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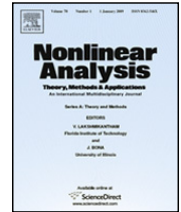
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On the continuity of residuals of triangular norms

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ABSTRACT

In this work a long-standing problem related to the continuity of R -implications, i.e., implications obtained as the residuum of t -norms, has been solved. A complete characterization of the class of continuous R -implications obtained from any arbitrary t -norm is given. In particular, it is shown that an R -implication I_T is continuous if and only if T is a nilpotent t -norm. Using this result, the exact intersection between the continuous subsets of R -implications and (S, N) -implications has been determined, by showing that the only continuous (S, N) -implication that is also an R -implication obtained from any t -norm, not necessarily left-continuous, is the Łukasiewicz implication up to an isomorphism.

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1. Introduction

Fuzzy implications were introduced and studied in the literature as a generalization of the classical implication operation. Following are the two main ways of defining an implication in the Boolean lattice $(L, \wedge, \vee, \rightarrow)$:

$$p \rightarrow q \equiv \neg p \vee q, \quad (1)$$

$$p \rightarrow q \equiv \max\{t \in L \mid p \wedge t \leq q\}, \quad (2)$$

where $p, q \in L$ and the relation \leq is defined in the usual way, i.e., $p \leq q$ iff $p \vee q = q$, for every $p, q \in L$. The implication (1) is usually called the material implication, while (2) is from the intuitionistic logic framework, where the implication is obtained as the residuum of the conjunction, and is often called as the pseudocomplement of p relative to q (see [1]). It is important to note that, despite their different formulas, the expressions (1) and (2) are equivalent in the Boolean lattice $(L, \wedge, \vee, \rightarrow)$. Interestingly, in the fuzzy logic framework, where the truth values can vary in the unit interval $[0, 1]$, the natural generalizations of the above definitions, viz., (S, N) - and R -implications, are not equivalent. This variety has led to some intensive research on fuzzy implications for nearly three decades.

1.1. Main focus of this work

Quite understandably then, the most established and well-studied classes of fuzzy implications are the above (S, N) - and R -implications (cf. [2–4]). Still, many open problems remain unsolved, see [5–7]. One of them is related to the continuous subsets of these families. Only recently a characterization of continuous (S, N) -implications was given in [8].

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However, a similar complete characterization regarding the continuous subset of R -implications has not been available so far. It is only known that in the class of R -implications obtained from *left-continuous* t -norms, the only continuous elements are those that are isomorphic to the Łukasiewicz implication, i.e., those R -implications obtained as residuals of nilpotent t -norms. In particular, the following question has remained unanswered so far:

Does there exist a continuous R -implication obtained from a non-left-continuous t -norm?

In this work we answer the above poser in the negative, by showing that the continuity of an R -implication forces the left-continuity of the underlying t -norm and hence show that an R -implication I_T is continuous if and only if T is a nilpotent t -norm.

Further, using this result, we are also able to resolve another question related to the intersections between the families of continuous R - and (S, N) -implications, which is also a generalization of an original result of SMETS and MAGREZ [9], see also [2,3]. In particular, we show that the only continuous (S, N) -implication that is also an R -implication obtained from any t -norm, not necessarily left-continuous, is the Łukasiewicz implication up to an isomorphism.

Note that this result also has applications in other areas of fuzzy logic and fuzzy set theory. For instance, in the many fuzzy logics based on t -norms, viz., BL-fuzzy logics [10], MTL-algebras [11] and their other variants, the negation is obtained from the t -norm itself and is not always involutive. However, the continuity of the residuum immediately implies that the corresponding negation is continuous, and hence involutive (see Theorem 2.8 and Remark 3.4).

Fuzzy inference mechanisms that use t -norms and their residual fuzzy implications as part of their inference scheme have many desirable properties (see, for instance, [12,13]). Based on the results contained in this paper one can choose this pair of operations appropriately to ensure the continuity of the ensuing inference.

