Four momentum

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Interval

- Consider the norm of the position 4-vector of a particle

$$s^2 = X^\mu X_\mu = g_{\mu\nu} X^\mu X^\nu = x^2 + y^2 + z^2 - c^2 t^2 \tag{1}$$

- We have seen earlier that this quantity remains the same across different inertial reference frames. Now consider an interval between 2 events $\Delta X = (c\Delta t, \Delta x, \Delta y, \Delta z)$. Its norm is

$$\Delta s^2 = \Delta X^\mu \Delta X_\mu = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^2 \tag{2}$$

- If $\Delta s^2 > 0$ then the interval is called space-like, if < 0 its called time like, and $= 0$ is called lightlike.

Differential

- Now define its differential

$$ds^2 = dX^\mu dX_\mu = dx^2 + dy^2 + dz^2 - c^2 dt^2 \tag{3}$$

- This will also be an invariant. Now define the "proper-time differential"

$$d\tau^2 = -\frac{ds^2}{c^2} = dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2) \tag{4}$$

- The importance of invariant quantities is their value does not depend on frame of reference

Proper time interval

- In a frame comoving with the particle, the particle will be at rest and its $dx = dy = dz = 0$. Therefore this invariant proper-time differential is the time interval measured in a frame co-moving with the particle. Hence its name.

$$d\tau^2 = dt^2 \left\{ 1 - \frac{1}{c^2} \left[(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2 \right] \right\} \tag{5}$$

$$= dt^2 \left(1 - \frac{u^2}{c^2} \right) \tag{6}$$

- here u is the velocity of the particle in the frame where the time interval is measured as dt
- $d\tau = dt\sqrt{1 - u^2/c^2}$, this shows the time dilation effect in the frame of observer compared to frame of particle.

4-velocity

- Define the 4-velocity as the derivative of the 4-position with proper time differential

$$U^{\mu} = \frac{dX^{\mu}}{d\tau} = \frac{d}{d\tau}\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \frac{dt}{d\tau}\begin{pmatrix} c \\ u_x \\ u_y \\ u_z \end{pmatrix} \tag{7}$$

- Here $u_{x,y,z}$ are the velocity of the particle measured in the observer frame. The norm of this 4-velocity is

$$g_{\mu\nu} U^{\mu} U^{\nu} = \frac{1}{1 - \frac{u^2}{c^2}}(u^2 - c^2) = -c^2 \tag{8}$$

- This norm is invariant in different reference frames

Momentum conservation

- Momentum should be defined such that it is conserved in all inertial reference frames
- Consider a group of n particles in a frame

$$\sum_{j=1}^{n} m_j \frac{dX_j^\mu}{dt} = const. \tag{9}$$

- Here j is not a tensor index, it is just a label of the particle, so we are explicitly writing the summation sign. When $\mu = 0$, this equation reduces to $\sum_{j=1}^{n} cm_j = const.$, which is just conservation of mass
- When $\mu = 1, 2, 3$, this expresses conservation of momentum along the three space dimensions.

Momentum conservation

- Expressing this in terms of proper time differential

$$\sum_{j=1}^{n} m_j \frac{dX_j^{\mu}}{d\tau}\sqrt{1-\frac{u_j^2}{c^2}} = const. \tag{10}$$

- In terms of the co-ordinates in another frame S', this becomes

$$\sum_{j=1}^{n} m_j (\Lambda^{-1})^{\mu}_{\nu}\left(\frac{dX_j^{'\nu}}{d\tau}\right)\sqrt{1-\frac{u_j^2}{c^2}} = const. \tag{11}$$

- Considering the fact that Λ^{-1} is a constant matrix (not changing in time) and can be inverted, it implies this matrix can be absorbed into the r.h.s. constant

$$\sum_{j=1}^{n} m_j \left(\frac{dX_j^{'\mu}}{d\tau}\right)\sqrt{1-\frac{u_j^2}{c^2}} = const. \tag{12}$$

