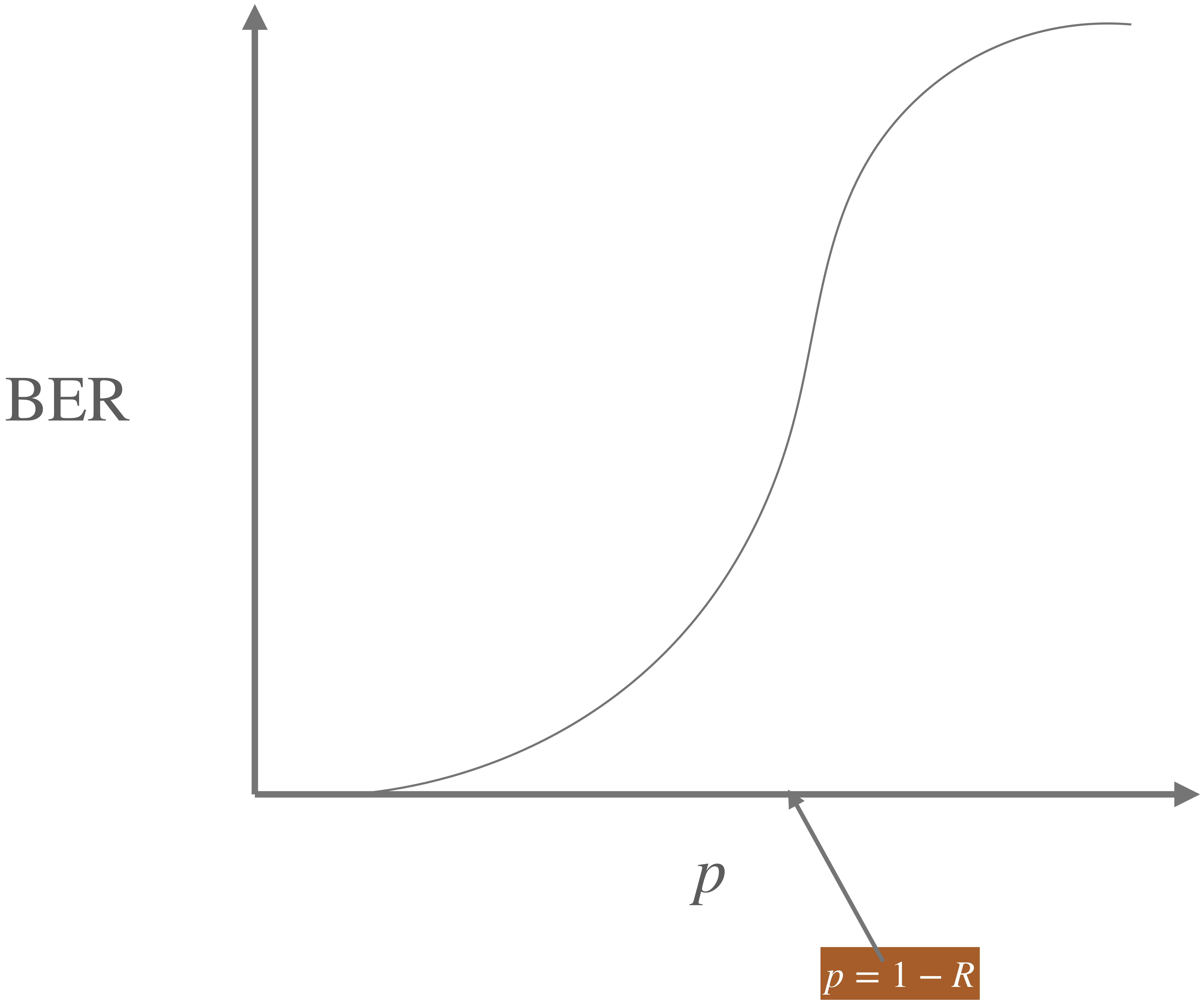
- ***Capacity-Achieving Codes for the Erasure Channel:***

  a sequence of codes for some rate R with increasing values of N such that

$$\mathrm{BER}, \mathrm{CER} \to 0 \quad \text{as} \quad N \to \infty \qquad\qquad \text{for all} \quad p < 1 - R$$



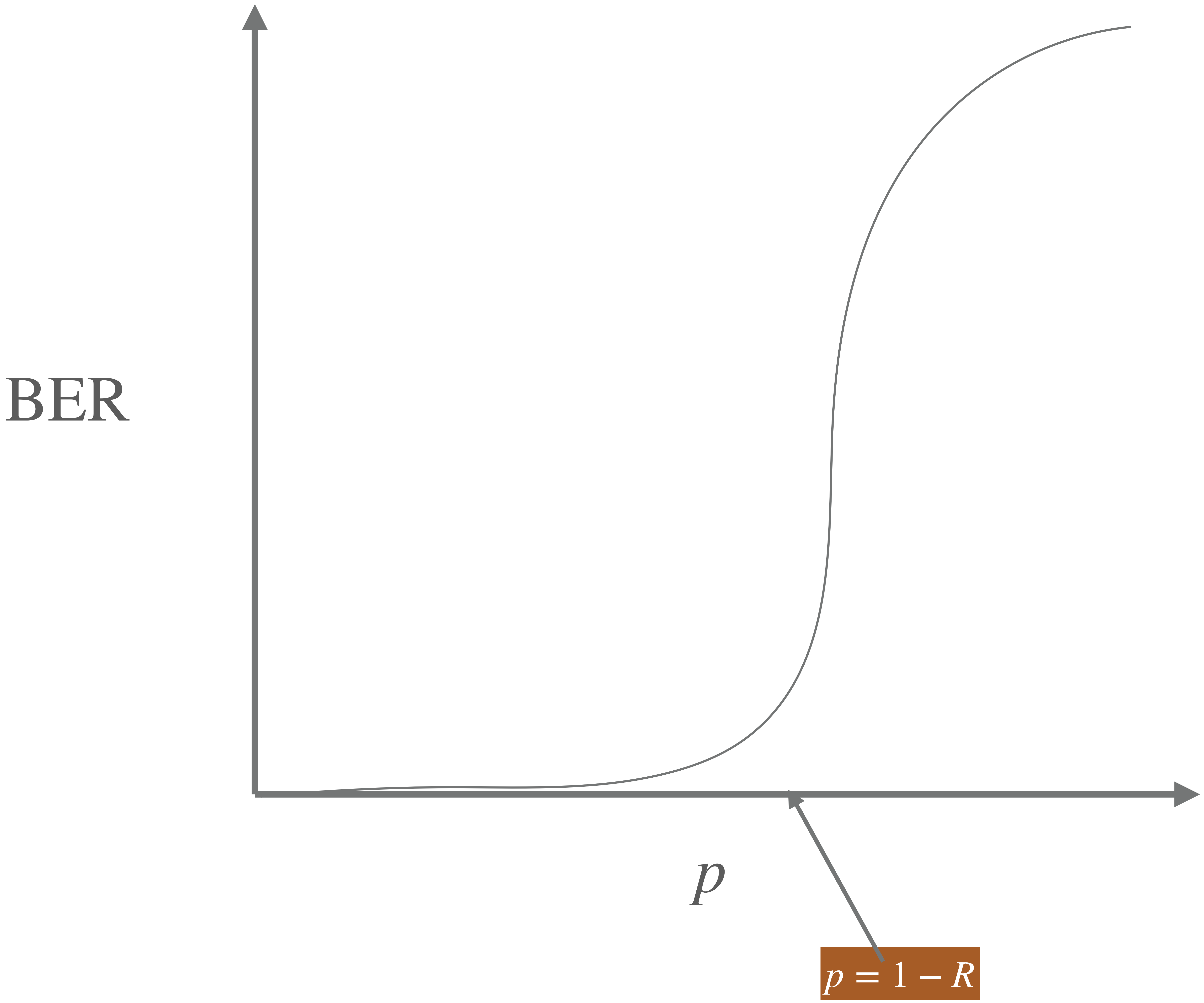
Code $\mathscr{C}_1$
Blocklength N_1
Rate R



