Four momentum

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September 24, 2020
Consider the norm of the position 4-vector of a particle

\[ s^2 = X^\mu X_\mu = g_{\mu\nu} X^\mu X^\nu = x^2 + y^2 + z^2 - c^2 t^2 \]  

(1)

We have seen earlier that this quantity remains the same across different inertial reference frames. Now consider an interval between 2 events \( \Delta X = (c\Delta t, \Delta x, \Delta y, \Delta z) \). Its norm is

\[ \Delta s^2 = \Delta X^\mu \Delta X_\mu = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^2 \]  

(2)

If \( \Delta s^2 > 0 \) then the interval is called space-like, if \( < 0 \) its called time like, and \( = 0 \) is called lightlike.
Now define its differential

\[ ds^2 = dX^\mu dX_\mu = dx^2 + dy^2 + dz^2 - c^2 dt^2 \]  \hspace{1cm} (3)

This will also be an invariant. Now define the “proper-time differential”

\[ d\tau^2 = -\frac{ds^2}{c^2} = dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2) \]  \hspace{1cm} (4)

The importance of invariant quantities is their value does not depend on frame of reference.
Proper time interval

- In a frame comoving with the particle, the particle will be at rest and its \( dx = dy = dz = 0 \). Therefore this invariant proper-time differential is the time interval measured in a frame co-moving with the particle. Hence its name.

\[
d\tau^2 = dt^2 \left\{ 1 - \frac{1}{c^2} \left[ \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2 \right] \right\}
\]

(5)

\[
d\tau^2 = dt^2 \left( 1 - \frac{u^2}{c^2} \right)
\]

(6)

- here \( u \) is the velocity of the particle in the frame where the time interval is measured as \( dt \)

- \( d\tau = dt \sqrt{1 - u^2/c^2} \), this shows the time dilation effect in the frame of observer compared to frame of particle.
**4-velocity**

Define the 4-velocity as the derivative of the 4-position with proper time differential

\[
U^\mu = \frac{dX^\mu}{d\tau} = \frac{d}{d\tau} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \frac{dt}{d\tau} \begin{pmatrix} c \\ u_x \\ u_y \\ u_z \end{pmatrix}
\]  

(7)

Here \(u_x, y, z\) are the velocity of the particle measured in the observer frame. The norm of this 4-velocity is

\[
g_{\mu\nu} U^\mu U^\nu = \frac{1}{1 - \frac{u^2}{c^2}} (u^2 - c^2) = -c^2
\]  

(8)

This norm is invariant in different reference frames
Momentum conservation

- Momentum should be defined such that it is conserved in all inertial reference frames
- Consider a group of \( n \) particles in a frame

\[
\sum_{j=1}^{n} m_j \frac{dX_j^\mu}{dt} = \text{const.} \tag{9}
\]

- Here \( j \) is not a tensor index, it is just a label of the particle, so we are explicitly writing the summation sign. When \( \mu = 0 \), this equation reduces to \( \sum_{j=1}^{n} cm_j = \text{const.} \), which is just conservation of mass
- When \( \mu = 1, 2, 3 \), this expresses conservation of momentum along the three space dimensions.
Momentum conservation

- Expressing this in terms of proper time differential

\[ \sum_{j=1}^{n} m_j \frac{dX^\mu_j}{d\tau} \sqrt{1 - \frac{u_j^2}{c^2}} = \text{const.} \tag{10} \]

- In terms of the co-ordinates in another frame \( S' \), this becomes

\[ \sum_{j=1}^{n} m_j (\Lambda^{-1})^\mu_\nu \left( \frac{dX'_\nu_j}{d\tau} \right) \sqrt{1 - \frac{u_j^2}{c^2}} = \text{const.} \tag{11} \]

- Considering the fact that \( \Lambda^{-1} \) is a constant matrix (not changing in time) and can be inverted, it implies this matrix can be absorbed into the r.h.s. constant

\[ \sum_{j=1}^{n} m_j \left( \frac{dX'_\mu_j}{d\tau} \right) \sqrt{1 - \frac{u_j^2}{c^2}} = \text{const.} \tag{12} \]
Momentum

