Lorentz transformations

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Postulates of relativity

- In the absence of a preferred "ether" frame of reference, Einstein decided to make all inertial frames equivalent resulting in the first postulate of relativity
- 1st postulate All laws of physics remain invariant in all inertial frames
- This includes Maxwell's laws of electrodynamics as well. This implied a constant speed of light in vacuum, which was the second postulate of Einstein
- 2nd postulate The velocity of light in empty space is a constant, independent of direction of propagation and also of the relative velocity between between source and observer

Consequences of these postulates

• The constancy of speed of light in all inertial frames has tremendous consequences



- Observer at O is stationary while O' is on a moving train. A' and B' are the ends of the train. Suppose in frame of O sees the endpoints of the train A' and B' coincide with points A and B, When this happens lightning strikes at these co-incident points. It leaves a mark on both the train tracks and the train.
- Light from these lightning reaches O simultaneously since O is at the midpoint of A B and therefore O concludes that the lightning strikes happened simultaneously, since speed of light is constant.

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Lorentz transformations

Consequences of these postulates

• Now look from the point of view of O' who sees that the lightning marks are equidistant from him. However, light from B' reaches O' before the light from A'.



- However, if he thinks speed of light is constant irrespective of direction of motion, then O' concludes that the strike at B' happened before the strike at A'
- Simultaneity of time is frame-dependent. t' = t does not hold. Similarly the Galilean transformation x' = x - vt also does not hold

Consider the space-time coordinates of an event (x, y, z, t) in frame S, which is measured at coordinates (x', y', z', t') in frame S'. There is some transformation law between these coordinates

$$x' = \Phi_1(x, y, z, t) \tag{1}$$

$$y' = \Phi_2(x, y, z, t) \tag{2}$$

$$z' = \Phi_3(x, y, z, t) \tag{3}$$

$$t' = \Phi_4(x, y, z, t) \tag{4}$$

- Consider a particle moving along the x axis in frame S with a constant velocity. Its trajectory will be x = c₀ + c₁t
- By Newton's first law and first postulate of relativity, the particle should have a constant velocity in x direction in frame S' also.

Linearity of the transform

• The x' velocity in frame S' will then be

$$\frac{dx'}{dt'} = \frac{dx'}{dt} / \left(\frac{dt'}{dt}\right) = \frac{\frac{\partial \Phi_1}{\partial x} \frac{dx}{dt} + \frac{\partial \Phi_1}{\partial t}}{\frac{\partial \Phi_4}{\partial x} \frac{dx}{dt} + \frac{\partial \Phi_4}{\partial t}}$$

$$= \frac{c_1 \frac{\partial \Phi_1}{\partial x} + \frac{\partial \Phi_1}{\partial t}}{c_1 \frac{\partial \Phi_4}{\partial x} + \frac{\partial \Phi_4}{\partial t}}$$
(5)
(6)

• In order for this to be constant, we require that $\frac{\partial \Phi_1}{\partial x}$ and the other partial derivatives are constant, implying that the transform is linear. In other words the transform is given by

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t + a_{15}$$
(7)

$$y' = a_{21}x + a_{22}y + a_{23}z + a_{24}t + a_{25}$$
(8)

$$z' = a_{31}x + a_{32}y + a_{33}z + a_{34}t + a_{35}$$
(9)

$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t + a_{45} \tag{10}$$

where all the a's are constants

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- By our setup, the origin O(x = 0, y = 0, z = 0) at t = 0 transforms to the origin O'(x' = 0, y' = 0, z' = 0) at t' = 0. This means the constants $a_{15} = a_{25} = a_{35} = a_{35} = 0$.
- Consider the motion of O in S' frame. Put x = y = z = 0 in Eq.7-10. This gives

$$x' = a_{14}t; y' = a_{24}t; z' = a_{34}t; t' = a_{44}t$$
(11)

The velocity of O in frame S' should simply be along the negative x axis. Thus a₂₄ = a₃₄ = 0

