Relativistic Electrodynamics

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Motivation and strategy

- We began by saying that Maxwell's laws of electrodynamics predict speed of light to be a constant, something that does hold across all inertial frames in Galilean relativity
- In order to make these laws hold true in all inertial frames, we had to introduce Lorentz transformations
- We introduced tensors that reside in Minkowski space and obey Lorentz transformation rules
- But Newton's old laws are not invariant under Lorentz transformation. So we redefined momentum and energy such that they become tensors and hence satisfy Lorentz transforms
- Now we express the Maxwell's laws in tensor form so that it can be seen that they satisfy Lorentz transform

Maxwell's equation

- The postulates of relativity were inspired by the need of Maxwell's equations and the constancy of speed of light

$$\nabla \cdot \mathbf{E} = 4\pi\rho \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \tag{3}$$

$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{j} \tag{4}$$

- These are the Maxwell equations in vacuum. We can use the vector potential $\mathbf{A}$ to express the magnetic field

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{5}$$

- because $\nabla \cdot (\nabla \times \mathbf{A}) = 0 \implies \nabla \cdot \mathbf{B} = 0$, so Eq. 2 is satisfied

Potential

- The electric field can be expressed in terms of potential as follows

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \tag{6}$$

- It can be checked that this satisfies Eq. 3

$$\nabla \times \mathbf{E} = -\nabla \times \nabla\phi - \frac{1}{c}\frac{\nabla \times \mathbf{A}}{\partial t} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \tag{7}$$

- If we plug them into the other 2 inhomogeneous equations (eq. 1 and 4), then we get

$$\nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -4\pi\rho \tag{8}$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \phi}{\partial t}\right) - \frac{4\pi}{c}\mathbf{j} \tag{9}$$

- Here we have used the vector identity $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

Gauge transformation

- There is an inherent freedom in choosing the potentials. For ex. the vector potential can be changed by an arbitrary field $\nabla\Psi$, because $\nabla \times \nabla\Psi = 0$. So lets define a new vector potential $\mathbf{A}' = \mathbf{A} + \nabla\Psi$
- We can see that this still gives us the old magnetic field

$$\nabla \times \mathbf{A}' = \nabla \times (\mathbf{A} + \nabla\Psi) = \nabla \times \mathbf{A} = \mathbf{B} \tag{10}$$

- In order to get the same electric field as Eq. 6, the scalar potential will have to be changed by $\phi' = \phi - \frac{1}{c}\frac{\partial\Psi}{\partial t}$. Lets check if this gives back the same electric field

$$-\nabla\phi' - \frac{1}{c}\frac{\partial\mathbf{A}'}{\partial t} = -\nabla\phi + \frac{1}{c}\frac{\partial\nabla\psi}{\partial t} - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} - \frac{1}{c}\frac{\partial\nabla\psi}{\partial t} \tag{11}$$

$$= -\nabla\phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} = \mathbf{E} \tag{12}$$

- Changing from $(\phi, \mathbf{A}) \rightarrow (\phi', \mathbf{A}')$ is called a gauge transformation

Lorentz gauge

- Let Ψ be adjusted such that the potential fields satisfy "Lorentz gauge" condition

$$\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0 \tag{13}$$

- Can we always find a Ψ which can satisfy this condition?
- Suppose we start with potentials $(\phi, \mathbf{A})$ which do not obey Lorentz gauge. Then we need a Ψ such that

$$\nabla \cdot \mathbf{A}' + \frac{1}{c}\frac{\partial \phi'}{\partial t} = \nabla \cdot \mathbf{A} + \nabla^2\Psi + \frac{1}{c}\frac{\partial \phi}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2} = 0 \tag{14}$$

$$\implies \nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2} = -\nabla \cdot \mathbf{A} - \frac{1}{c}\frac{\partial \phi}{\partial t} \tag{15}$$

- Solving Eq. 15 (which should be possible) will provide us with such a Ψ such that our potentials satisfy the Lorentz gauge.