- The momentum should be conserved in $S'$ frame as well, which can be written as

$$
\sum_{j=1}^{n} m'_j dX'_j \sqrt{1 - \frac{u'^2_j}{c^2}} = \text{const.} \quad (13)
$$

- Note here that we have allowed for the mass of the particle $m_j$ to be different in the $S'$ frame ($m'_j$). This is required as we will see

- The constants in Eq. 12 and 13 need not be the same. We multiply by a constant $\lambda$ to make them equal such that

$$
\sum_{j=1}^{n} \left\{ m_j \sqrt{1 - \frac{u^2_j}{c^2}} - \lambda m'_j \sqrt{1 - \frac{u'^2_j}{c^2}} \right\} \frac{dX'_j}{d\tau} = 0 \quad (14)
$$
Mass

- Since the particle velocities \( \frac{dX'_j}{d\tau} \) are independent of each other, this means

\[
m_j \sqrt{1 - \frac{u_j^2}{c^2}} = \lambda m'_j \sqrt{1 - \frac{u'_j^2}{c^2}} = m_{0j}
\]  

(15)

- \( m_{0j} \) is the mass of a particle where the particle velocity \( u_j \) is zero (rest mass).

- Either frame \( S \) or \( S' \) can be the rest frame and in those cases the rest mass should come out same. This means \( \lambda = 1 \).

- The momentum conservation law becomes

\[
\sum_{j=1}^{n} m_{0j} \frac{1}{\sqrt{1 - \frac{u_j^2}{c^2}}} \frac{dX'_j}{dt} = \text{const.}
\]  

(16)
Newton’s law

- Now we know that the following 4-vector is conserved (each component)
  \[ P^\mu = m_0 U^\mu \] (17)

- The spatial components 1-3 are $\gamma m_0 u$, where $\gamma$ is the Lorentz factor derived from the 3-velocity of the particle $u$. This can be interpreted as the relativistic momentum

- The time component is $\gamma m_0 c$. What is this?

- Classical Newton’s law is $F = dp/dt$. Let’s try to write Newton’s law in 4-vector form
  \[ F^\mu = \frac{d}{d\tau}(P^\mu) \] (18)

- Here $F^\mu$ is a 4-vector force. Now consider the dot-product of this force with $U^\mu$
  \[ g_{\mu\nu} F^\mu U^\nu = m_0 g_{\mu\nu} \frac{d}{d\tau}(U^\mu) U^\nu \] (19)
Newton’s law

- It follows that

\[ g_{\mu \nu} F^\mu U^\nu = m_0 \frac{d}{d\tau} (g_{\mu \nu} U^\mu U^\nu) - m_0 g_{\mu \nu} U^\mu \frac{d}{d\tau} (U^\nu) \]  \hspace{1cm} (20)

- Now we have seen that \( g_{\mu \nu} U^\mu U^\nu = -c^2 \), which is a constant so

\[ \frac{d}{d\tau} (g_{\mu \nu} U^\mu U^\nu) = 0 \]

\[ g_{\mu \nu} F^\mu U^\nu = m_0 g_{\mu \nu} \frac{d}{d\tau} (U^\mu) U^\nu \]  \hspace{1cm} (21)

\[ = -m_0 g_{\mu \nu} U^\mu \frac{d}{d\tau} (U^\nu) \]  \hspace{1cm} (22)

\[ = -m_0 g_{\mu \nu} U^\nu \frac{d}{d\tau} (U^\mu) \]  \hspace{1cm} (23)

- The last step just involves interchanging \( \mu \) and \( \nu \) since \( g_{\mu \nu} \) is symmetric. From 21 and 23, we can see that \( g_{\mu \nu} F^\mu U^\nu = 0 \)
Separating the space and time components of this equation gives

\[ F^1 U^1 + F^2 U^2 + F^3 U^3 = F^0 U^0 \]  \hspace{1cm} (24)