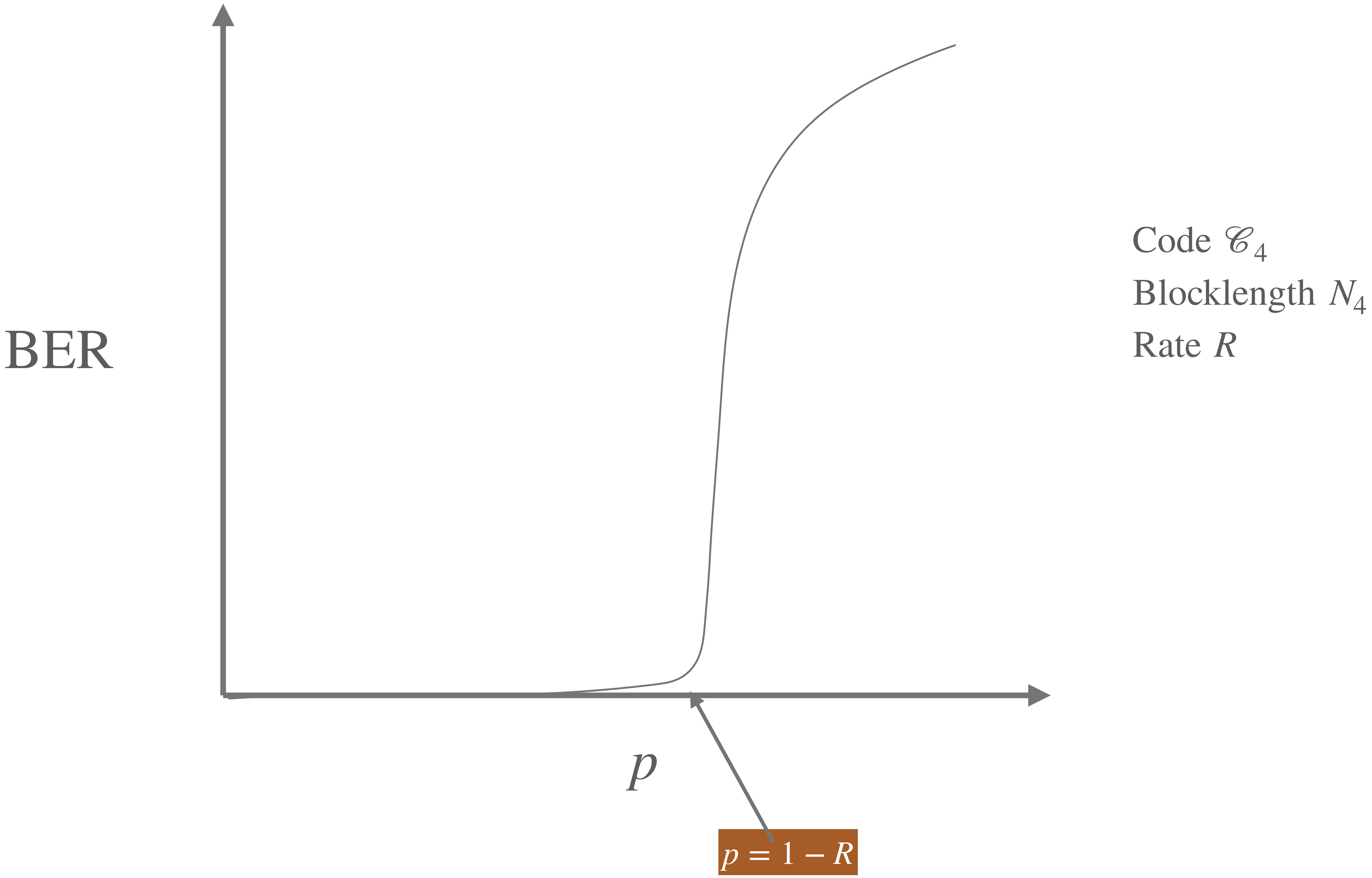
Code $\mathscr{C}_2$

Blocklength N_2

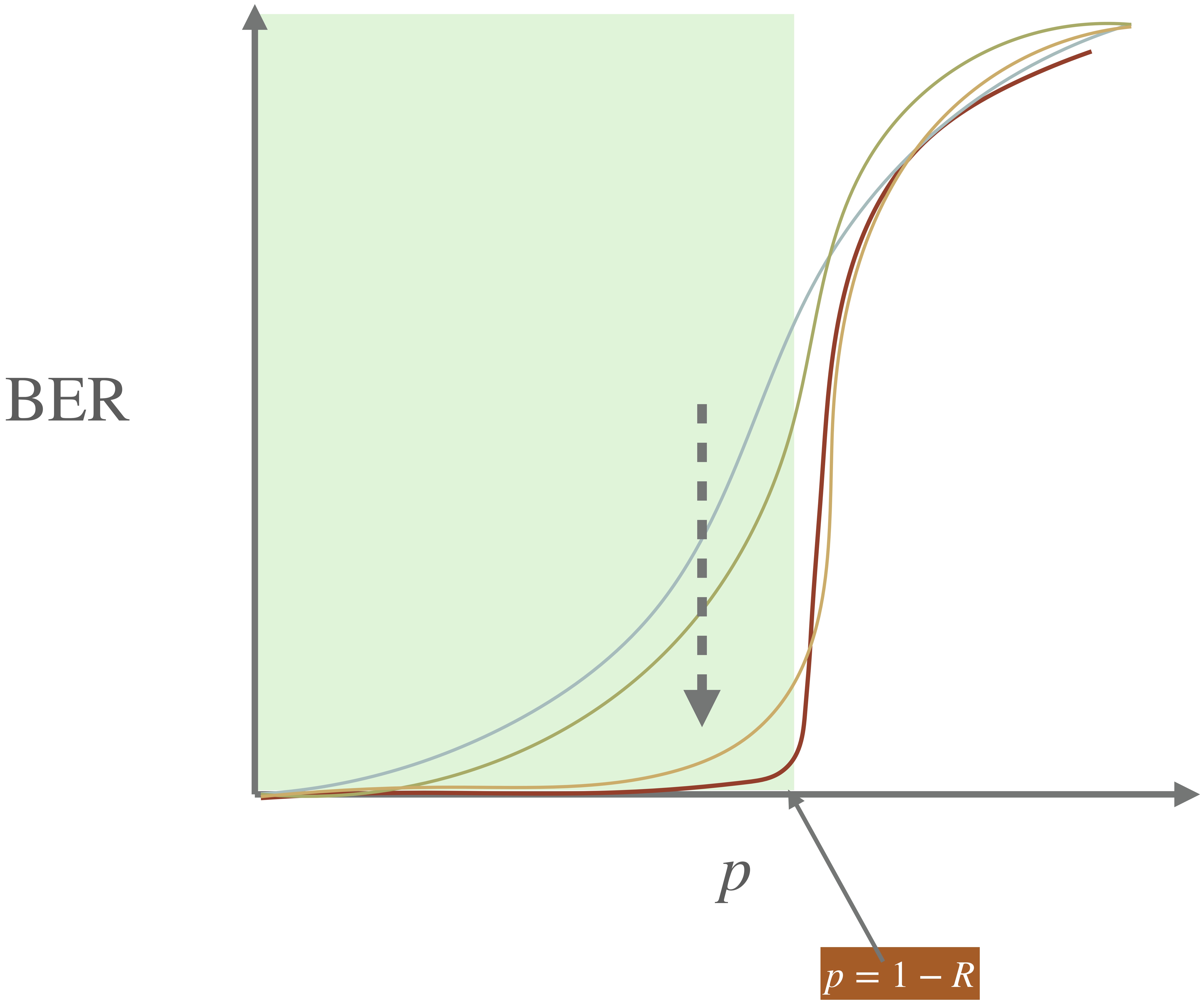
Rate R



Code $\mathscr{C}_3$
Blocklength N_3
Rate R



BER
p
$p = 1 - R$
Code $\mathscr{C}_4$
Blocklength N_4
Rate R



Codes $\mathscr{C}_1, \ldots, \mathscr{C}_4$

Blocklengths $N_1, \ldots, N_4$

Rate R

KNOWN CAPACITY-ACHIEVING CODES

<table>
<tr><td>Irregular LDPC Codes</td><td>1997</td><td>Luby, Mitzenmacher, Shokrollahi, Spielman, Stemann</td><td>BEC</td><td rowspan="5">CER → 0</td></tr>
<tr><td>Polar Codes</td><td>2009</td><td>Arıkan</td><td>BMS</td></tr>
<tr><td>Spatially-Coupled LDPC Codes</td><td>2011</td><td>Kudekar, Richardson, Urbanke</td><td>BMS</td></tr>
<tr><td>Reed-Muller Codes</td><td>2023</td><td>Abbe and Sandon</td><td>BMS</td></tr>
<tr><td>BCH Codes</td><td rowspan="2">2017</td><td rowspan="2">Kudekar, Kumar, Mondelli, Pfister, Şaşoğlu, Urbanke</td><td>BEC</td></tr>
<tr><td>Quadratic Residue Codes</td><td>BEC</td><td rowspan="3">BER → 0</td></tr>
<tr><td>Some more sequences of Cyclic Codes</td><td>2016</td><td>Kumar, Calderbank, Pfister</td><td>BEC</td></tr>
<tr><td>Berman Codes & their duals, A family of Abelian Codes</td><td>2022</td><td>Natarajan, Krishnan</td><td>BEC</td></tr>