Charge

- Consider that we have some volume element with charge density ρ, moving with velocity $\mathbf{u}$ in a frame of reference.
- Let this element have a charge density ρ_0 in it's rest frame (which is an invariant)
- When observed from a reference frame in which this volume element is moving with velocity u, parallel length of this volume will be contracted, while the perpendicular lengths will remain unchanged. So the volume will be

$$dV = dV_0\sqrt{1 - \frac{u^2}{c^2}} \tag{16}$$

- However, it is experimentally verified that the total charge is an invariant. So $\rho dV = \rho_0 dV_0$. This implies

$$\rho = \frac{\rho_0}{\sqrt{1 - \frac{u^2}{c^2}}} \tag{17}$$

Current density

- The current density of a moving charge is given by $\mathbf{j} = \rho \mathbf{u}$.
- Now consider a 4-vector defined as $J^\mu = \rho_0 U^\mu$. ρ_0 is the rest charge density which is an invariant. U^μ we have defined before as $dX^\mu/d\tau$.
- We can see that J^μ will be a contravariant vector.
- The time element of J^μ is

$$J^0 = c\rho_0(dt/d\tau) = \frac{c\rho_0}{\sqrt{1 - \frac{u^2}{c^2}}} = c\rho \tag{18}$$

- So the time component is directly proportional to the charge density
- The space component is

$$J^i = \rho_0(dX^i/d\tau) = \rho_0 u^i(dt/d\tau) = \rho u^i; \quad i = 1, 2, 3 \tag{19}$$

Charge conservation

- So J^μ is the charge-current density 4-vector

$$J^\mu = \begin{pmatrix} c\rho \\ \rho\mathbf{u} \end{pmatrix} = \begin{pmatrix} c\rho \\ \mathbf{j} \end{pmatrix} \tag{20}$$

- Now consider the charge conservation equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \tag{21}$$

- This can be simply written as

$$\frac{\partial (c\rho)}{\partial (ct)} + \frac{\partial j^i}{\partial X^i} = \frac{\partial J^\mu}{\partial X^\mu} = 0; \qquad i = 1, 2, 3 \tag{22}$$

- We have written this equation in tensor form. This means that this equation will hold in any inertial frame, as the quantities will obey Lorentz transformations, i.e. $\frac{\partial J'^\mu}{\partial X'^\mu} = 0$

Potential 4-vector

- Using Lorentz gauge Eqs. 8 and 9 can we written as

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho \tag{23}$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{j} \tag{24}$$

- The operator on the lhs $\nabla^2 - (1/c^2)\partial^2/\partial t^2$ is called as the d'Alembertian operator and represented by $\Box^2$.
- We have already created a charge-current density 4-vector $J^\mu = (c\rho, \mathbf{j})$.
- Now we can define the potential 4-vector $A^\mu = (\phi, \mathbf{A})$

d'Alembertian

- In 4-vector notation, the d'Alembertian operator can be written as

$$\Box^2 = g^{\mu\nu} \frac{\partial}{\partial X^\mu} \frac{\partial}{\partial X^\nu} \tag{25}$$

- If we transform it to frame S' then

$$\Box'^2 = g^{\mu\nu} \frac{\partial}{\partial X'^\mu} \frac{\partial}{\partial X'^\nu} = g^{\mu\nu} \left(\frac{\partial X^\alpha}{\partial X'^\mu} \right) \left(\frac{\partial X^\beta}{\partial X'^\nu} \right) \frac{\partial}{\partial X^\alpha} \frac{\partial}{\partial X^\beta} \tag{26}$$

$$= g^{\mu\nu} (\Lambda^{-1})^\alpha_\mu (\Lambda^{-1})^\beta_\nu \frac{\partial}{\partial X^\alpha} \frac{\partial}{\partial X^\beta} \tag{27}$$

- Consider $g^{\mu\nu}(\Lambda^{-1})^\alpha_\mu(\Lambda^{-1})^\beta_\nu$. If $\alpha = \beta = 0$ then this becomes $-\gamma^2 + \gamma^2 v^2/c^2 = -1$. If $\alpha = \beta = 1$ then this is $\gamma^2 - \gamma^2 v^2/c^2 = 1$. If $\alpha = \beta = 2, 3$, then this is 1. In rest all combinations it can be checked that this is 0. This shows that

$$\Box'^2 = g^{\alpha\beta} \frac{\partial}{\partial X^\alpha} \frac{\partial}{\partial X^\beta} \tag{28}$$

d'Alembertian

- So the d'Alembertian is a scalar operator. Using this we can express the Maxwell's equations in a 4-vector form as

$$\Box^2 A^\mu = -\frac{4\pi}{c} J^\mu \tag{29}$$

- Since J^μ is contravariant, so A^μ is also a contravariant vector.
- The Lorentz gauge condition can be written as

$$\frac{\partial A^\mu}{\partial X^\mu} = 0 \tag{30}$$

- What about the electric and magnetic field vectors?