1.2. Outline of the work

In Section 2 we cover some relevant preliminaries related to basic fuzzy logic connectives. In Section 3 we introduce in detail the family of R -implications, also called residual implications, and list out all the important results leading up to the main characterization result, whose generalization forms the main focus of this work. Section 4 contains some analysis required for the main results in this work, wherein we discuss the relationship between continuity and monotonicity of partial functions of residual implications. Section 5 contains the main results of this work, wherein we show that if the R -implication obtained from a t -norm is continuous then the underlying t -norm is necessarily left-continuous and hence is nilpotent. In Section 6, after giving a very short but necessary introduction to the family of (S, N) -implications, this main result is made use of in determining the exact intersection between the continuous subsets of R -implications and (S, N) -implications.

2. Preliminaries

We assume that the reader is familiar with the classical results concerning basic fuzzy logic connectives, but to make this work more self-contained, we introduce basic notations used in the text and we briefly mention some of the concepts and results employed in the rest of the work.

By Φ we denote the family of all increasing bijections $\varphi: [0, 1] \rightarrow [0, 1]$. We say that functions $f, g: [0, 1]^n \rightarrow [0, 1]$, where $n \in \mathbb{N}$, are Φ -conjugate, if there exists $\varphi \in \Phi$ such that $g = f_\varphi$, where

$$f_\varphi(x_1, \dots, x_n) := \varphi^{-1}(f(\varphi(x_1), \dots, \varphi(x_n))),$$

for all $x_1, \dots, x_n \in [0, 1]$. Equivalently, g is said to be the Φ -conjugate of f or isomorphic to f .

2.1. T -norms and T -conorms

Definition 2.1 (cf. [2,14,3]).

- (i) A function $T: [0, 1]^2 \rightarrow [0, 1]$ is called a t -norm, if it is monotonic increasing in both variables, commutative, associative and has 1 as the neutral element.
- (ii) A function $S: [0, 1]^2 \rightarrow [0, 1]$ is called a t -conorm, if it is monotonic increasing in both variables, commutative, associative and has 0 as the neutral element.
- (iii) A t -norm T is said to be *border continuous*, if it is continuous on the boundary of the unit square $[0, 1]^2$, i.e., on the set $[0, 1]^2 \setminus (0, 1)^2$.
- (iv) A t -norm T is said to be *left-continuous*, if it is left-continuous in each component.
- (v) A t -norm T is said to be *nilpotent*, if it is continuous and if each $x \in (0, 1)$ is a nilpotent element of T , i.e., if there exists an $n \in \mathbb{N}$ such that $x_T^{[n]} = 0$, where

$$x_T^{[n]} := \begin{cases} 1, & \text{if } n = 0, \\ T(x, x_T^{[n-1]}), & \text{if } n \geq 1. \end{cases}$$

- (vi) A t -norm T is said to be *Archimedean* if for every $x, y \in (0, 1)$ there is $n \in \mathbb{N}$ such that $x_T^{[n]} < y$.

Remark 2.2 (See [14], p. 17). For the border continuity of a t -norm T , it is sufficient to require the continuity on the upper right boundary, since from the monotonicity we get

$$\lim_{x \rightarrow 0^+} T(x, y) \leq \lim_{x \rightarrow 0^+} T(x, 1) = \lim_{x \rightarrow 0^+} x = 0 = T(0, y), \quad y \in [0, 1].$$

Remark 2.3. From the commutativity, the left-continuity of a t -norm T is equivalent to the left-continuity of T with respect to the first or the second variable. Moreover, $T(x, 1) = 1$ and $T(x, 0) = 0$ for every $x \in [0, 1]$, thus a t -norm T is left-continuous if and only if for any $y \in (0, 1)$ and every increasing sequence $(x_n)_{n \in \mathbb{N}}$, where $x_n \in [0, 1]$, we have

$$\lim_{n \rightarrow \infty} T(x_n, y) = T(\lim_{n \rightarrow \infty} x_n, y).$$

Proposition 2.4. If T is an Archimedean t -norm, then $T(x, y) < \min(x, y)$, for all $x, y \in (0, 1)$.