The first term on the LHS can be written as

\[ F^1 U^1 = \frac{d}{d\tau} (P^1) U^1 = \frac{dx}{d\tau} \frac{d}{d\tau} \left( m_0 U^1 \right) \]  \hspace{1cm} (25)

\[ = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} u_x \frac{d}{d\tau} \left( \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}} u_x \right) \]  \hspace{1cm} (26)

The RHS is

\[ F^0 U^0 = \frac{dP^0}{d\tau} \frac{d}{d\tau} (ct) = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \frac{d}{d\tau} \left( \frac{m_0 c^2}{\sqrt{1 - \frac{u^2}{c^2}}} \right) \]  \hspace{1cm} (27)
Work

- Equating the 2 sides we get

\[ \mathbf{u} \cdot \frac{d}{dt} \left( \frac{m_0 \mathbf{u}}{\sqrt{1 - \frac{u^2}{c^2}}} \right) = \frac{d}{dt} \left( \frac{m_0 c^2}{\sqrt{1 - \frac{u^2}{c^2}}} \right) \]  (28)

- Here \( \mathbf{u} \) represents the 3D velocity of the particle. Also we have replace \( d/d\tau \) with \( d/dt \) as they are present on both sides of the equation.

- The time derivative on the LHS is the 3D force vector, and so the LHS can be written as \( \mathbf{F} \cdot \mathbf{u} \)

- This is the rate at which work is done by the force on the particle and should be the rate of change of its energy. So we can write

\[ \frac{dE}{dt} = \frac{d}{dt} \left( \frac{m_0 c^2}{\sqrt{1 - \frac{u^2}{c^2}}} \right) \]  (29)
Energy

- Integrating this w.r.t. time we get

\[ E = \frac{m_0 c^2}{\sqrt{1 - \frac{u^2}{c^2}}} + E_0 \]  

(30)

- If we want this to represent the kinetic energy (lets call it T) of the particle, then T=0 when \( u = 0 \), so \( E_0 = -m_0 c^2 \).

\[ T = m_0 c^2 \left( \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} - 1 \right) \]

(31)

- When \( \frac{u}{c} \ll 1 \) (non-relativistic limit), we can Taylor expand around \( \frac{u^2}{c^2} = 0 \) to get

\[ T \approx m_0 c^2 \left( 1 + \frac{1}{2} \frac{u^2}{c^2} + \frac{3}{8} \frac{u^4}{c^4} + ... - 1 \right) \]

(32)

- Ignoring higher order terms, we get back the non-relativistic kinetic energy \( (1/2)m_0 u^2 \)
Rest mass

- The kinetic energy $T$ is defined as $\gamma m_0 c^2 - m_0 c^2$. This constant can be thought of as a rest mass energy (the energy in a particle at rest). Then the total energy is

$$E = T + m_0 c^2 = \gamma m_0 c^2$$  \hspace{1cm} (33)

- Going back to the momentum vector which can be written as

$$P^\mu = m_0 U^\mu = \begin{pmatrix} m_0 \gamma c \\ m_0 \gamma u_x \\ m_0 \gamma u_y \\ m_0 \gamma u_z \end{pmatrix} = \begin{pmatrix} E/c \\ m_0 \gamma u \end{pmatrix}$$  \hspace{1cm} (34)

- This shows that the 4-momentum vector consists of 2 parts. The time-like component is proportional to the total energy, while the space-like part is the 3D momentum vector. So this can also be called as the energy-momentum 4-vector
Energy-momentum relation

- We can now write the following relation between the 3D momentum and energy

\[ p = m_0 \gamma u = \frac{Eu}{c^2} \]  

\[ \implies p^2 c^2 = E^2 \frac{u^2}{c^2} \]  

\[ = E^2 - E^2 \left(1 - \frac{u^2}{c^2}\right) = E^2 - E^2 / \gamma^2 \]

- Since \( E^2 = \gamma^2 m_0^2 c^4 \), we get \( p^2 c^2 = E^2 - m_0^2 c^4 \) or \( E^2 = p^2 c^2 + m_0^2 c^4 \)

- The energy of a particle tends to infinity as \( u \) tends to \( c \). That is why it is not possible to accelerate a material particle to speed of light or beyond.
In some books, the term $\gamma m_0$ is represented simply as mass $m$. This gives the relation $E = mc^2$

In some treatments, $m_0$ is simply written as $m$, so there is only one mass and that is the rest mass. In that case $E = \gamma mc^2$.