Momentum

- The momentum should be conserved in S' frame as well, which can be written as

$$\sum_{j=1}^{n} m'_j \frac{dX_j'^{\mu}}{d\tau}\sqrt{1-\frac{u_j'^2}{c^2}} = const. \tag{13}$$

- Note here that we have allowed for the mass of the particle m_j to be different in the S' frame (m'_j). This is required as we will see
- The constants in Eq. 12 and 13 need not be the same. We multiply by a constant λ to make them equal such that

$$\sum_{j=1}^{n}\left\{m_j\sqrt{1-\frac{u_j^2}{c^2}} - \lambda m'_j\sqrt{1-\frac{u_j'^2}{c^2}}\right\}\frac{dX_j'^{\mu}}{d\tau} = 0 \tag{14}$$

Mass

- Since the particle velocities $\frac{dX_j^{'\mu}}{d\tau}$ are independent of each other, this means

$$m_j\sqrt{1-\frac{u_j^2}{c^2}} = \lambda m'_j\sqrt{1-\frac{u_j'^2}{c^2}} = m_{0j} \tag{15}$$

- m_{0j} is the mass of a particle where the particle velocity u_j is zero (rest mass).
- Either frame S or S' can be the rest frame and in those cases the rest mass should come out same. This means $\lambda = 1$.
- The momentum conservation law becomes

$$\sum_{j=1}^{n} m_{0j}\frac{1}{\sqrt{1-\frac{u_j^2}{c^2}}}\frac{dX_j^{\mu}}{dt} = const. \tag{16}$$

Newton's law

- Now we know that the following 4-vector is conserved (each component)
$$P^\mu = m_0 U^\mu \tag{17}$$
- The spatial components 1-3 are $\gamma m_0 \mathbf{u}$, where γ is the Lorentz factor derived from the 3-velocity of the particle $\mathbf{u}$. This can be interpreted as the relativistic momentum
- The time component is $\gamma m_0 c$. What is this?
- Classical Newton's law is $\mathbf{F} = d\mathbf{p}/dt$. Lets try to write Newton's law in 4-vector form
$$F^\mu = \frac{d}{d\tau}(P^\mu) \tag{18}$$
- Here F^μ is a 4-vector force. Now consider the dot-product of this force with U^μ
$$g_{\mu\nu} F^\mu U^\nu = m_0 g_{\mu\nu} \frac{d}{d\tau}(U^\mu) U^\nu \tag{19}$$

Newton's law

- It follows that

$$g_{\mu\nu} F^\mu U^\nu = m_0 \frac{d}{d\tau}(g_{\mu\nu} U^\mu U^\nu) - m_0 g_{\mu\nu} U^\mu \frac{d}{d\tau}(U^\nu) \tag{20}$$

- Now we have seen that $g_{\mu\nu} U^\mu U^\nu = -c^2$, which is a constant so $\frac{d}{d\tau}(g_{\mu\nu} U^\mu U^\nu) = 0$

$$\begin{aligned} g_{\mu\nu} F^\mu U^\nu &= m_0 g_{\mu\nu} \frac{d}{d\tau}(U^\mu) U^\nu \quad (21) \\ &= -m_0 g_{\mu\nu} U^\mu \frac{d}{d\tau}(U^\nu) \quad (22) \\ &= -m_0 g_{\mu\nu} U^\nu \frac{d}{d\tau}(U^\mu) \quad (23) \end{aligned}$$

- The last step just involves interchanging μ and ν since $g_{\mu\nu}$ is symmetric. From 21 and 23, we can see that $g_{\mu\nu} F^\mu U^\nu = 0$

Work

- Separating the space and time components of this equation gives

$$F^1 U^1 + F^2 U^2 + F^3 U^3 = F^0 U^0 \tag{24}$$

- The first term on the LHS can be written as

$$F^1 U^1 = \frac{d}{d\tau}(P^1) U^1 = \frac{dx}{d\tau} \frac{d}{d\tau}\left(m_0 U^1 \right) \tag{25}$$

$$= \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} u_x \frac{d}{d\tau}\left(\frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}} u_x \right) \tag{26}$$