Proof. Let T be an Archimedean t -norm. If, on the contrary, there exist some $x_0, y_0 \in (0, 1)$ such that $x_0 \geq y_0$ and $T(x_0, y_0) = y_0 = \min(x_0, y_0)$, then we will prove, by induction, that for every $n \in \mathbb{N}$ we have

$$x_{0T}^{[n]} \geq y_0. \quad (3)$$

Indeed, firstly see that

$$\begin{aligned} x_{0T}^{[0]} &= 1 > T(x_0, y_0) = y_0, \\ x_{0T}^{[1]} &= x_0 \geq T(x_0, y_0) = y_0. \end{aligned}$$

Let us assume that (3) holds for some $n \in \mathbb{N}$. Then by the monotonicity of T and our inductive assumption we get

$$x_{0T}^{[n+1]} = T(x_0, x_{0T}^{[n]}) \geq T(x_0, y_0) = y_0,$$

which implies that T is not Archimedean, a contradiction. \square

2.2. Negations from t -conorms and t -norms

Fuzzy negations are, once again, generalizations of classical negations.

Definition 2.5. A function $N: [0, 1] \rightarrow [0, 1]$ is called a *fuzzy negation*, if it is decreasing and satisfies the boundary conditions $N(1) = 0$ and $N(0) = 1$. A fuzzy negation N is called *strong*, if it is an involution, i.e., $N \circ N = \text{id}_{[0, 1]}$.

One can associate a fuzzy negation to any t -norm or t -conorm as given in the definition below.

Definition 2.6 (See [14], p. 232 or [6]). Let T be a t -norm. A function $N_T: [0, 1] \rightarrow [0, 1]$ defined as

$$N_T(x) := \sup\{t \in [0, 1] \mid T(x, t) = 0\}, \quad x \in [0, 1] \quad (4)$$

is called the *natural negation* of T .

Remark 2.7. (i) It is easy to prove that N_T is a fuzzy negation. In the literature N_T is also called the contour line C_0 of T (see [15,16]).

- (ii) Since for any t -norm T we have $T(x, 0) = 0$ for all $x \in [0, 1]$, the appropriate set in (4) is non-empty.
- (iii) Notice that if $T(x, y) = 0$ for some $x, y \in [0, 1]$, then $y \leq N_T(x)$. Moreover, if $z < N_T(x)$, then $T(x, z) = 0$. If T is left-continuous then $T(x, y) = 0$ for some $x, y \in [0, 1]$ if and only if $y \leq N_T(x)$.

The next result will be required later on.

Theorem 2.8 ([17, Theorem 2.14]). Let T be any t -norm.

- (i) If N_T is continuous, then it is strong.
- (ii) If N_T is discontinuous, then it is not strictly decreasing.

2.3. Fuzzy implications

In the literature, especially at the beginnings, we can find several different definitions of fuzzy implications. In this article we will use the following one, which is equivalent to the definition introduced by Fodor and Roubens [2], Definition 1.15 (see also [18], p. 50).

Definition 2.9 ([2,7]). A function $I: [0, 1]^2 \rightarrow [0, 1]$ is called a *fuzzy implication* if it satisfies the following conditions:

$$I \text{ is decreasing in the first variable,} \quad (I1)$$

$$I \text{ is increasing in the second variable,} \quad (I2)$$

$$I(0, 0) = 1, \quad I(1, 1) = 1, \quad I(1, 0) = 0. \quad (I3)$$

The set of all fuzzy implications will be denoted by \mathcal{FI} .

Additional properties of fuzzy implications were postulated in many works (see, for example, [19,2,3]). The most important of them are presented below.

Definition 2.10. A fuzzy implication I is said to have

(i) the *left neutrality property*, if

$$I(1, y) = y, \quad y \in [0, 1]. \quad (NP)$$

(ii) the *exchange principle*, if for all $x, y, z \in [0, 1]$,

$$I(x, I(y, z)) = I(y, I(x, z)). \quad (EP)$$

(iii) the *identity principle*, if

$$I(x, x) = 1, \quad x \in [0, 1]. \quad (IP)$$

(iv) the *ordering property*, if

$$I(x, y) = 1 \iff x \leq y, \quad x, y \in [0, 1]. \quad (OP)$$

Just as in the case of t -norms or t -conorms, a fuzzy negation can be obtained from fuzzy implications too as follows.