The important quantity is $E^2 - p^2c^2$ which is a Lorentz invariant and equal to $m_0^2c^4$.

A photon has energy $E = h\nu$, and its mass is zero. So it has a momentum $p = h\nu/c$. 

Transformation of 4-momentum

- The transformation of 4-vectors follows the Lorentz transform. So

\[ P'_\mu = \Lambda^\mu_\nu P^\nu; \quad P^\mu = (\Lambda^{-1})^\mu_\nu P'^\nu \quad (39) \]

- This gives

\[ p'_x = \gamma_v (p_x - \frac{vE}{c^2}) \quad (40) \]
\[ p'_y = p_y \quad (41) \]
\[ p'_z = p_z \quad (42) \]
\[ \frac{E'}{c} = \gamma_v \left( \frac{E}{c} - \frac{v}{c} p_x \right) \quad (43) \]

- Here \( \nu \) is the velocity of the frame \( S' \) w.r.t. \( S \) and \( \gamma_v \) is used to denote the gamma factor that is defined using this velocity \( \nu \).
The inverse transformation is given by

\[ p_x = \gamma_v (p_x' + \frac{v E'}{c^2}) \]  \hspace{1cm} (44)
\[ p_y = p_y' \]  \hspace{1cm} (45)
\[ p_z = p_z' \]  \hspace{1cm} (46)
\[ \frac{E}{c} = \gamma_v \left( \frac{E'}{c} + \frac{v}{c} p_x' \right) \]  \hspace{1cm} (47)
Energy-mass relation

- Consider a body at rest in $S'$ frame so that $p' = 0$, and $E' = m_0 c^2$.
- So in frame $S$, using the inverse transform, its momentum and energy is

$$p_x = \gamma_v m_0 v$$

$$E = \gamma_v m_0 c^2$$

(48)

(49)

- Now let's say this body emits some radiation of energy $E'_r$ in frame $S'$ while staying at rest in $S'$. So the total momentum of the radiation should be 0 in order to conserve momentum.

- In frame $S$ the momentum and energy of this radiation will be

$$p_{xr} = \gamma_v \frac{v}{c^2} E'_r; \quad E_r = \gamma_v E'_r$$

(50)
Energy-mass relation

- Since the particle continues to stay at rest in frame $S'$, it means that the particle continues to move with speed $v$ in frame $S$.
- Before emission of radiation, momentum was $\gamma v m_0 v$
- After emission, total momentum is $\gamma v m'_0 v + \gamma v \frac{v}{c^2} E'_r$, equating the 2 we get
  \[ \gamma v m_0 v = \gamma v \frac{v}{c^2} E'_r + \gamma v m'_0 v \]  \hspace{1cm} (51)
- Here, the third term is the momentum of the body in frame $S$ after radiation is emitted. Since the body continues to be at rest in $S'$, this implies its velocity continues to remain $v$ in frame $S$. Now if Eq. 51 is to hold, we have to take a different $m'_0$ for the body after the radiation is emitted.
Energy-mass relation

- The change in the rest mass of the body is given by
  \[ \Delta m = m'_0 - m_0 \]
  \[ \Delta m = -\frac{E'_r}{c^2} \] \hspace{1cm} (52)

- The energy lost as radiation results in a decrease of the rest mass. This is the basis of nuclear physics.
Nuclear reactions

- Mass of nuclear particles is measured in terms of a.m.u (atomic mass units) also called as Dalton

\[ 1a.m.u = 1D = 1.66053907 \times 10^{-27} \text{kg} \quad (53) \]

- 1 a.m.u is equivalent to

\[ E = mc^2 \quad (54) \]

\[ = (1.66053907 \times 10^{-27}) \times (3 \times 10^8)^2 \quad (55) \]

\[ = 1.4924 \times 10^{-10} \text{J} \quad (56) \]

\[ = 1.4924 \times 10^{-10}/(1.6E - 19) = 931.5 \times 10^6 \text{eV} \quad (57) \]

\[ = 931.5\text{MeV} \quad (58) \]
Radioactive decay

Consider the reaction

\[
\frac{235}{92} U + \frac{1}{0} n \rightarrow \frac{144}{56} Ba + \frac{89}{36} Kr + 3\frac{1}{0} n
\]  