Electromagnetic tensor

- Define the 2nd-rank antisymmetric tensor

$$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} \tag{31}$$

$$= \frac{\partial A^{\nu}}{\partial X_{\mu}} - \frac{\partial A^{\mu}}{\partial X_{\nu}} \tag{32}$$

- $X_{\nu} = g_{\nu\mu} X^{\mu} = (-ct, x, y, z)$. Lets calculate some elements of $F^{\mu\nu}$.

$$F^{00} = \frac{\partial A^0}{\partial X_0} - \frac{\partial A^0}{\partial X_0} = 0 = F^{11} = F^{22} = F^{33} \tag{33}$$

$$F^{01} = \frac{\partial A^1}{\partial X_0} - \frac{\partial A^0}{\partial X_1} = \frac{\partial A_x}{\partial(-ct)} - \frac{\partial \phi}{\partial x} = E_x \tag{34}$$

$$F^{02} = \frac{\partial A^2}{\partial X_0} - \frac{\partial A^0}{\partial X_2} = \frac{\partial A_y}{\partial(-ct)} - \frac{\partial \phi}{\partial y} = E_y \tag{35}$$

Electromagnetic tensor

- Continue with calculation of $F^{\mu\nu}$ tensor

$$F^{03} = \frac{\partial A^3}{\partial X_0} - \frac{\partial A^0}{\partial X_3} = \frac{\partial A_z}{\partial(-ct)} - \frac{\partial \phi}{\partial z} = E_z \tag{36}$$

$$F^{12} = \frac{\partial A^2}{\partial X_1} - \frac{\partial A^1}{\partial X_2} = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = B_z \tag{37}$$

$$F^{13} = \frac{\partial A^3}{\partial X_1} - \frac{\partial A^1}{\partial X_3} = \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} = -B_y \tag{38}$$

$$F^{23} = \frac{\partial A^3}{\partial X_2} - \frac{\partial A^2}{\partial X_3} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = B_x \tag{39}$$

- Since $F^{\mu\nu}$ is an anti-symmetric tensor, we have derived all its components

Electromagnetic tensor

- $F^{\mu\nu}$ can be represented as a matrix as follows, where the first index is row number and second index is column number

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \tag{40}$$

- $F^{\mu\nu}$ is derived from the tensors ∂^μ and A^ν, and so we can be certain that it is also a 2nd rank covariant tensor
- How to write Maxwell's equations in terms of $F^{\mu\nu}$?

Covariant ED

- Consider the equation

$$\frac{\partial F^{\alpha\beta}}{\partial X^{\beta}} = \frac{4\pi}{c} J^{\alpha} \tag{41}$$

- If $\alpha = 0$, this becomes

$$\frac{\partial F^{0\beta}}{\partial X^{\beta}} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{4\pi}{c}(c\rho) \tag{42}$$

- This is nothing but Poissons equations $\nabla \cdot \mathbf{E} = 4\pi\rho$ (one of the Maxwell equations)
- If $\alpha = 1$ then it becomes

$$\frac{\partial F^{1\beta}}{\partial X^{\beta}} = \frac{-1}{c}\frac{\partial E_x}{\partial t} + \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) = \frac{4\pi}{c} j_x \tag{43}$$

- One can see that this is nothing but the x-component of the Ampere's law $\nabla \times \mathbf{B} - (1/c)(\partial \mathbf{E}/\partial t) = (4\pi/c)\mathbf{j}$. $\alpha = 2, 3$ give the other 2 components