Definition 2.11. Let I be a fuzzy implication. The function N_I defined as $N_I(x) := I(x, 0)$ for all $x \in [0, 1]$, is a fuzzy negation and is called the *natural negation of I* .

3. R -implications

From the isomorphism that exists between classical two-valued logic and classical set theory one can immediately note the following set theoretic identity:

$$\overline{P} \cup Q = \overline{P \setminus Q} = \cup \{T \mid P \cap T \subseteq Q\},$$

where P, Q are subsets of some universal set. The above identity gives one way of defining the Boolean implication and is employed in the intuitionistic logic. Fuzzy implications obtained as the generalization of the above identity form the family of residuated implications, usually called as R -implications in the literature. It is important to note that the name ' R -implication' is a short version of 'residual implication', and I_T is also called as the residuum of T (see [2,14,3]).

Definition 3.1 (cf. [19,2,3,7]). A function $I: [0, 1]^2 \rightarrow [0, 1]$ is called an R -implication, if there exists a t -norm T such that

$$I(x, y) = \sup \{t \in [0, 1] \mid T(x, t) \leq y\}, \quad (5)$$

for all $x, y \in [0, 1]$. If an R -implication is generated from a t -norm T , then we will often denote this by I_T .

Example 3.2. The Łukasiewicz implication

$$I_{LK}(x, y) = \min(1, 1 - x + y), \quad x, y \in [0, 1],$$

is an R -implication obtained from the nilpotent (Łukasiewicz) t -norm

$$T_{LK}(x, y) = \max(x + y - 1, 0), \quad x, y \in [0, 1]. \quad (6)$$

For more well-known R -implications along with their t -norms from which they have been obtained, we refer the readers to other sources, notably [2,14,7].

Proposition 3.3 ([7, Proposition 2.5.10]). If I_T is an R -implication based on some t -norm T , then the Φ -conjugate of I_T is also an R -implication generated from the Φ -conjugate t -norm of T , i.e., if $\varphi \in \Phi$, then $(I_T)_\varphi = I_{T_\varphi}$.

Hence an R -implication obtained from any nilpotent t -norm is a Φ -conjugate of the Łukasiewicz implication I_{LK} .

Remark 3.4. Note, firstly, that the natural negation obtained from an R -implication is also the associated negation of the underlying t -norm, i.e., $N_T(x) = I_T(x, 0)$ for all $x \in [0, 1]$. Moreover, if I_T is continuous then N_T is continuous and hence is strong by Theorem 2.8.

Theorem 3.5 (cf. [2], [6, Theorem 5.5]). If T is any t -norm, then $I_T \in \mathcal{FI}$ and it satisfies (NP) and (IP). Moreover, if T is left-continuous, then I_T satisfies (EP) and (OP).

Proposition 3.6 ([6, Proposition 5.8]). For a t -norm T the following statements are equivalent:

- (i) T is border continuous.
- (ii) I_T satisfies the ordering property (OP).

For R -implications generated from left-continuous t -norms many characterization results are available, see for example, [2,7]. Now we state the following main characterization result whose generalization forms the focus of this work.

Theorem 3.7 (cf. [20], Corollary 2; [7], Theorem 2.5.33). For a function $I: [0, 1]^2 \rightarrow [0, 1]$ the following are equivalent:

- (i) I is a continuous R -implication based on some left-continuous t -norm.
- (ii) I is Φ -conjugate with the Łukasiewicz implication, i.e., there exists $\varphi \in \Phi$, which is uniquely determined, such that

$$I(x, y) = \varphi^{-1}(\min(1 - \varphi(x) + \varphi(y), 1)), \quad (7)$$

for all $x, y \in [0, 1]$.

We would like to note that the proof of the above result is dependent on many other equivalence results concerning fuzzy implications, especially concerning R -implications and their contrapositivity, see, for instance, the corresponding proofs in [20,7]. For more facts related to R -implications see [2,6,7]. Later on we show that from the obtained results in this work, we obtain what is perhaps the first direct proof of the above result (see Corollary 5.4).