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KNOWN CAPACITY-ACHIEVING CODES

Proof is based on code symmetry (automorphism group)

Irregular LDPC Codes	1997	Luby, Mitzenmacher, Shokrollahi, Spielman, Stemann	BEC	
Polar Codes	2009		BMS	
Spatially-Coupled LDPC Codes	2011	Kudekar, Richardson, Urbanke	BMS	CER → 0
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BERMAN CODES AND THEIR DUALS

- Family of binary linear codes parametrised by 3 integers

$$n \geq 2, \quad m \geq 2, \quad r \in \{0,1,\ldots,m\}$$

- Berman Codes $\quad \mathscr{B}_n(r,m) = \left[n^m,\ \sum_{i=r+1}^{m} \binom{m}{i} (n-1)^i,\ 2^{r+1} \right]$

- Dual Berman Codes $\mathscr{D}_n(r,m) = \mathscr{B}_n(r,m)^\perp = \left[n^m,\ \sum_{i=0}^{r} \binom{m}{i} (n-1)^i,\ n^{m-r} \right]$

  - Reed-Muller codes correspond to $n = 2$: $\quad \mathscr{D}_2(r,m) = \mathrm{RM}(r,m)$

CONSTRUCTION OF BERMAN CODES AND THEIR DUALS

- **Berman Codes for n = odd prime:**
  S.D. Berman, 'Semisimple Cyclic and Abelian Codes,' *Kibernetika,* 1967
  - As ideals in commutative group algebras: *abelian codes*
  - To show that there exist abelian codes with larger min distance than cyclic codes