- The RHS is

$$F^0 U^0 = \frac{dP^0}{d\tau} \frac{d}{d\tau}(ct) = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \frac{d}{d\tau}\left(\frac{m_0 c^2}{\sqrt{1 - \frac{u^2}{c^2}}} \right) \tag{27}$$

Work

- Equating the 2 sides we get

$$\mathbf{u} \cdot \frac{d}{dt}\left(\frac{m_0 \mathbf{u}}{\sqrt{1-\frac{u^2}{c^2}}}\right) = \frac{d}{dt}\left(\frac{m_0 c^2}{\sqrt{1-\frac{u^2}{c^2}}}\right) \tag{28}$$

- Here $\mathbf{u}$ represents the 3D velocity of the particle. Also we have replace $d/d\tau$ with d/dt as they are present on both sides of the equation
- The time derivative on the LHS is the 3D force vector, and so the LHS can be written as $\mathbf{F} \cdot \mathbf{u}$
- This is the rate at which work is done by the force on the particle and should be the rate of change of its energy. So we can write

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{m_0 c^2}{\sqrt{1-\frac{u^2}{c^2}}}\right) \tag{29}$$

Energy

- Integrating this w.r.t. time we get

$$E = \frac{m_0 c^2}{\sqrt{1 - \frac{u^2}{c^2}}} + E_0 \tag{30}$$

- If we want this to represent the kinetic energy (lets call it T) of the particle, then T=0 when $u = 0$, so $E_0 = -m_0 c^2$.

$$T = m_0 c^2 \left(\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} - 1 \right) \tag{31}$$

- When $u/c \ll 1$ (non-relativistic limit), we can Taylor expand around $u^2/c^2 = 0$ to get

$$T \approx m_0 c^2 \left(1 + \frac{1}{2}\frac{u^2}{c^2} + \frac{3}{8}\frac{u^4}{c^4} + ... - 1 \right) \tag{32}$$

- Ignoring higher order terms, we get back the non-relativisitic kinetic energy $(1/2) m_0 u^2$

Rest mass

- The kinetic energy T is defined as $\gamma m_0 c^2 - m_0 c^2$. This constant can be thought of as a rest mass energy (the energy in a particle at rest). Then the total energy is

$$E = T + m_0 c^2 = \gamma m_0 c^2 \tag{33}$$

- Going back to the momentum vector which can be written as

$$P^\mu = m_0 U^\mu = \begin{pmatrix} m_0 \gamma c \\ m_0 \gamma u_x \\ m_0 \gamma u_y \\ m_0 \gamma u_z \end{pmatrix} = \begin{pmatrix} \frac{E}{c} \\ m_0 \gamma \mathbf{u} \end{pmatrix} \tag{34}$$

- This shows that the 4-momentum vector consists of 2 parts. The time-like component is proportional to the total energy, while the space-like part is the 3D momentum vector. So this can also be called as the energy-momentum 4-vector

Energy-momentum relation

- We can now write the following relation between the 3D momentum and energy

$$\mathbf{p} = m_0\gamma\mathbf{u} = \frac{E\mathbf{u}}{c^2} \tag{35}$$

$$\implies p^2c^2 = E^2u^2/c^2 \tag{36}$$

$$= E^2 - E^2(1 - \frac{u^2}{c^2}) = E^2 - E^2/\gamma^2 \tag{37}$$

$$\tag{38}$$

- Since $E^2 = \gamma^2 m_0^2 c^4$, we get $p^2c^2 = E^2 - m_0^2c^4$ or $E^2 = p^2c^2 + m_0^2c^4$
- The energy of a particle tends to infinity as u tends to c. That is why it is not possible to accelerate a material particle to speed of light or beyond.

Energy

- In some books, the term γm_0 is represented simply as mass m. This gives the relation $E = mc^2$
- In some treatments, m_0 is simply written as m, so there is only one mass and that is the rest mass. In that case $E = \gamma mc^2$.
- The important quantity is $E^2 - p^2c^2$ which is a Lorentz invariant and equal to $m_0^2c^4$.
- A photon has energy $E = h\nu$, and its mass is zero. So it has a momentum $p = h\nu/c$.