4. Continuous partial functions of R -implications

Note that from Theorem 3.5 we can consider, for any fixed $\alpha \in [0, 1)$, the decreasing partial function $I_T(\cdot, \alpha): [\alpha, 1] \rightarrow [\alpha, 1]$, which we will denote by g_α^T . Observe that g_α^T is decreasing and such that $g_\alpha^T(\alpha) = 1$ and $g_\alpha^T(1) = \alpha$.

Remark 4.1. If the domain of g_α^T is extended to $[0, 1]$, then this is exactly what are called *contour lines* by Maes and De Baets in [15,21]. If $\alpha = 0$, then g_0^T is the natural negation associated with the t -norm T (see [6]).

Theorem 4.2. Let T be any t -norm. For any fixed $\alpha \in [0, 1)$, if g_α^T is continuous, then g_α^T is strictly decreasing.

Proof. Let T be any t -norm and $\alpha \in [0, 1)$ be fixed. We know that g_α^T is decreasing. On the contrary, let us assume that g_α^T is constant on some interval $[x_0, y_0]$ for some $\alpha < x_0 < y_0 < 1$, i.e., there exists $p \in [\alpha, 1]$ such that

$$g_\alpha^T(x_0) = g_\alpha^T(y_0) = p.$$

Let us fix arbitrarily $z \in (x_0, y_0)$.

Firstly, consider the case $p = 1$. Then

$$g_\alpha^T(z) = I_T(z, \alpha) = \sup\{t \in [0, 1] \mid T(z, t) \leq \alpha\} = 1,$$

thus $T(z, 1 - \varepsilon) \leq \alpha$ for any $\varepsilon \in (0, 1)$. Hence

$$g_\alpha^T(1 - \varepsilon) = \sup\{t \in [0, 1] \mid T(1 - \varepsilon, t) \leq \alpha\} \geq z,$$

for all $\varepsilon \in (0, 1 - \alpha)$. However, by the continuity of g_α^T , as $\varepsilon \rightarrow 0^+$, we get

$$\alpha = g_\alpha^T(1) = g_\alpha^T(\lim_{\varepsilon \rightarrow 0^+} 1 - \varepsilon) = \lim_{\varepsilon \rightarrow 0^+} g_\alpha^T(1 - \varepsilon) \geq \lim_{\varepsilon \rightarrow 0^+} z = z,$$

a contradiction to the fact that $\alpha < x_0 < z$.

If $p = \alpha$, then

$$g_\alpha^T(z) = I_T(z, \alpha) = \sup\{t \in [0, 1] \mid T(z, t) \leq \alpha\} = \alpha,$$

thus $T(z, \alpha + \varepsilon) > \alpha$ for all $\varepsilon \in (0, 1 - \alpha)$. Hence

$$g_\alpha^T(\alpha + \varepsilon) = \sup\{t \in [0, 1] \mid T(\alpha + \varepsilon, t) \leq \alpha\} \leq z,$$

for all $\varepsilon \in (0, 1 - \alpha)$. Once again, by the continuity of g_α^T we have, as $\varepsilon \rightarrow 0^+$, that

$$1 = g_\alpha^T(\alpha) = g_\alpha^T(\lim_{\varepsilon \rightarrow 0^+} \alpha + \varepsilon) = \lim_{\varepsilon \rightarrow 0^+} g_\alpha^T(\alpha + \varepsilon) \leq \lim_{\varepsilon \rightarrow 0^+} z = z,$$

a contradiction to the fact that $z < 1$.

Finally, let $p \in (\alpha, 1)$. Then, by the definition of g_α^T , we have

$$T(z, p + \varepsilon) > \alpha \geq T(z, p - \varepsilon),$$

for any $\varepsilon > 0$ such that $p + \varepsilon \leq 1$ and $p - \varepsilon \geq \alpha$. Therefore

$$I_T(p + \varepsilon, \alpha) \leq z \leq I_T(p - \varepsilon, \alpha),$$

hence

$$g_\alpha^T(p + \varepsilon) \leq z \leq g_\alpha^T(p - \varepsilon).$$

Since g_α^T is continuous, we have, as $\varepsilon \rightarrow 0^+$, that

$$g_\alpha^T(p) = z.$$

Now this happens for every $z \in (x, y)$, which contradicts the fact that g_α^T is a function itself. Hence g_α^T is strictly decreasing. \square

In fact, the above result essentially states that, if the “generalized” inverse of a monotone function is continuous, then it is strictly decreasing (see Remark 3.4(ii), [5], also Theorem 11, [15]).