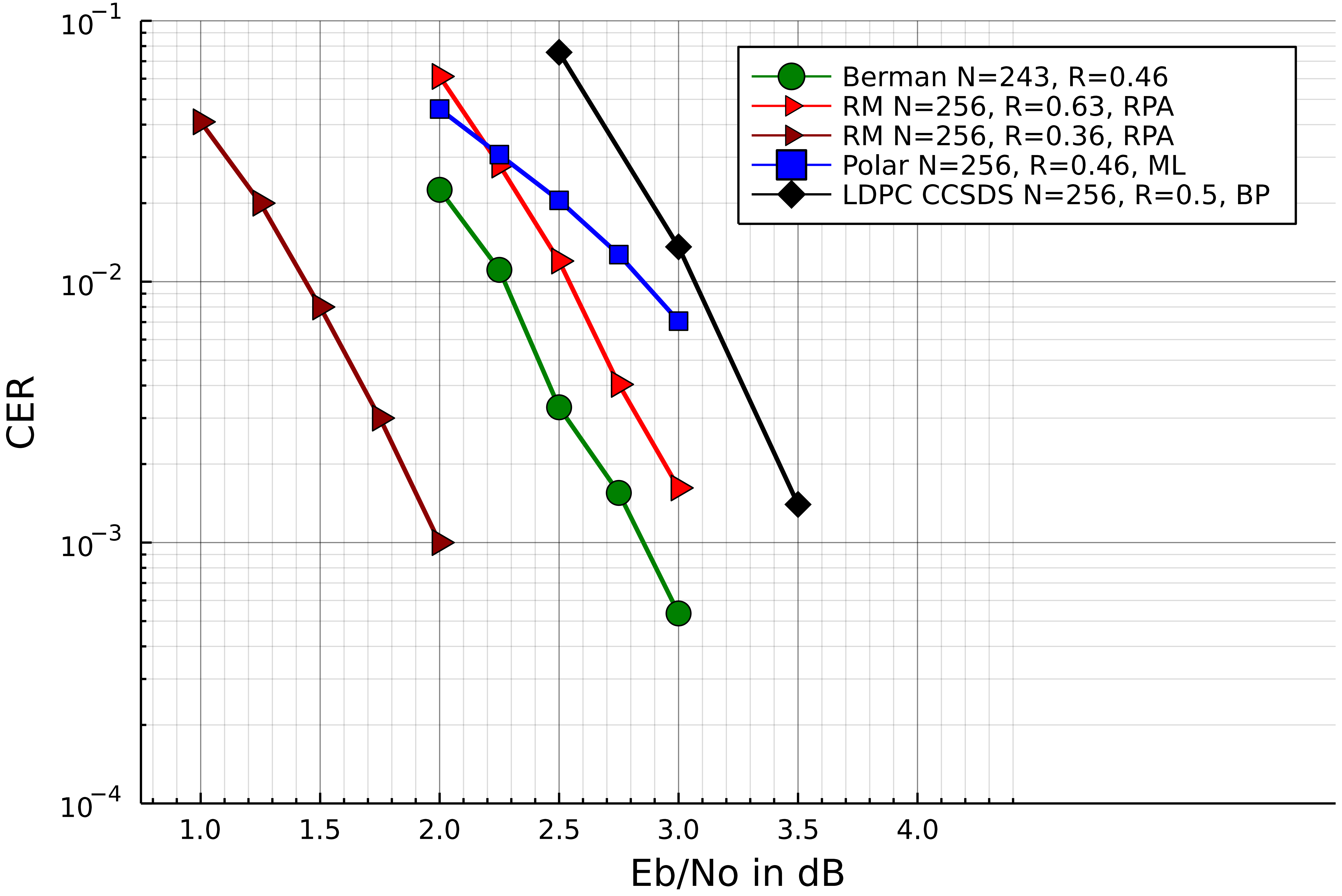
- **Dual Berman Codes**
  Blackmore and Norton, 'On a Family of Abelian Codes and their State Complexities,' *IEEE-IT,* 2001
  - As abelian codes
  - Identify an iterative group algebraic construction that works for all $n \geq 2$
  - Recognise that $n = 2$ yields RM codes

CONSTRUCTION OF BERMAN CODES AND THEIR DUALS

- **Achieve capacity of the binary erasure channel** [N. & Krishnan, *IEEE-IT,* 2023]

  - Simple Plotkin-like construction of Berman codes and their duals (like RM codes)
  - Efficient decoding up to half the minimum distance (like RM codes)
  - Identify some automorphisms
  - Use a result of [Kumar, Calderbank, Pfister, ISIT 2016] that relates automorphism group and capacity-achievability in erasure channels

BERMAN CODE $(n = 3)$ IN ADDITIVE WHITE GAUSSIAN NOISE CHANNEL



$\mathscr{B}_3(3,5)$:

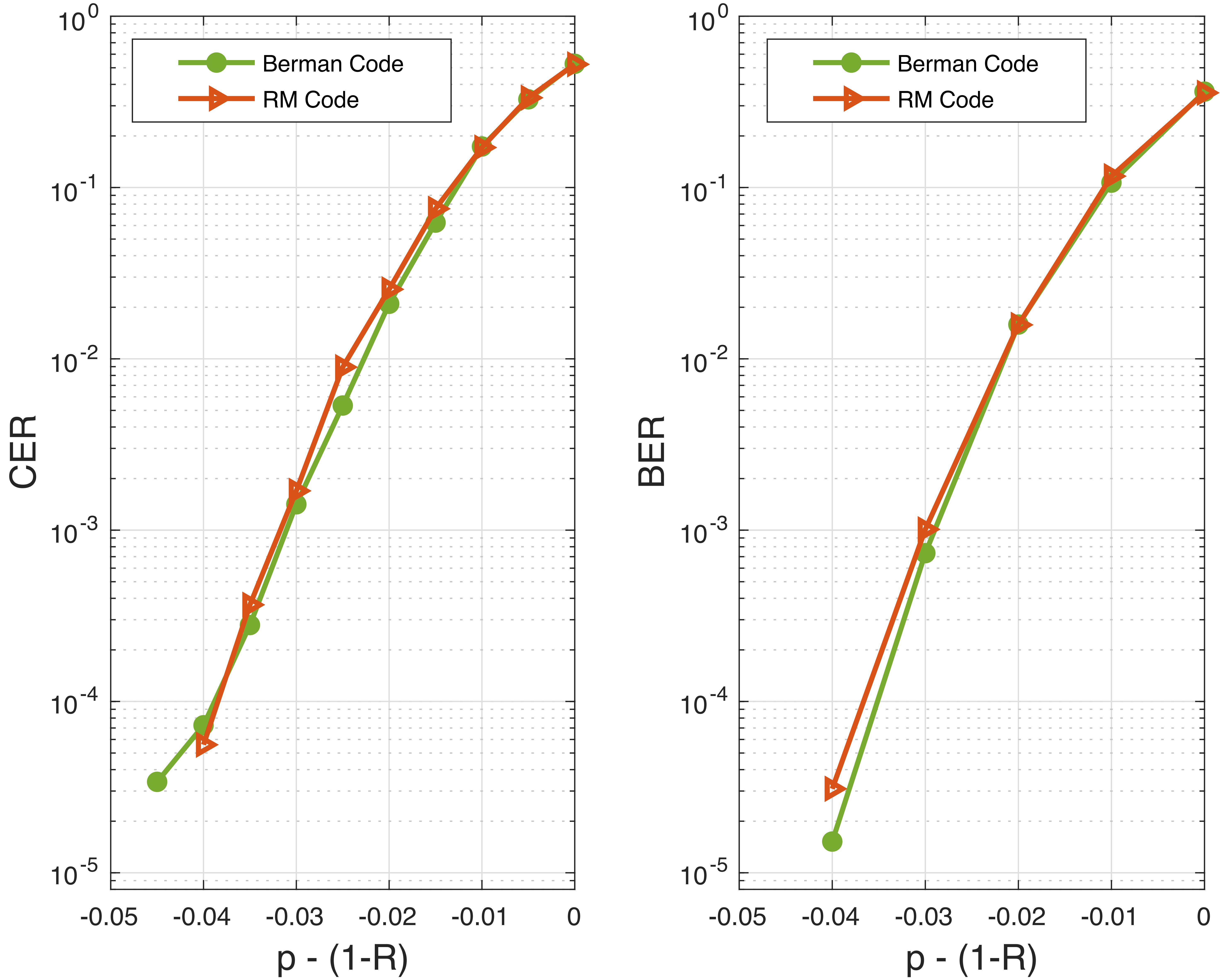
$N = 243,\ R = 0.46,\ d_{\min} = 16$

Compared to RM codes:

- Richer options for rate and length
- Seems to provide graceful trade-off between rate and CER

(Currently working on efficient decoders for Berman codes and their duals)

BERMAN CODE $(n = 3)$ VS. REED-MULLER CODE IN $\mathrm{BEC}(p)$



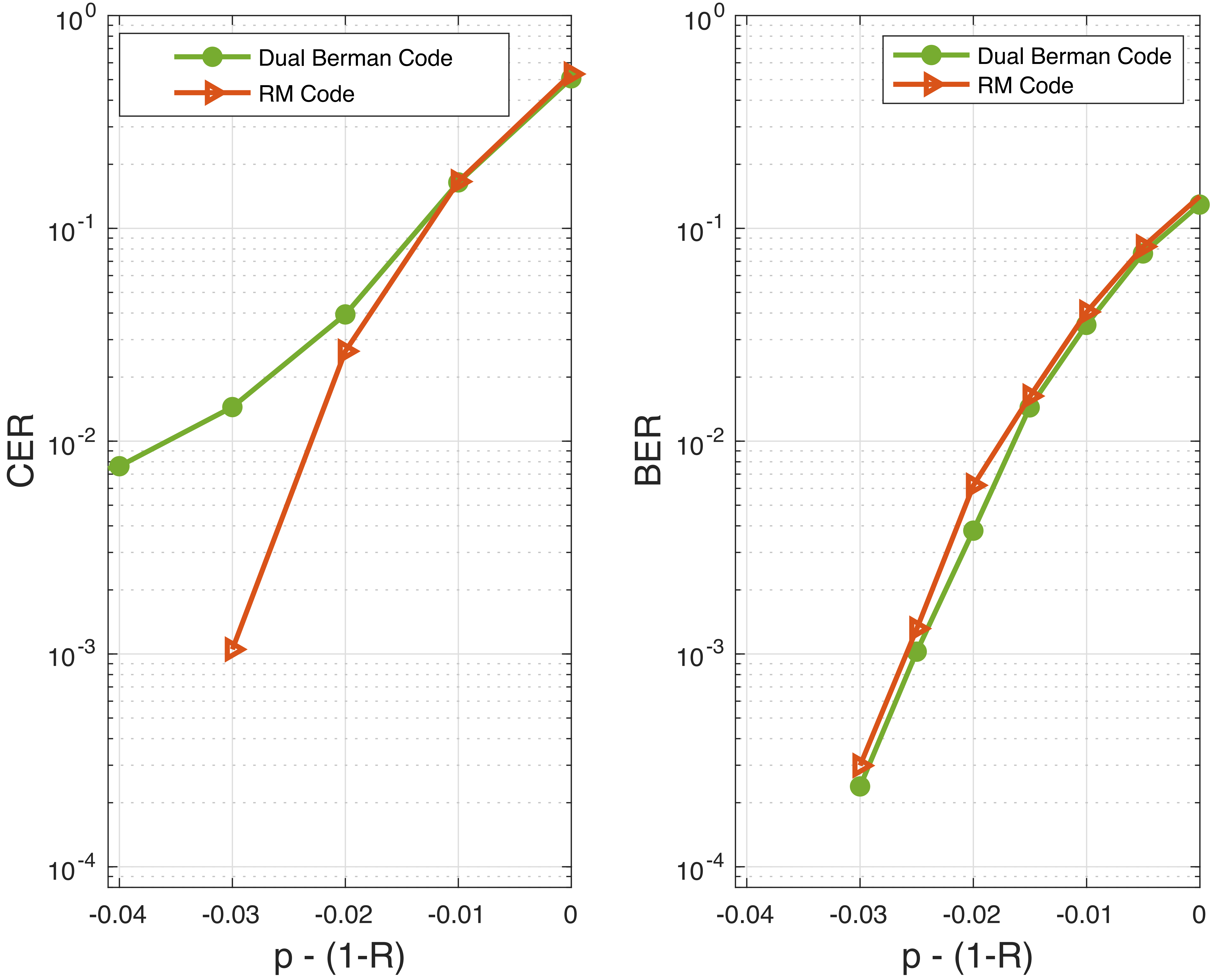
- $\mathscr{B}_3(5,7)$:

  $N = 2187,\ R = 0.26,\ d_{\min} = 64$

- $\mathrm{RM}(4,11)$:

  $N = 2048,\ R = 0.27,\ d_{\min} = 128$

DUAL BERMAN CODE $(n = 3)$ VS. REED-MULLER CODE IN $\mathrm{BEC}(p)$



- $\mathscr{D}_3(5,7)$:

  $N = 2187,\ R = 0.74,\ d_{\min} = 9$

- $\mathrm{RM}(6,11)$:

  $N = 2048,\ R = 0.73,\ d_{\min} = 32$

BERMAN CODES: CONSTRUCTION & PROPERTIES

LINEAR CODES: SUBSPACES OF HAMMING SPACE $\{0,1\}^N$

- $\mathbb{F}_2 = \{0,1\}$ is the binary field
  - addition is XOR, multiplication is AND
- $\mathbb{F}_2^N$ is a vector space
  - Contains 2^N binary vectors of length N
- $\mathscr{C} \subset \mathbb{F}_2^N$ is a **subspace/linear code** if closed under addition
  - $\mathbf{a}, \mathbf{b} \in \mathscr{C} \Rightarrow \mathbf{a} + \mathbf{b} \in \mathscr{C}$
- $|\mathscr{C}| = 2^K$ if dimension of $\mathscr{C}$ is K
  - encode K message bits into a codeword of length N

$$\mathbb{F}_2^3 = \{(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)\}$$

$$\mathscr{C} = \{(0,0,0), (0,1,1), (1,1,0), (1,0,1)\}$$

EXAMPLES OF LINEAR CODES (BUILDING BLOCKS OF BERMAN CODES)