Covariant ED

- Now consider the equation

$$\frac{\partial F^{\mu\nu}}{\partial X_\lambda} + \frac{\partial F^{\nu\lambda}}{\partial X_\mu} + \frac{\partial F^{\lambda\mu}}{\partial X_\nu} = 0 \tag{44}$$

- If all 3 indices (μ,ν, λ) are same, then this is just $0 = 0$
- Consider $\mu = \nu = 1$, $\lambda = 0$. Then this becomes

$$\frac{\partial F^{10}}{\partial X_1} + \frac{\partial F^{01}}{\partial X_1} = 0 \tag{45}$$

- This is trivially true due to anti-symmetric property of F. Any combination involving 2 indices same will have this property
- In all three indices are different but none of them is 0, then we get

$$\frac{\partial F^{12}}{\partial X_3} + \frac{\partial F^{23}}{\partial X_1} + \frac{\partial F^{31}}{\partial X_2} = \frac{\partial B_z}{\partial z} + \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = \nabla \cdot \mathbf{B} = 0 \tag{46}$$

Covariant ED

- Now consider all 3 indices different, but one of them is 0. Ex. $\mu = 1, \nu = 2,\ \lambda = 0$

$$
\frac{\partial F^{12}}{\partial X_0} + \frac{\partial F^{20}}{\partial X_1} + \frac{\partial F^{01}}{\partial X_2} = -\frac{1}{c}\frac{\partial B_z}{\partial t} - \frac{\partial E_y}{\partial x} + \frac{\partial E_x}{\partial y} = 0 \tag{47}
$$

$$
\implies \frac{1}{c}\frac{\partial B_z}{\partial t} + (\nabla \times \mathbf{E})_z = 0 \tag{48}
$$

- This is the last of the Maxwell's equations
- The point of this formulation is that Maxwell's equations are written in terms of tensors which by definition transform according to the Lorentz transform. This implies that Maxwell's equations are automatically invariant under Lorentz transform, as required by first postulate of relativity

Transform of E and B

- Electric and magnetic fields do not become 4-vectors by themselves. Instead they are components of a 2nd rank contravariant tensor $F^{\mu\nu}$. So we can write their transformation rules for a 2nd rank tensor

$$F'^{\alpha\beta} = \left(\frac{\partial X'^{\alpha}}{\partial X^{\delta}}\right)\left(\frac{\partial X'^{\beta}}{\partial X^{\eta}}\right) F^{\delta\eta} = \boxed{\Lambda^{\alpha}_{\delta}\Lambda^{\beta}_{\eta}F^{\delta\eta}} \tag{49}$$

- Lets try some fields

$$E'_x = F'^{01} = \Lambda^0_\delta \Lambda^1_\eta F^{\delta\eta} \tag{50}$$
$$= \Lambda^0_0\Lambda^1_1 F^{01} + \Lambda^0_1\Lambda^1_0 F^{10} = \gamma^2 E_x - (\gamma^2 v^2/c^2)E_x = E_x \tag{51}$$

$$E'_y = F'^{02} = \Lambda^0_\delta \Lambda^2_\eta F^{\delta\eta} \tag{52}$$
$$= \Lambda^0_0\Lambda^2_2 F^{02} + \Lambda^0_1\Lambda^2_2 F^{12} \tag{53}$$
$$= \gamma E_y + (-\gamma v/c)B_z = \gamma(E_y - \beta B_z) \tag{54}$$

- Here $\beta = v/c$

Transform of E and B

- We can calculate in the same way for all the fields to get

$$E'_x = E_x \tag{55}$$

$$E'_y = \gamma(E_y - \beta B_z) \tag{56}$$

$$E'_z = \gamma(E_z + \beta B_y) \tag{57}$$

$$B'_x = B_x \tag{58}$$

$$B'_y = \gamma(B_y + \beta E_z) \tag{59}$$

$$B'_z = \gamma(B_z - \beta E_y) \tag{60}$$

General transform

- Consider a general 3D velocity **v** between the two reference frames gives

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}, \qquad \mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}$$

$$\mathbf{E}'_{\perp} = \gamma\left(\mathbf{E}_{\perp} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right), \qquad \mathbf{B}'_{\perp} = \gamma\left(\mathbf{B}_{\perp} - \frac{\mathbf{v}}{c}\times\mathbf{E}\right)$$