5. Main results: Continuous R -implications

The main result of this work is the generalization of Theorem 3.7, viz., we show that the *left-continuity* of the underlying t -norm is implied and need not be assumed. Thus we give a complete characterization of the class of all continuous R -implications by showing that it is equivalent to the class of fuzzy implications which are Φ -conjugate to the Łukasiewicz implication.

Theorem 5.1. *Let T be a t -norm and I_T the R -implication obtained from it. If I_T is continuous, then T is border continuous.*

Proof. On the contrary, let us assume that I_T is continuous and T is not border continuous. Then, by Remark 2.2, there exist $y_0 \in (0, 1)$ and an increasing sequence $(x_n)_{n \in \mathbb{N}}$, where $x_n \in [0, 1]$, such that $\lim_{n \rightarrow \infty} x_n = 1$, but

$$\lim_{n \rightarrow \infty} T(x_n, y_0) = y' < y_0 = T(1, y_0).$$

This implies, in particular, that

$$I_T(y_0, y') = \sup\{t \in [0, 1] \mid T(y_0, t) \leq y'\} = 1.$$

Now, by (I1) and (IP) of I_T (cf. Theorem 3.5) we have that

$$1 = I_T(y_0, y') \leq I_T(y', y') = 1,$$

i.e., $I_T(x, y') = 1$ for all $x \in [y', y_0]$. Note that $I_T(\cdot, y') = g_{y'}^T$. Since I_T is continuous we have that $g_{y'}^T$ is also continuous and from Theorem 4.2 we see that it is strictly decreasing. However, from the above, we see that $g_{y'}^T$ is constant on $[y', y_0]$, a contradiction. Thus T is border continuous. \square

Theorem 5.2. *Let T be a t -norm and I_T the R -implication obtained from it. If I_T is continuous, then T is Archimedean.*

Proof. Let T be a t -norm. On the contrary, let us assume that I_T is continuous and T is non-Archimedean. Then, by the Definition 2.1(viii) there exist $x_0, y_0 \in (0, 1)$ such that for all $n \in \mathbb{N}$ we have that $x_{0T}^{[n]} \geq y_0$.

Let us denote

$$X_0 := \{z \in [0, 1] \mid x_{0T}^{[n]} > z \text{ for all } n \in \mathbb{N}\}.$$

Observe, that $X_0 \neq \emptyset$ since for all $y < y_0$ we have that $x_{0T}^{[n]} > y$ for all $n \in \mathbb{N}$. Further, let

$$z_0 := \sup X_0.$$

See that $0 < z_0 \leq x_0$ and $z_0 - \varepsilon \in X_0$ for all $\varepsilon \in (0, z_0]$. Also, if $t > z_0$, then there exists $m \in \mathbb{N}$ such that $x_{0T}^{[m]} \leq t$, which implies that

$$z_0 - \varepsilon < x_{0T}^{[m+1]} = T(x_0, x_{0T}^{[m]}) \leq T(x_0, t),$$

for any $t > z_0$. Hence

$$I_T(x_0, z_0 - \varepsilon) = \sup\{t \in [0, 1] \mid T(x_0, t) \leq z_0 - \varepsilon\} \leq z_0,$$

for all $\varepsilon \in (0, z_0]$. From the continuity of I_T we get

$$I_T(x_0, z_0) = I_T(x_0, \lim_{\varepsilon \rightarrow 0^+} z_0 - \varepsilon) = \lim_{\varepsilon \rightarrow 0^+} I_T(x_0, z_0 - \varepsilon) \leq \lim_{\varepsilon \rightarrow 0^+} z_0 = z_0.$$