- Repetition Code of length N

$$\mathscr{C}_{\mathrm{rep},N} = \{(0,\ldots,0),\ (1,\ldots,1)\}$$

- Single Parity Check Code of length N

$$\mathscr{C}_{\mathrm{spc},N} = \left\{\mathbf{a} \in \mathbb{F}_2^N \ :\ a_1 + \cdots + a_N = 0\right\}$$

- Trivial Linear Codes

$$\mathscr{C} = \mathbb{F}_2^N \text{ (no redundancy)}, \qquad \mathscr{C} = \{\ (0,\ldots,0)\ \} \text{ (no information transmission)}$$

ORIGINAL CONSTRUCTION OF BERMAN

- Let n **be an odd prime**, and $m \geq 2$ and $r \in \{0,1,\ldots,m\}$

- Consider the quotient ring $\mathscr{R}_{n,m} = \mathbb{F}_2[X_1, \ldots, X_m] \,/\, \langle X_1^n - 1, \ldots, X_m^n - 1 \rangle$
  - This is the group algebra $\mathbb{F}_2\left[C_n \times \cdots \times C_n\right]$
  - Natural $\mathbb{F}_2$-basis of $\mathscr{R}_{n,m}$: $\left\{ X_1^{i_1} \ldots X_m^{i_m} \;:\; 0 \leq i_1, \ldots, i_m \leq n-1 \right\}$

- Natural $\mathbb{F}_2$-linear map $\rho : \mathscr{R}_{n,m} \rightarrow \mathbb{F}_2^{n^m}$

$$\sum_{i_1=0}^{n-1} \cdots \sum_{i_m=0}^{n-1} a_{(i_1,\ldots,i_m)} X_1^{i_1} \cdots X_m^{i_m} \rightarrow \left(a_{(i_1,\ldots,i_m)} \;:\; 0 \leq i_1, \ldots, i_m \leq n-1 \right)$$

ORIGINAL CONSTRUCTION OF BERMAN

- $\mathscr{R}_{n,m} = \mathbb{F}_2[X_1, \ldots, X_m] \,/\, \langle X_1^n - 1, \ldots, X_m^n - 1 \rangle$

- For any $S \subset \{1,2,\ldots,m\}$ define $u_S \in \mathscr{R}_{n,m}$ as $u_S = \prod_{j \notin S} \frac{X_j^n + 1}{X_j + 1} = \prod_{j \notin S} \left(1 + X_j + X_j^2 + \cdots + X_j^{n-1}\right)$

- Ideal $\mathscr{I}_n(r,m) = \langle u_S : |S| = r \rangle$

  its annihilator $\mathscr{A}_n(r,m) = \left\{y \in \mathscr{R}_{n,m} \ : \ yz = 0 \ \forall z \in \mathscr{I}_n(r,m)\right\}$

Berman

Berman Code $\mathscr{B}_n(r,m) = \rho\left(\mathscr{A}_n(r,m)\right)$

Blackmore & Norton

Dual Berman Code $\mathscr{D}_n(r,m) = \rho\left(\mathscr{I}_n(r,m)\right)$

PLOTKIN-LIKE CONSTRUCTION OF BERMAN CODES

- Let $n \geq 2, \quad m \geq 2, \quad r \in \{0,1,\ldots,m\}$

- When $r = m$: $\quad \mathscr{B}_n(r = m, m) = \{(0,\ldots,0)\}$

- When $r = 0$: $\quad \mathscr{B}_n(r = 0, m) = \left\{ (a_1, \ldots, a_{n^m}) \; : \; \sum_i a_i = 0 \right\}$

- When $1 \leq r \leq m-1$:

$$\mathscr{B}_n(r, m) = \left\{ (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) \; : \; \mathbf{v}_i \in \mathscr{B}_n(r-1, m-1), \sum_i \mathbf{v}_i \in \mathscr{B}_n(r, m-1) \right\}$$

EXAMPLE: $n = 3, r = 1, m = 2$

- $\mathscr{B}_n(r,m) \subset \mathbb{F}^9$ $\quad\mathscr{B}_n(r-1, m-1) = \mathscr{C}_{\text{spc},3}$ $\quad\mathscr{B}_n(r, m-1) = \{\,(0,0,0)\,\}$

- $\mathscr{B}_3(1,2) = \left\{(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \;:\; \mathbf{v}_i \in \mathscr{C}_{\text{spc},3},\;\; \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = (0,0,0)\right\}$

- (1,0,1, 0,1,1, 1,1,0) is a codeword of $\mathscr{B}_3(1,2)$:

  $(1,0,1)\,,\,(0,1,1),\;(1,1,0)\;\in \mathscr{C}_{\text{spc},3}$

  $(1,0,1) + (0,1,1) + (1,1,0) = (0,0,0) \in \mathscr{B}_3(1,1) = \{(0,0,0)\}$

PLOTKIN-LIKE CONSTRUCTION OF DUAL BERMAN CODES

- Let $n \geq 2, \quad m \geq 2, \quad r \in \{0,1,\ldots,m\}$
- When $r = m$: $\quad \mathcal{D}_n(r = m, m) = \mathbb{F}_2^{n^m}$
- When $r = 0$: $\quad \mathcal{D}_n(r = 0, m) = \{(0,\ldots,0), (1,\ldots,1)\}$
- When $1 \leq r \leq m-1$:

$$\mathcal{D}_n(r,m) = \Big\{ (\mathbf{u}+\mathbf{u}_1, \mathbf{u}+\mathbf{u}_2, \ldots, \mathbf{u}+\mathbf{u}_{n-1}, \mathbf{u}) \ : \ \mathbf{u}_i \in \mathcal{D}_n(r-1, m-1), \mathbf{u} \in \mathcal{D}_n(r, m-1) \Big\}$$

WHEN $n = 2$ DUAL BERMAN CODES ARE RM CODES (PLOTKIN CONSTRUCTION)