Transformation of 4-momentum

- The transformation of 4-vectors follows the Lorentz transform. So

$$P'^{\mu} = \Lambda^{\mu}_{\nu} P^{\nu}; \qquad P^{\mu} = (\Lambda^{-1})^{\mu}_{\nu} P'^{\nu} \tag{39}$$

- This gives

$$p'_x = \gamma_v (p_x - vE/c^2) \tag{40}$$
$$p'_y = p_y \tag{41}$$
$$p'_z = p_z \tag{42}$$
$$\frac{E'}{c} = \gamma_v \left(\frac{E}{c} - \frac{v}{c} p_x\right) \tag{43}$$

- Here v is the velocity of the frame S' w.r.t. S and γ_v is used to denote the gamma factor that is defined using this velocity v.

Inverse Transformation

- The inverse transformation is given by

$$p_x = \gamma_v(p'_x + vE'/c^2) \tag{44}$$
$$p_y = p'_y \tag{45}$$
$$p_z = p'_z \tag{46}$$
$$\frac{E}{c} = \gamma_v(\frac{E'}{c} + \frac{v}{c}p'_x) \tag{47}$$

Energy-mass relation

- Consider a body at rest in S' frame so that $\mathbf{p}' = 0$, and $E' = m_0 c^2$.
- So in frame S, using the inverse transform, its momentum and energy is

$$p_x = \gamma_v m_0 v \tag{48}$$
$$E = \gamma_v m_0 c^2 \tag{49}$$

- Now lets say this body emits some radiation of energy E'_r in frame S' while staying at rest in S'. So the total momentum of the radiation should be 0 in order to conserve momentum
- In frame S the momentum and energy of this radiation will be

$$p_{xr} = \gamma_v \frac{v}{c^2} E'_r; \qquad E_r = \gamma_v E'_r \tag{50}$$

Energy-mass relation

- Since the particle continues to stay at rest in frame S', it means that the particle continues to move with speed v in frame S.
- Before emission of radiation, momentum was $\gamma_v m_0 v$
- After emission, total momentum is $\gamma_v m_0' v + \gamma_v \frac{v}{c^2} E_r'$, equating the 2 we get

$$\gamma_v m_0 v = \gamma_v \frac{v}{c^2} E_r' + \gamma_v m_0' v \tag{51}$$

- Here, the third term is the momentum of the body in frame S after radiation is emitted. Since the body continues to be at rest in S', this implies its velocity continues to remain v in frame S. Now if Eq. 51 is to hold, we have to take a different m_0' for the body after the radiation is emitted.

Energy-mass relation

- The change in the rest mass of the body is given by $\Delta m = m_0' - m_0$

$$\Delta m = -\frac{E_r'}{c^2} \tag{52}$$

- The energy lost as radiation results in a decrease of the rest mass. This is the basis of nuclear physics

Nuclear reactions

- Mass of nuclear particles is measured in terms of a.m.u (atomic mass units) also called as Dalton

$$1a.m.u = 1D = 1.66053907 \times 10^{-27} kg \tag{53}$$

- 1 a.m.u is equivalent to

$$\begin{aligned} E &= mc^2 & (54)\\ &= (1.66053907 \times 10^{-27}) * (3 \times 10^8)^2 & (55)\\ &= \boxed{1.4924 \times 10^{-10} J} & (56)\\ &= 1.4924 \times 10^{-10}/(1.6E - 19) = 931.5 \times 10^6 eV & (57)\\ &= 931.5 MeV & (58) \end{aligned}$$