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t$$
 (12)

$$y' = a_{21}x + a_{22}y + a_{23}z \tag{13}$$

$$z' = a_{31}x + a_{32}y + a_{33}z \tag{14}$$

$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t \tag{15}$$



• Consider the same physical system but just inverting the *x*, *x*' and *z*, *z*' axes. The *y* transformation will become

$$y' = -a_{21}x + a_{22}y - a_{23}z \tag{16}$$

• But y' should remain unaffected by rotation of x' and z' axis. So $a_{21} = a_{23} = 0$. Similarly $a_{12} = a_{13} = a_{31} = a_{32} = 0$

$$x' = a_{11}x + a_{14}t \tag{17}$$

$$y' = a_{22}y \tag{18}$$

$$z' = a_{33}z \tag{19}$$

$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t \tag{20}$$

• Inverting the matrix of linear tranformations Eqs. 18-19, we get the inverse transform

$$\begin{aligned} x &= \Delta^{-1} (a_{22}a_{33}a_{44}x' + a_{14}a_{33}a_{42}y' + a_{14}a_{22}a_{43}z' - a_{14}a_{22}a_{33}t') \\ (21) \\ y &= a_{22}^{-1}y' \\ z &= a_{33}^{-1}z' \\ t &= \Delta^{-1} (-a_{22}a_{33}a_{41}x') + \Delta^{-1} (a_{11}a_{22}a_{33}t') \end{aligned}$$

• By first law of relativity, the transformation from x to x' should follow similar equations as from x' to x. So $a_{42} = a_{43} = 0$.

 Now consider the following setup where frame S is moving with velocity v' relative to S'. They are related by rotation of x and z axis



- In fig (b) physically S is just the frame S' of Fig. (a). Therefore v' = v.
- Now consider the point y = 1, x = z = 0. This will transform to y' = a₂₂. Now imagine inverting the x, x' and z, z' axis like before and consider going from S to S' as the inverse transform. So now y = 1 and use the inverse transform equations 18-21, i.e. y' = a₂₂⁻¹. This means a₂₂ = ±1, and keeping non-relativistic limit in mind a₂₂ = a₃₃ = 1.

• So the transform equations now are

$$x' = a_{11}x + a_{14}t \tag{25}$$

$$y' = y \tag{26}$$

$$z' = z \tag{27}$$

$$t' = a_{41}x + a_{44}t \tag{28}$$

- The transform of O (putting x = 0) will be $x' = a_{14}t$. Also x' = -vt'and $t' = a_{44}t$. This means $a_{14} = -va_{44}$
- Now consider the inverse transforms (22-25) and the motion of O'(x'=0) in S. Then $x = vt = -\Delta^{-1}a_{14}t'$, $t = \Delta^{-1}a_{11}t'$. This gives $a_{14} = -va_{11}$

• So
$$a_{11} = a_{44} \equiv \alpha$$
, and $a_{14} = -v\alpha$

• We are left with the following transformation laws

$$x' = \alpha(x - vt) \tag{29}$$

$$y' = y \tag{30}$$

$$z' = z \tag{31}$$

$$t' = a_{41}x + \alpha t \tag{32}$$

 Now consider that as the 2 frames coincide at t = 0, a flash is emitted in all directions from the point O. This will give rise to spherical wavefronts in both frames S and S' as per the second postulate of Einsteins relativity.



- Consider the event of light reaching a detector at point (x_1, y_1, z_1) at time t_1 in frame S. This event will satisfy the condition $x_1^2 + y_1^2 + z_1^2 c^2 t_1^2 = 0$
- The transform of this event in frame S' will have coordinates (x'_1, y'_1, z'_1, t'_1) . But due to constant speed of light, in frame S' also these coordinates will obey $x'^2_1 + y'^2_1 + z'^2_1 c^2 t'^2_1 = 0$.