(59)

\[m_0(\frac{235}{92} U) = 235.043943a.m.u.\] (60)

\[m_0(\frac{1}{0} n) = 1.008665a.m.u\] (61)

\[m_0(\frac{144}{56} Ba) = 143.922953a.m.u\] (62)

\[m_0(\frac{89}{36} Kr) = 88.91763a.m.u\] (63)

\[\Delta m = 235.043943 + 1.008665 - 143.922953 - 88.91763 \]

\[- 3 \times 1.008665\] (64)

\[= 0.18603a.m.u\] (66)

\[E = \Delta mc^2 = 0.18603 \times 931.5 = 173MeV\] (67)
PROBLEM 3.8 An excited atom of total mass $M$, at rest with respect to the inertial system chosen, goes over into a lower state with an energy smaller by $\Delta W$. It emits a photon, and thereby undergoes a recoil. The frequency of the photon will, therefore, not be exactly $\nu = \Delta W/\hbar$ but smaller. Compute this frequency.

**Solution** Let the original rest mass of the excited atom be $M_0$ and the rest mass after transition to the lower state be $M'_0$. So according to the problem

$$ (M_0c^2 - M'_0c^2) = \Delta W $$

(3.127)

The emitted photon with momentum $\hbar \nu/c$ will produce a recoil of equal momentum in the atom and hence from the conservation of momentum

$$ \frac{\hbar \nu}{c} = p' $$

(3.128)

where

$$ p = \left( \frac{M_0 \omega}{1 - v^2/c^2} \right)^{1/2} = M' p $$

Again using the principle of conservation of energy before and after emission, we obtain

$$ M_0c^2 = M' c^2 + \hbar \nu $$

(3.129)

Further the total energy of the atom after it goes over to the lower state is

$$ (M' c^2)^2 = p'^2 c^2 + (M_0 c^2)^2 $$

(3.130)

Where the momentum at this stage is $p' = M' p$

Now from (3.128), we get

$$ M' c^2 = M_0 c^2 - \hbar \nu $$

Hence using (3.129) and (3.130) we obtain

$$ (M' c^2)^2 = (M_0 c^2 - \hbar \nu)^2 = p'^2 c^2 + M'_0 c^4 $$

It follows therefore from (3.127) and (3.128)

$$ M'_0 c^4 + \hbar^2 \nu^2 - 2 \hbar \nu M_0 c^2 = \hbar^2 \nu^2 - (M_0 c^2 - \Delta W)^2 $$

which finally leads to the simple relation

$$ \hbar \nu = \Delta W \left( 1 - \frac{\Delta W}{2M_0 c^2} \right) $$
PROBLEM 3.11 An unstable neutral particle decays into two charged particles of kinetic energies 190 MeV and 30 MeV and momenta 300 MeV/c and 240 MeV/c, respectively. Determine the masses of the decay products. If the angle between the decay particles is 45°, determine (a) the rest mass of the neutral particle, (b) its momentum, and (c) its kinetic energy.

Solution We have

\[ E^2 = m_0 c^4 + p^2 c^2 \]  
\[ T = \frac{m_0 c^2}{\sqrt{1 - v^2 c^2}} - m_0 c^2 = E - m_0 c^2 \]

or

\[ T^2 = E^2 + m_0 c^4 - 2m_0 c^2 E \]
\[ = p^2 c^2 + 2m_0 c^4 - 2m_0 c^2 E \]

or

\[ p^2 c^2 - T^2 = 2m_0 c^2 E - 2m_0 c^4 \]
\[ = 2m_0 c^2 (E - m_0 c^2) = 2T m_0 c^2 \]

Hence

\[ m_0 \approx \frac{p^2 c^2 - T^2}{2T c^2} \]