Now, by (11) and (IP) of I_T (cf. Theorem 3.5) we have that

$$z_0 \geq I_T(x_0, z_0) \geq I_T(1, z_0) = z_0,$$

i.e., $I_T(x, z_0) = z_0$ for all $x \in [x_0, 1]$. Note that $I_T(\cdot, z_0) = g_{z_0}^T$. Since I_T is continuous we have that $g_{z_0}^T$ is also continuous and from Theorem 4.2 we see that it is strictly decreasing. However, from the above, we see that $g_{z_0}^T$ is constant on $[x_0, 1]$, a contradiction. Thus T is Archimedean. \square

We note that based on Theorem 5.2 we get what is perhaps – to the best of the author's knowledge – the first independent proof of Theorem 3.7. Before, proving this result we recall some known results in the following remark.

Remark 5.3 (cf. [14]).

- (i) A left-continuous T that is Archimedean is necessarily continuous (see [14], Proposition 2.16).
- (ii) A continuous Archimedean t -norm T is either strict or nilpotent.
- (iii) If a continuous Archimedean t -norm T has zero divisors then it is nilpotent (see [14], Theorem 2.18).
- (iv) A nilpotent t -norm T is a Φ -conjugate of (isomorphic to) the Łukasiewicz t -norm (6).

Corollary 5.4. Let T be a left-continuous t -norm and I_T the R -implication obtained from it. Then the following are equivalent:

- (i) I_T is continuous.
- (ii) T is isomorphic to T_{LK} .

Proof. (i) \implies (ii): Let T be left-continuous and I_T be continuous. Then, from Theorem 5.2, we see that T is Archimedean and hence by Remark 5.3(i) T is necessarily continuous. Further, by Remark 5.3(ii), T is either strict or nilpotent. Now, from Remark 3.4, since I_T is continuous N_T is strong and hence from Remark 2.7(iii) we see that T has zero divisors. Finally, from Remark 5.3(iii) and (iv), it follows that T is nilpotent and hence is isomorphic to T_{LK} .

(ii) \implies (i): The converse is obvious, since the R -implication obtained from any nilpotent t -norm is a Φ -conjugate of the Łukasiewicz implication I_{LK} . Since I_{LK} is continuous, any Φ -conjugate of it is also continuous. \square

Based on the above results, we are ready to present our main result showing that if I_T is continuous, then the left-continuity of T need not be assumed but follows as a necessity.

Theorem 5.5. Let T be a t -norm and I_T the R -implication obtained from it. If I_T is continuous, then T is left-continuous.

Proof. Let T be a t -norm such that I_T is continuous. From Theorems 5.1 and 5.2 we see that T is border continuous and Archimedean.

On the contrary, let us assume that T is non-left-continuous. From Remark 2.3 there exist $x_0 \in (0, 1]$, $y_0 \in (0, 1)$ and an increasing sequence $(x_n)_{n \in \mathbb{N}}$, where $x_n \in [0, 1]$, such that $\lim_{n \rightarrow \infty} x_n = x_0$, but

$$\lim_{n \rightarrow \infty} T(x_n, y_0) = z' < z_0 = T(x_0, y_0).$$

Since T is border continuous it suffices to consider the case when $x_0 \in (0, 1)$.

Firstly observe that

$$I_T(y_0, z') = \sup\{t \in [0, 1] \mid T(y_0, t) \leq z'\} = x_0, \quad (8)$$

since from the monotonicity of T we have $T(y_0, x_n) \leq z'$ for every $n \in \mathbb{N}$ and $T(y_0, x_0) = z_0 > z'$.

Next, from Proposition 2.4, by the Archimedeaness and monotonicity of T , we see that for any arbitrary $\varepsilon \in (0, 1 - x_0)$ we have that

$$T(x_0, 1 - \varepsilon) < \min(x_0, 1 - \varepsilon) = x_0. \quad (9)$$

Now, by (8) and (9) we get

$$T(x_0, 1 - \varepsilon) < I_T(y_0, z'),$$

for any $\varepsilon \in (0, 1 - x_0)$, thus

$$T(x_0, 1 - \varepsilon) < \sup\{t \in [0, 1] \mid T(y_0, t) \leq z'\},$$

hence

$$T(y_0, T(x_0, 1 - \varepsilon)) \leq z'$$

By the associativity of T we get

$$T(T(x_0, y_0), 1 - \varepsilon) \leq z',$$

i.e.,

$$T(z_0, 1 - \varepsilon) \leq z'.$$

for any $\varepsilon \in (0, 1 - x_0)$. This implies that

$$\lim_{\varepsilon \rightarrow 0^+} T(z_0, 1 - \varepsilon) \leq z' < z_0 = T(z_0, 1),$$

i.e., T is not border continuous, a contradiction to [Theorem 5.1](#) and hence T is left-continuous. \square