- Now consider another point in spacetime that does not fall on the wavefront (x_2, y_2, z_2, t_2) . Let this point obey $x_2^2 + y_2^2 + z_2^2 c^2 t_2^2 = A_0$, where A_0 is some non-zero value.
- It can be seen that we can displace this point by $y_2 + \delta y$ and $z_2 + \delta_z$, such that $x_2^2 + (y_2 + \delta y)^2 + (z_2 + \delta z)^2 - c^2 t_2^2 = 0$
- This displaced point lies on the wavefront and its transform will also obey $x_2'^2 + (y_2' + \delta y')^2 + (z_2' + \delta z')^2 c^2 t_2'^2 = 0$
- But we already know that y and z coordinates do not change under the transformation.
- Using this, we can show that $x_2'^2 + y_2'^2 + z_2'^2 c^2 t_2'^2 = A_0$
- What this means is that $x^2 + y^2 + z^2 c^2 t^2$ remains same under the Lorentz transformation for points not just on the wavefront but for all points. This is called Lorentz invariance

Invariance

$$x^{2} + y^{2} + z^{2} - c^{2}t^{2} = x^{\prime 2} + y^{\prime 2} + z^{\prime 2} - c^{2}t^{\prime 2}$$
(33)

• Putting y' = y and z' = z, we get

$$x^2 - c^2 t^2 = x^{\prime 2} - c^2 t^{\prime 2}$$
(34)

Substituting the Lorentz transform on the RHS gives

$$x^{2} - c^{2}t^{2} = (\alpha^{2} - c^{2}a_{41}^{2})x^{2} - c^{2}\alpha^{2}(1 - \frac{v^{2}}{c^{2}})t^{2} - 2\alpha(\alpha v + a_{41}c^{2})xt$$
(35)

• Since this is true for all x and t, the coefficients of each independent term should be equal, implying

$$\alpha = (1 - \frac{v^2}{c^2})^{-1/2} \tag{36}$$

$$a_{41} = \frac{-\alpha v}{c^2} = \frac{-v/c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$
(37)

• Thus the Lorentz transform equations become

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$
(38)

$$y' = y$$
(39)

$$z' = z$$
(40)

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$
(41)

Non-relativistic limit

• The non-relativistic limit is when the relative velocity is much smaller than speed of light. This is simply obtained by putting v/c = 0 in the equations

$$x' = x - vt \tag{42}$$

$$y' = y \tag{43}$$

$$z' = z \tag{44}$$

$$t' = t \tag{45}$$

• In this limit we recover the Galilean transformation

• The inverse transforms are obtained by plugging these constants in Eqs. 21-24

$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}$$
(46)

$$y = y'$$
(47)

$$z = z'$$
(48)

$$t = \frac{t' + \frac{vx'}{c^2}}{(49)}$$

As expected this is just replacing v with -v

3. In a certain inertial frame, two light pulses are emitted at points 5 km apart and separated in time by 5 μ s. An observer moving at a speed v along the line joining these points notes that the pulses are simultaneous. Find v. [JEST 2012]

 $\sqrt{1-\frac{v^2}{c^2}}$

Homework problem solution

3. Let us take an inertial frame S in which x_1 and x_2 are the space coordinates of the sources of the two light pulses. According to the problem, $x_2 - x_1 = 5$ km, $t_2 - t_1 = 5 \ \mu s = 5 \times 10^{-6}$ s. Let S' be the frame of reference moving with velocity v along the x-axis where the pulses of light appear to be simultaneous. Let t'_1 and t'_2 be the instants of time of the two pulses as observed by the observer on S'. By Lorentz transformation, we have

$$t_1' = \left(t_1 - \frac{vx_1}{c^2}\right) \left(1 - \frac{v^2}{c^2}\right)^{-1/2}, \qquad t_2' = \left(t_2 - \frac{vx_2}{c^2}\right) \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

Hence,
$$t'_2 - t'_1 = \left[(t_2 - t_1) - \frac{v}{c^2} (x_2 - x_1) \right] \left(1 - \frac{v^2}{c^2} \right)^{-1/2}$$

We must have $t'_2 - t'_1 = 0$, $(t_2 - t_1) = 5 \times 10^{-6}$ s, $x_2 - x_1 = 5$ km $c = 3 \times 10^5$ km/s.

Hence,

$$0 = \left[5 \times 10^{-6} - \frac{(5v)}{(9 \times 10^{10})} \right] \left(1 - \frac{v^2}{c^2} \right)^{-1/2}$$

Hence, $5v = 5 \times 10^{-6} \times 9 \times 10^{10}$, i.e. $v = 9 \times 10^4$ km/s.