- Use $n = 2$
- When $r = m$: $\quad \mathscr{D}_2(r = m, m) = \mathbb{F}_2^{2^m}$
- When $r = 0$: $\quad \mathscr{D}_2(r = 0, m) = \{(0,\ldots,0), (1,\ldots,1)\}$
- When $1 \leq r \leq m - 1$:

$$\mathscr{D}_2(r, m) = \left\{ (\mathbf{u} + \mathbf{u}_1, \mathbf{u}) \ : \ \mathbf{u}_1 \in \mathscr{D}_2(r-1, m-1), \mathbf{u} \in \mathscr{D}_2(r, m-1) \right\}$$

BASE-n INDEXING OF ENTRIES OF A CODEWORD

- Replace natural indexing that uses $i \in \{1,2,\ldots,n^m\}$ with
  the base-n expansion of $(i-1) \in \{1,\ldots,n^m\}$

$$(i-1) \;=\; \sum_{k=1}^{m} i_k\, n^{(k-1)}, \quad \text{where } i_k \in [n] = \{0,1,\ldots,n-1\}$$

- Use the vector $\mathbf{i} = (i_1, i_2, \ldots, i_m)$ as an index instead of the integer i
  (note that $\mathbf{i} \in \{0,1,\ldots,n-1\}^m = [n]^m$)

- Codeword $\mathbf{a} = (\, a_{\mathbf{i}} \;:\; \mathbf{i} \in [n]^m\,)$, where each $a_{\mathbf{i}} \in \mathbb{F}_2$

BASES FOR BERMAN AND DUAL BERMAN CODES

- Recall: $[n] = \{0,1,\ldots,n-1\}$, $[n]^m = \{0,1,\ldots,n-1\}^m$

  Index the n^m coordinates of a codeword using elements of $[n]^m$

- Define a partial order on $[n]^m$

  $$(i_1, \ldots, i_m) \preceq (j_1, \ldots, j_m) \quad \text{if and only if} \quad i_k \in \{j_k\} \cup \{0\} \quad \text{for all } k$$

  Example: $(0,0,1) \preceq (0,2,1) \preceq (1,2,1)$

- Define the Hamming weight $w\big((i_1, \ldots, i_m)\big)$ as usual.

BASES FOR BERMAN AND DUAL BERMAN CODES

- Recall partial order: $(i_1, \ldots, i_m) \preceq (j_1, \ldots, j_m)$ iff $i_k \in \{j_k\} \cup \{0\}$ for all k.

- For each $(j_1, \ldots, j_m) \in [n]^m$ define $\mathbf{b}(j_1, \ldots, j_m), \mathbf{d}(j_1, \ldots, j_m) \in \mathbb{F}_2^{n^m}$

$$\text{Support of } \mathbf{b}(j_1, \ldots, j_m) = \text{set of all } (i_1, \ldots, i_m) \preceq (j_1, \ldots, j_m)$$

$$\text{Support of } \mathbf{d}(j_1, \ldots, j_m) = \text{set of all } (i_1, \ldots, i_m) \succeq (j_1, \ldots, j_m)$$

An $\mathbb{F}_2$-basis for $\mathscr{B}_n(r, m) = \left\{ \mathbf{b}(j_1, \ldots, j_m) \ : \ w\left((j_1, \ldots, j_m)\right) \geq r + 1 \right\}$

An $\mathbb{F}_2$-basis for $\mathscr{D}_n(r, m) = \left\{ \mathbf{d}(j_1, \ldots, j_m) \ : \ w\left((j_1, \ldots, j_m)\right) \leq r \right\}$

BASES FOR BERMAN AND DUAL BERMAN CODES

Consider $A_n^{\otimes m}$, where $A_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathbb{F}_2^{n \times n}$

Rows of $A_n^{\otimes m}$ with Hamming weight $\geq 2^{r+1}$ generate $\mathscr{B}_n(r, m)$

Columns of $A_n^{\otimes m}$ with Hamming weight $\geq n^{m-r}$ generate $\mathscr{D}_n(r, m)$

SPECTRAL DESCRIPTION (ODD n)

- Use $[n]^m$ to index the coordinates of n^m-length codewords

$$\mathbf{c} = \left(c_{(i_1,\ldots,i_m)} \; : \; (i_1, \ldots, i_m) \in [n]^m \right)$$

- Let α be a primitive n^{th} root of unity (from the appropriate field extension of $\mathbb{F}_2$)

- The m-dimensional discrete Fourier transform of $\mathbf{c}$ is

$$\hat{\mathbf{c}} = \left(\hat{c}_{(j_1,\ldots,j_m)} \; : \; (j_1, \ldots, j_m) \in [n]^m \right), \quad \text{where } \hat{c}_{(j_1,\ldots,j_m)} = \sum_{(i_1,\ldots,i_m)} c_{(i_1,\ldots,i_m)} \, \alpha^{i_1 j_1 + \cdots + i_m j_m}$$

SPECTRAL DESCRIPTION (ODD n)

- Recall: $\hat{c}_{(j_1,\ldots,j_m)} = \sum_{(i_1,\ldots,i_m)} c_{(i_1,\ldots,i_m)} \alpha^{i_1 j_1 + \cdots + i_m j_m}$

- Berman Code $\mathscr{B}_n(r,m) = \left\{ \mathbf{c} \in \mathbb{F}_2^{n^m} \ : \ \hat{c}_{(j_1,\ldots,j_m)} = 0, \ \ w((j_1,\ldots,j_m)) \le r \right\}$

- Dual Berman Code. $\mathscr{D}_n(r,m) = \left\{ \mathbf{c} \in \mathbb{F}_2^{n^m} \ : \ \hat{c}_{(j_1,\ldots,j_m)} = 0, \ \ w((j_1,\ldots,j_m)) > r \right\}$

AUTOMORPHISMS & CAPACITY-ACHIEVABILITY

AUTOMORPHISMS OF A CODE

- Suppose $\pi : [n]^m \to [n]^m$ is a permutation (an invertible map)

- Apply π on the indices of a codeword to permute its coordinates:

$$\pi(\mathbf{a}) = (a_{\pi(\mathbf{i})} \ : \ \mathbf{i} \in [n]^m)$$

$$\mathbf{i}^{\text{th}} \text{ entry of } \pi(\mathbf{a}) = \pi(\mathbf{i})^{\text{th}} \text{ entry of } \mathbf{a}$$

Automorphism Group of a Code $\mathscr{C}$

$$\mathrm{Aut}(\mathscr{C}) \triangleq \{ \pi \ : \ \pi(\mathbf{a}) \in \mathscr{C} \text{ for all } \mathbf{a} \in \mathscr{C} \}$$