Radioactive decay

- Consider the reaction

$$ {}^{235}_{92}U + {}^{1}_{0}n \rightarrow {}^{144}_{56}Ba + {}^{89}_{36}Kr + 3{}^{1}_{0}n \tag{59} $$

$$ m_0({}^{235}_{92}U) = 235.043943 a.m.u. \tag{60} $$

$$ m_0({}^{1}_{0}n) = 1.008665 a.m.u \tag{61} $$

$$ m_0({}^{144}_{56}Ba) = 143.922953 a.m.u \tag{62} $$

$$ m_0({}^{89}_{36}Kr) = 88.91763 a.m.u \tag{63} $$

$$ \Delta m = 235.043943 + 1.008665 - 143.922953 - 88.91763 \tag{64} $$

$$ - 3 \times 1.008665 \tag{65} $$

$$ = 0.18603 a.m.u \tag{66} $$

$$ E = \Delta mc^2 = 0.18603 * 931.5 = 173 MeV \tag{67} $$

HW

PROBLEM 3.8 An excited atom of total mass M, at rest with respect to the inertial system chosen, goes over into a lower state with an energy smaller by ΔW. It emits a photon, and thereby undergoes a recoil. The frequency of the photon will, therefore, not be exactly $\nu = \Delta W/h$ but smaller. Compute this frequency.

Solution Let the original rest mass of the excited atom be M_0 and the rest mass after transition to the lower state be M'_0. So according to be problem

$$(M_0c^2 - M'_0c^2) = \Delta W \tag{3.127}$$

The emitted photon with momentum hv/c will produce a recoil of equal momentum in the atom and hence from the conservation of momentum

$$\frac{h\nu}{c} = p' \tag{3.128}$$

where

$$p = \left(\frac{M'_0 v}{1 - v^2/c^2}\right)^{1/2} = M'v$$

Again using the principle of conservation of energy before and after emission, we obtain

$$M_0c^2 = M'c^2 + h\nu \tag{3.129}$$

Further the total energy of the atom after it goes over to the lower state is

$$(M'c^2)^2 = p'^2c^2 + (M'_0c^2)^2 \tag{3.130}$$

Where the momentum at this stage is $p' = M'v$
Now from (3.128), we get

$$M'c^2 = M_0c^2 - h\nu$$

Hence using (3.129) and (3.130) we obtain

$$(M'c^2)^2 = (M_0c^2 - h\nu)^2 = p'^2c^2 + M'_0{}^2c^4$$

It follows therefore from (3.127) and (3.128)

$$M_0^2c^4 + h^2\nu^2 - 2h\nu M_0c^2 = h^2\nu^2 + (M_0c^2 - \Delta W)^2$$

which finally leads to the simple relation

$$h\nu = \Delta W\left(1 - \frac{\Delta W}{2M_0c^2}\right)$$

HW

PROBLEM 3.11 An unstable neutral particle decays into two charged particles of kinetic energies 190 MeV and 30 MeV and momenta 300 MeV/c and 240 MeV/c, respectively. Determine the masses of the decay products. If the angle between the decay particles is 45°, determine (a) the rest mass of the neutral particle, (b) its momentum, and (c) its kinetic energy.

Solution We have

$$E^2 = m_0{}^2c^4 + p^2c^2 \tag{3.143}$$

$$T = \frac{m_0c^2}{\sqrt{1-v^2/c^2}} - m_0c^2 = E - m_0c^2 \tag{3.144}$$

or

$$T^2 = E^2 + m_0^2c^4 - 2m_0c^2E$$
$$= p^2c^2 + 2m_0{}^2c^4 - 2m_0c^2E$$

or

$$p^2c^2 - T^2 = 2m_0c^2E - 2m_0^2c^4$$
$$= 2m_0c^2(E - m_0c^2) = 2Tm_0c^2$$

Hence

$$m_0 = \frac{p^2c^2 - T^2}{2Tc^2} \tag{3.145}$$

Therefore, the rest mass of the first particle is

$$m_{01} = \frac{\{(300)^2 - (190)^2\} \times 1.6 \times 10^{-6}}{2 \times 190 \times 9 \times 10^{20}}\ \text{g} \qquad (\because\ 1\ \text{MeV} = 1.6 \times 10^{-6}\ \text{erg})$$

$$= 25.2 \times 10^{-26}\ \text{g}$$

$$= \frac{25.2 \times 10^{-26}}{9.1 \times 10^{-28}}\, m_e = 277m_e$$

Similarly,

$$m_{02} = \frac{\{(240)^2 - (30)^2\} \times 1.6 \times 10^{-6}}{2 \times 30 \times 9 \times 10^{20} \times 9.1 \times 10^{-28}}\, m_e$$

$$□\ 1844m_e$$

If p is the momentum of the parent particle, then from the principle of conservation of momentum, we have