General 3D transform

- In general , if the reference frame S' is moving with some 3D velocity $\mathbf{v}$ w.r.t. S then the transformation can be derived as

$$\mathbf{B}' = \gamma\left[\mathbf{B} - \frac{1}{c}(\mathbf{v} \times \mathbf{E}) - \frac{(\mathbf{B}\cdot\mathbf{v})\mathbf{v}}{v^2}(\frac{1}{\gamma} - 1)\right] \tag{61}$$

$$\mathbf{E}' = \gamma\left[\mathbf{E} + \frac{1}{c}(\mathbf{v} \times \mathbf{B}) + \frac{(\mathbf{E}\cdot\mathbf{v})\mathbf{v}}{v^2}(\frac{1}{\gamma} - 1)\right] \tag{62}$$

- Inverse transform is obtained by simply interchange primed and unprimed quantities, and changing sign of $\mathbf{v}$.

HW

PROBLEM 4.5 The electric field at a point P which is situated at a distance r perpendicular to an infinite linear charge distribution with the density σ per unit length is $2\sigma\mathbf{r}/r^2$. Apply suitable Lorentz transformation formulae to show that the magnetic field **B** at the same location due to an infinitely long rectilinear current **j** is $\mathbf{B} = 2(\mathbf{j} \times \mathbf{r})/cr^2$.

Solution Let the velocity with which the linear charge distribution moves to give rise to the rectilinear current be **v**. Then $\mathbf{j} = \gamma\sigma_0\mathbf{v}$, where σ_0 is the linear rest charge density, that is, the density per unit length measured in the rest frame. In the frame where the observer moves at a velocity **v** along with the current direction, the electro-static field is transverse and is equal to

$$\mathbf{E}'_\perp = \frac{2\sigma_0\mathbf{r}}{r^2}$$

Thus,

$$\mathbf{B}_\perp = \gamma\left(\mathbf{B}'_\perp + \frac{\mathbf{v}}{c} \times \mathbf{E}'\right) = \frac{\gamma\mathbf{v}}{c} \times \frac{2\sigma_0\mathbf{r}}{r^2} = \frac{2(\gamma\sigma_0\mathbf{v}) \times \mathbf{r}}{cr^2}$$

That is,

$$\mathbf{B}_\perp = \frac{2(\mathbf{j}\times\mathbf{r})}{cr^2}$$

where $\mathbf{j} = \sigma\mathbf{v}$ with σ representing the linear charge density as measured by the moving observer.

HW

PROBLEM 4.8 Starting from Maxwell's equation, prove that $\partial J_\mu/\partial x_\mu = 0$.

Solution The Maxwell's equation is

$$\frac{\partial F_{\mu\alpha}}{\partial x_\alpha} = \frac{4\pi}{c} J_\mu$$

So

$$\frac{\partial^2 F_{\mu\alpha}}{\partial x_\mu \partial x_\alpha} = \frac{4\pi}{c}\frac{\partial J_\mu}{\partial x_\mu}$$

Since $F_{\mu\alpha}$ is antisymmetric, whereas $\partial^2/\partial x_\mu \partial x_\alpha$ is symmetric in μ and α, the left-hand side is zero. It is therefore proved that $\partial J_\mu/\partial x_\mu = 0$.

PROBLEM 4.9 An observer O finds himself to be in an electric field $\mathbf{E} = (0, E, 0)$ with no magnetic field. Another observer O' moves at a velocity $\mathbf{v} = (v, 0, 0)$ relative to O. Show that O' measures electric and magnetic fields $\mathbf{E}'$, $\mathbf{B}'$ which are connected by the relation

$$\mathbf{B}'_\perp + \frac{\mathbf{v}}{c} \times \mathbf{E}' = 0.$$

Solution We have

$$\mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel} = 0$$

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel} = 0$$

since the electric field vector is perpendicular to the relative velocity vector. But

$$\mathbf{B}_\perp = \gamma\left(\mathbf{B}'_\perp + \frac{\mathbf{v}}{c} \times \mathbf{E}'\right) = 0$$

since there is no magnetic field in the frame of O. So, we get

$$\mathbf{B}'_\perp + \frac{\mathbf{v}}{c} \times \mathbf{E}' = 0$$

HW

8. Describe the motion of a particle with charge e and mass m_0 in a uniform electric field E_0. The initial velocity of the particle is v_0 perpendicular to the field. Eliminate the time t_0, obtain the trajectory of the particle in space and discuss the shape of the path