From [Theorems 3.7](#) and [5.5](#) we obtain the following result.

Corollary 5.6. For a function $I: [0, 1]^2 \rightarrow [0, 1]$ the following statements are equivalent:

- (i) I is a continuous R -implication based on some t -norm.
- (ii) I is Φ -conjugate with the Łukasiewicz implication, i.e., there exists $\varphi \in \Phi$, which is uniquely determined, such that I has the form (7) for all $x, y \in [0, 1]$.

6. Intersection between continuous R - and (S, N) -implications

It is well-known in the classical logic that the unary negation operation \neg can combine with any other binary operation to generate rest of the binary operations. This distinction of the unary \neg is also shared by the Boolean implication \rightarrow , if defined in the following usual way: $p \rightarrow q \equiv \neg p \vee q$. A generalization of this formula to the fuzzy logic gives the family of (S, N) -implications.

Definition 6.1 (cf. [19,22,8]). A function $I: [0, 1]^2 \rightarrow [0, 1]$ is called an (S, N) -implication, if there exist a t -conorm S and a fuzzy negation N such that

$$I(x, y) = S(N(x), y), \quad x, y \in [0, 1].$$

Moreover, if an (S, N) -implication is generated from S and N , then we will denote this by $I_{S,N}$. Firstly note that $I_{S,N} \in \mathcal{FI}$ for any t -conorm S and any fuzzy negation N . In the class of continuous (S, N) -implications we have the following important result.

Proposition 6.2 ([8, Proposition 5.4]). For a function $I: [0, 1]^2 \rightarrow [0, 1]$ the following statements are equivalent:

- (i) I is a continuous (S, N) -implication.
- (ii) I is an (S, N) -implication generated from some continuous t -conorm S and some continuous fuzzy negation N .

The intersections between the families and subfamilies of R - and (S, N) -implications have been studied by many authors, see e.g. [23,9,2,6]. As regards the intersection between their continuous subsets only the following result has been known so far:

Theorem 6.3. The only continuous (S, N) -implications that are also R -implications obtained from left-continuous t -norms are the fuzzy implications which are Φ -conjugate with the Łukasiewicz implication.

Now, from [Corollary 5.6](#) and [Proposition 6.2](#) the following equivalences follow immediately:

Theorem 6.4. For a function $I: [0, 1]^2 \rightarrow [0, 1]$ the following statements are equivalent:

- (i) I is a continuous (S, N) -implication that is also an R -implication obtained from a left-continuous t -norm.
- (ii) I is a continuous (S, N) -implication that is also an R -implication.
- (iii) I is an (S, N) -implication that is also a continuous R -implication.
- (iv) I is Φ -conjugate with the Łukasiewicz implication, i.e., there exists $\varphi \in \Phi$, which is uniquely determined, such that I has the form (7).

7. Conclusions

In this paper, we have shown that the continuous R -implications cannot be obtained from purely left-continuous t -norms and that the only continuous R -implications are those that are Φ -conjugate to the Łukasiewicz implication. Using this result we have been able to answer another question related to the intersection between the continuous sub-families of (S, N) - and R -implications. Also, from the results obtained during the course of proving the main theorem, we are able to give a direct proof of the fact that if the residual I_T obtained from a left-continuous T is continuous, then T is necessarily continuous and hence nilpotent.

We believe that this work will have positive implications in the research areas of t -norm based fuzzy logics and fuzzy inference mechanisms. We also hope that this work will further help in solving many of the open problems still remaining with regard to these two basic families of fuzzy implications.

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