Therefore, the rest mass of the first particle is

\[ m_{01} = \frac{(300)^2 - (190)^2}{2 \times 190 \times 9 \times 10^{-24}} \times 1.6 \times 10^{-6} \text{ MeV} = 5.2 \times 10^{-36} \text{ g} \]

\[ \approx \frac{25.2 \times 10^{-36}}{9.1 \times 10^{-28}} \times m_e = 277 m_e \]

Similarly,

\[ m_{02} = \frac{(240)^2 - (30)^2}{2 \times 30 \times 9 \times 10^{-24}} \times 1.6 \times 10^{-6} \times 1844 m_e \]

If \( p \) is the momentum of the parent particle, then from the principle of conservation of momentum, we have

\[ p = \sqrt{p_1^2 + p_2^2 + 2p_1 p_2 \cos 45^\circ} \]

\[ = \sqrt{(300)^2 + (240)^2 + 2 \times 30 \times 240 \times \frac{1}{\sqrt{2}}} \text{ MeV/c} \]

\[ \square 500 \text{ MeV/c} \]

Total energy of the first particle = kinetic energy + rest energy

\[ (190 + 277 \times 0.511) \text{ MeV} \]

since the rest energy of an electron is 0.511 MeV. Similarly, the total energy of the second particle

\[ (30 + 1844 \times 0.511) \text{ MeV} \]

So the total energy of the parent particle

\[ (190 + 277 \times 0.511) + (30 + 1844 \times 0.511) = 1304 \text{ MeV} \]

Now

\[ E^2 - p^2 c^2 = m_0 c^4 \]

So (rest mass)^2 of the parent particle = \( m_0^2 \)

\[ = \frac{(1304)^2 - (500)^2}{81 \times 10^{40}} \times 1.6 \times 10^{-12} \text{ g}^2 \]

Hence

\[ m_{01}^2 = \frac{(1304)^2 - (500)^2}{81 \times 10^{40} \times (9.1)^2 \times 10^{-28}} \times 552.3 \times 10^4 \text{ g}^2 \]

Therefore,

\[ m_0 = 2350 m_e \]

The kinetic energy = total energy - rest energy

\[ = 1304 - 2350 \times 0.511 = 103 \text{ MeV} \]
11. If a charged $\pi$-meson of rest mass 273.2 electron masses decays at rest into a muon of rest mass 206.8 electron masses and a neutrino of zero rest mass, show that the kinetic energies of the muon and neutrino are 4.1 MeV and 29.8 MeV, respectively.

11. Since the $\pi$-meson decays at rest, the total momentum is zero. Hence $p_\mu$ and $p_\nu$ are equal and opposite.

From the principle of conservation of energy

$$m_\pi c^2 = \left( p_\mu^2 c^2 + m_\mu c^2 \right)^{1/2} + p_\nu c$$

In terms of MeV

$$273.2 \times 0.511 = \left[ p_\mu^2 c^2 + (206.8 \times 0.511)^2 \right]^{1/2} = p_\mu c$$

or

$$\left[ 273.2 \times 0.511 - p_\mu c \right]^2 = (206.8 \times 0.511)^2 + p_\mu^2 c^2$$

or

$$(273.2 \times 0.511)^2 + p_\mu^2 c^2 - 2 \times 273.2 \times 0.511 p_\mu c = (206.8 \times 0.511)^2 + p_\mu^2 c^2$$

or

$$2 \times 273.2 \times 0.511 p_\mu c = \left( (273.2)^2 - (206.8)^2 \right) \times (0.511)^2$$

Hence

$$p_\mu c = p_\nu c = \frac{\left( (273.2)^2 - (206.8)^2 \right) \times (0.511)^2}{2 \times 273.2}$$

$$= 29.8 \text{ MeV}$$

This is the energy of the neutrino. The total energy of the muon is

$$\left( (206.8 \times 0.511)^2 + (29.8)^2 \right)^{1/2}$$

Hence the kinetic energy = total energy - rest energy

$$= \left( (206.8 \times 0.511)^2 + (29.8)^2 \right)^{1/2} - 206.8 \times 0.511$$

$$= 4.1 \text{ MeV}$$