EXAMPLE: AN AUTOMORPHISM OF $\mathscr{B}_3(1,2)$

$\pi(\mathbf{i}) =$

0	2	1	0	2	1	0	2	1
1	1	1	2	2	2	0	0	0

$\mathbf{i} =$

0	1	2	0	1	2	0	1	2
0	0	0	1	1	1	2	2	2

$\mathbf{a} =$

1	0	1	0	1	1	1	1	0

$\in \mathscr{B}_3(1,2)$

$\pi(\mathbf{a}) =$

0	1	1	1	0	1	1	1	0

$\in \mathscr{B}_3(1,2)$

AUTOMORPHISMS OF $\mathscr{B}_n(r,m)$ AND $\mathscr{D}_n(r,m)$

- **Theorem:** The following maps are automorphisms of Berman codes and their duals

  - for any permutations $\sigma_1, \ldots, \sigma_m$ of the set $[n] = \{0,1,\ldots,n-1\}$

$$(i_1, i_2, \ldots, i_m) \rightarrow \left(\sigma_1(i_1), \sigma_2(i_2), \ldots, \sigma_m(i_m)\right)$$

  - for any permutation γ of the set $\{1,\ldots,m\}$

$$(i_1, i_2, \ldots, i_m) \rightarrow \left(i_{\gamma(1)}, i_{\gamma(2)}, \ldots, i_{\gamma(m)}\right)$$

ALL SUFFICIENTLY SYMMETRIC CODES ACHIEVE BEC CAPACITY

➤ [Kumar, Calderbank & Pfister, 2016] showed that code sequences with rich-enough automorphism groups achieve BEC capacity (BER $\to$ 0)

1. Codes must be ***transitive***

   for every $\mathbf{i} \neq \mathbf{j}$ there must exist a $\pi \in \mathrm{Aut}(\mathscr{C})$ such that $\pi(\mathbf{i}) = \mathbf{j}$

2. Minimum orbit size under a specific subgroup of automorphisms must grow unboundedly as $N \to \infty$

$$\mathcal{O}_{\min}(\mathscr{C}) = \min_{\mathbf{i} \neq \mathbf{0}} \left| \left\{ \pi(\mathbf{i}) \ : \ \pi \in \mathrm{Aut}(\mathscr{C}),\ \pi(\mathbf{0}) = \mathbf{0} \right\} \right|$$

1. BERMAN CODES ARE TRANSITIVE

- Suppose $(i_1, \ldots, i_m) \neq (j_1, \ldots, j_m)$

- Pick any $\sigma_1, \ldots, \sigma_m$ (permutations on $[n]$) such that

$$\sigma_1(i_1) = j_1, \;\; \ldots, \; \sigma_m(i_m) = j_m$$

- Consider the automorphism

$$\pi \;:\; \left(i_1, i_2, \ldots, i_m \right) \;\rightarrow\; \left(\sigma_1\left(i_1\right), \sigma_2\left(i_2\right), \ldots, \sigma_m\left(i_m\right) \right)$$

- Then $\pi(\mathbf{i}) = \mathbf{j}$

2. ORBIT UNDER AUTOMORPHISMS THAT FIX $(0,\ldots,0)$

Consider $\mathbf{i} = (i_1, \ldots, i_m) \neq (0,\ldots,0)$

1. Apply all $\pi : \left(i_1, i_2, \ldots, i_m\right) \to \left(\sigma_1\left(i_1\right), \sigma_2\left(i_2\right), \ldots, \sigma_m\left(i_m\right)\right)$ such that
   - $\sigma_1(0) = 0, \ \ldots, \sigma_m(0) = 0$

   ➤ All vectors with the same support as $\mathbf{i}$ are in the orbit of $\mathbf{i}$

2. Apply all $\pi : \left(i_1, i_2, \ldots, i_m\right) \to \left(i_{\gamma(1)}, i_{\gamma(2)}, \ldots, i_{\gamma(m)}\right)$

   ➤ All vectors with the same Hamming weight as $\mathbf{i}$ are in the orbit of $\mathbf{i}$

2. ORBIT UNDER AUTOMORPHISMS THAT FIX $(0,\ldots,0)$

Consider $\mathbf{i} = (i_1, \ldots, i_m) \neq (0,\ldots,0)$

The size of the orbit of $\mathbf{i}$ under this subgroup of automorphisms:

$$\left| \left\{ \pi(\mathbf{i}) \ : \ \pi \in \mathrm{Aut}(\mathscr{C}),\ \pi(\mathbf{0}) = \mathbf{0} \right\} \right| \ \geq \ \#\text{vectors with weight } w(\mathbf{i}) = \binom{m}{w(\mathbf{i})}(n-1)^{w(\mathbf{i})}$$

$$\mathcal{O}_{\min} \geq \min_{\mathbf{i} \neq \mathbf{0}} \binom{m}{w(\mathbf{i})}(n-1)^{w(\mathbf{i})} = (n-1)m \geq 2m \qquad \text{if } n \geq 3.$$

ALGEBRAIC CRITERION TO ACHIEVE CAPACITY OF BEC (BER $\to 0$)

Consider a sequence of codes with strictly increasing block lengths

[Kumar, Calderbank, Pfister, 2016]: If the sequence of codes satisfies

1. The rate of this sequence of codes $\to R$
2. Each code in this sequence has a transitive group of automorphisms
3. $\mathcal{O}_{\min} \to \infty$

As long as $R < 1 - p$, then for this sequence of codes:

the probability of recovering each codeword bit in $\mathrm{BEC}(p) \to 1$.

APPLYING [KUMAR, CALDERBANK, PFISTER, 2016]

- Pick any $n \geq 3$.

- Choose any desired code rate $R \in (0,1)$.

- Construct sequence of Berman codes

  for each $m \geq 2$, choose order $r_m \in \{0,1,\ldots,m\}$ to be the integer closest to

$$m\left(\frac{n-1}{n}\right) + Q^{-1}(R)\sqrt{m\frac{(n-1)}{n^2}}$$

  where $Q(x) = \int_{t=x}^{+\infty} (2\pi)^{-1/2} e^{-t^2/2}\, dt$.

BERMAN CODES ACHIEVE VANISHING BER IN BEC FOR RATES UP TO CAPACITY

For $n \geq 3$, consider the sequence of Berman codes $\mathscr{B}_n(r_m, m)$, $m = 2,3,\ldots$

This sequence of codes satisfies the criteria of [Kumar, Calderbank, Pfister, 2016]

1. The rate of this sequence of codes $\rightarrow$ R
2. Each code in this sequence has a transitive group of automorphisms
3. $\mathscr{O}_{\min} \geq 2m \rightarrow \infty$

If $R < 1 - p$ then as $m \rightarrow \infty$,

the probability of recovering each codeword bit in BEC(p) $\rightarrow$ 1.

(a similar result holds for dual Berman codes)

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