8. Let the electric field E_0 be in the x-direction and the initial velocity of the charged particle be in the y-direction. Then the charge moves in the xy-plane. The equation of motion is given by

$$\frac{dp_x}{dt} = eE_0, \quad \frac{dp_y}{dt} = 0$$

so that $p_x = (eE_0t + a)$ and $p_y = p_0$. Here 'a' is a constant and p_0 is the initial momentum. Since initially the momentum is in the y-direction

$$p_x = 0 \quad \text{and} \quad p_y = mv_0 = p_0 \text{ at } t = 0.$$

Therefore $a = 0$, so that $p_x = eE_0t$ and $p_y = p_0$.

The total energy of such particle is

$$W = \left(p^2c^2 + m_0^2c^4\right)^{1/2} = \left(m_0^2c^4 + c^2p_0^2 + c^2e^2E_0^2t^2\right)$$

Now since $\mathbf{p} = m\mathbf{v}$ and $W = mc^2$, we have

$$\mathbf{v} = \frac{\mathbf{p}}{W}c^2$$

so that

$$v_x = \frac{dx}{dt} = \frac{p_xc^2}{W} \quad \text{and} \quad v_y = \frac{dy}{dt} = \frac{p_0c^2}{W}$$

Writing, $W_0^2 = \left(m_0^2c^4 + c^2p_0^2\right) =$ constant, we have the equations of motion

$$\frac{dx}{dt} = \frac{c^2eE_0t}{\left[W_0^2 + \left(ceE_0t\right)^2\right]^{1/2}} \tag{A–9}$$

and

$$\frac{dy}{dt} = \frac{p_0c^2}{\left[W_0^2 + \left(ceE_0t\right)^2\right]^{1/2}} \tag{A–10}$$

Integrating (A–9)

$$x = \frac{1}{eE_0}\left[W_0^2 + \left(ceE_0t\right)^2\right]^{1/2} \tag{A–11}$$

where for convenience the constant of integration is fixed to be zero choosing the suitable starting x-coordinate.

From (A–10) on integration, we further obtain

$$y = \frac{p_0c}{eE_0}\sinh^{-1}\left(\frac{ceE_0t}{W_0}\right) \tag{A–12}$$

It follows from (A–12) that

$$\cosh\left[\sinh^{-1}\left(\frac{ceE_0t}{W_0}\right)\right] = \frac{1}{W_0}\left[W_0^2 + \left(ceE_0t\right)^2\right]^{1/2}$$

Therefore from (A–11) we obtain the equation of the particle trajectory

$$x = \frac{W_0}{eE_0}\cosh\left(\frac{eE_0y}{p_0c}\right) \tag{A–13}$$

Equivalence principle

- Consider 2 objects of mass m_1 and m_2 falling freely under the influence of gravity

$$m_1 a_1 = \frac{GMm_1}{r^2}; \qquad m_2 a_2 = \frac{GMm_2}{r^2} \tag{63}$$

$$\implies a_1 = a_2 = \frac{GM}{r^2} \tag{64}$$

- Objects of different material, sizes, composition, etc. all fall at same rate in a gravitational field - weak equivalence principle

Equivalence principle

- Consider an observer inside an elevator that is falling freely. All objects inside the elevator will move at same acceleration, and so the observer will think that he is at rest
- Inversely, imagine the elevator in free space, but accelerating with constant acceleration g. An observer will think that he is under the influence of gravity
- Einstein's equivalence principle - An observer falling freely under gravity will find all non-gravitational physics indistinguishable from such physics in absence of gravity.
- Lead to discovery of general theory of relativity

Further courses

- EP3887 - General Relativity
- EP4258 - Gravitation and cosmology
- EP3277 - Relativistic Quantum Mechanics; and further particle physics courses
- EP3100 - Advanced Special Relativity
- EP2218 - Electrodynamics; and radiation physics courses