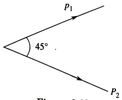

Figure 3.18

$$p = \sqrt{p_1^2 + p_2^2 + 2p_1p_2\cos 45°}$$

$$= \sqrt{(300)^2 + (240)^2 + 2 \cdot 300 \cdot 240 \cdot \frac{1}{\sqrt{2}}}\ \text{MeV/c}$$

$$□\ 500\ \text{MeV/c}$$

Total energy of the first particle = kinetic energy + rest energy
= (190 + 277 × 0.511) MeV

since the rest energy of an electron is 0.511 MeV. Similarly, the total energy of the second particle
= (30 + 1844 × 0.511) MeV

So the total energy of the parent particle
= (190 + 277 × 0.511) + (30 + 1844 × 0.511)
= 1304 MeV

Now

$$E^2 - p^2c^2 = m_0^2c^4$$

So (rest mass)^2 of the parent particle = m_0^2

$$= \frac{\{(1304)^2 - (500)^2\} \times (1.6)^2 \times 10^{-12}}{81 \times 10^{40}}\ \text{g}^2$$

Hence

$$\frac{m_0^2}{m_e^2} = \frac{\{(1304)^2 - (500)^2\} \times (1.6)^2 \times 10^{-12}}{81 \times 10^{40} \times (9.1)^2 \times 10^{-56}}\ □\ 552.3 \times 10^4\ \text{g}^2$$

Therefore,

$$m_0 = 2350\ m_e$$

The kinetic energy = total energy − rest energy
= 1304 − 2350 × 0.511 ≈ 103 MeV

HW

11. If a charged π-meson of rest mass 273.2 electron masses decays at rest into a muon of rest mass 206.8 electron masses and a neutrino of zero rest mass, show that the kinetic energies of the muon and neutrino are 4.1 MeV and 29.8 MeV, respectively.

11. Since the π-meson decays at rest, the total momentum is zero. Hence p_μ and p_ν are equal and opposite.

From the principle of conservation of energy

$$m_\pi c^2 = \left(p_\mu^2 c^2 + m_\mu^2 c^4\right)^{1/2} + p_\mu c$$

In terms of MeV

$$273.2 \times 0.511 = \left\{p_\mu^2 c^2 + (206.8 \times 0.511)^2\right\}^{1/2} = p_\mu c$$

or

$$\left\{273.2 \times 0.511 - p_\mu c\right\}^2 = (206.8 \times 0.511)^2 + p_\mu^2 c^2$$

or

$$(273.2 \times 0.511)^2 + p_\mu^2 c^2 - 2 \times 273.2 \times 0.511 p_\mu c$$
$$= (206.8 \times 0.511)^2 + p_\mu^2 c^2$$

or

$$2 \times 273.2 \times 0.511\, p_\mu c = \left\{(273.2)^2 - (206.8)^2\right\} \times (0.511)^2$$

Hence

$$p_\mu c = p_\nu c = \frac{\left\{(273.2)^2 - (206.8)^2\right\} \times (0.511)^2}{2 \times 273.2}$$
$$= 29.8 \text{ MeV}$$

This is the energy of the neutrino. The total energy of the muon is

$$\left\{(206.8 \times 0.511)^2 + (29.8)^2\right\}^{1/2}$$

Hence the kinetic energy = total energy − rest energy

$$= \left\{(206.8 \times 0.511)^2 + (29.8)^2\right\}^{1/2} - 206.8 \times 0.511$$
$$= 4.1 \text{ MeV}$$