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ON THE PECKING ORDER BETWEEN THOSE OF MITSCH AND CLIFFORD

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ABSTRACT. Order-theoretic explorations of algebraic structures are known to lead to hitherto hidden insights. Two such relations that have stood out are those of Mitsch and Clifford – the former for the generality in its application and the latter for the insights it offers. In this work, our motivation is to study the converse: we want to explore the extent of the utility of Mitsch's order and the applicability of Clifford's order. Firstly, we show that if the Mitsch's poset is either bounded or a chain, arguably a richer order theoretic structure, the semigroup reduces to one of a simple band. Secondly, noting that the special semigroups on which Clifford's relation does give rise to an order has not been characterised so far, we solve this problem by proposing a property called Quasi-Projectivity that is essential in this context and also give necessary and sufficient conditions for the Clifford's relation to give a total and compatible order, even if the semigroup is not commutative. Further, by showing some interesting connections between this relation and the orders obtained by Green's relations, we further reaffirm the importance and naturalness of the order proposed by Clifford. Finally, by discussing the Clifford's relations on ordered semigroups, we present some novel perspectives and also show that some of the assumptions in the often cited results of Clifford's are not necessary. On the whole, our study argues favourably towards Clifford's than that of the Mitsch's relation, in so far as the structural information gained about the underlying semigroup.

1. Introduction

The study of semigroups from an order-theoretic perspective has proven to be an effective and a complimentary tool to Green's relations in unearthing hitherto hidden insights on the underlying structure, see for instance the works of [1, 16, 18, 20] and of Clifford [2–5, 7]. Towards this end, many relations have been proposed by researchers [8, 14, 16, 18, 19] at various times and the special semigroups on which such relations impose a partial order have also been characterised.

Among the various order relations proposed, two stand out – that of Mitsch [14] and Clifford [4]. The former since it defines an order on any semigroup and the latter due to the many structural insights it offers[1], especially from the topological perspective, see, for instance, [13].

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[1] Also see Section 6.

1.1. Motivation for this work

On the one hand, while the relation defined by Mitsch [14] does lead to a partial order on any semigroup – albeit by making it a monoid if there is no suitable local left identity for each element, very few works have appeared presenting a deeper exploration on the algebraic properties that can be unearthed if one obtains richer order-theoretic structures. This forms the first of our twin motivations for this submission.

On the other hand, the relation proposed by Clifford [4] has had phenomenal impact, especially in the field of topological groups.[2] However, it should be noted that Clifford in his seminal work [4] begins by assuming this relation to be a partial order, and to the best of the authors' knowledge, there exist no works that discuss the conditions or the special semigroups for which the relation defined therein gives rise to an order. Further, in the main result of that work (see Theorem 6.2), the order is expected to be a total order. Once again, it is not known when the Clifford's order relation gives rise to a total order. These observations lead us to the second motivation for this work.

1.2. Outline of this submission

In this submission we begin by studying the class of semigroups which leads to a bounded poset or a chain with respect to Mitsch's order relation. Our study shows that such semigroups reduce to that of bands, where every element is idempotent.

Following this, we discuss a special kind of semigroup, which we call the *Quasi-Projective* semigroup. We then show that such special semigroups completely characterise the class of semigroups for which Clifford's relation becomes an order. Interestingly, we show that the Green's relations, which are only quasi-orders, become an order precisely on such semigroups, lending more credence to the claim of Clifford's order being *the* natural order on a semigroup.

Further, we also study the conditions under which Clifford's order (i) is compatible with the semigroup operation and (ii) gives rise to either a trivial or a total order. In the final section, we present some nascent perspectives that can be gained by imposing Clifford's order, even if there already exists an order on the underlying base set of the semigroup, by showing that some of the assumptions in Clifford's seminal results are only sufficient but not necessary.

2. Mitsch's relation and Richer posets

In the sequel, S will always denote a semigroup and S^1 denotes the set S if S has an identity and the set S with an identity adjoined in the other case.

Mitsch [14] proposed the following relation on a semigroup S and showed that it always leads to a partial order.

Definition 2.1. Let S be a semigroup. The relation $\le_{\mathcal{M}}$ on S defined as follows: For every $a, b \in S$,

$$a \le_{\mathcal{M}} b \iff \begin{cases} a = xb = by, \\ xa = a, \end{cases} \qquad \text{for some } x, y \in S^1 . \tag{1}$$

Theorem 2.2 ([14: Theorem 3]). *The relation $\le_{\mathcal{M}}$ defined in* (1) *is a partial order on S.*

[2] For an interesting review-cum-commentary of his works, we refer the readers to [12].

Consider the semigroup $S = ((0,1), \times)$ where $\times$ is the usual product. Clearly, for no $a \in (0,1)$ there exists an $x \in (0,1)$ such that $x \times a = a$. Thus we adjoin an identity, a two-sided one in this case, and let $S^1 = (0,1]$ to obtain a Mitsch poset[3], i.e., an order according to $\leq_{\mathcal{M}}$, on $(0,1)$. Clearly, this shows the importance of this property, in ensuring reflexivity, which we capture separately for use in the sequel.

Definition 2.3. A semigroup S is said to satisfy the *Local Left Identity* property

$$\text{if for every } a \in S, \text{ there exists an } x \in S \text{ s.t. } xa = a. \tag{LLI}$$

In a follow up paper, Mitsch [15] discussed some special posets obtainable from $\leq_{\mathcal{M}}$ and characterised the kind of semigroups that would give a total order w.r.t. $\leq_{\mathcal{M}}$. We refine this result further and, in fact, show that any bounded Mitsch poset necessarily implies that every element of the semigroup is idempotent.

Towards this end, we recall the following result of Mitsch [15]. Let E_S denote the set of all idempotent elements of S. The relation $\leq_{E_S}$ on E_S is defined as follows: For every $e, f \in E_S$,

$$e \leq_{E_S} f \iff e = ef = fe\ . \tag{2}$$

Clearly, the relation $\leq_{E_S}$ is a partial order on E_S. It can be easily seen that the Mitsch order $\leq_{\mathcal{M}}$ coincides with $\leq_{E_S}$ on the set of idempotents of any S.

Lemma 2.4 ([15: Lemma 2.1 (i)])**.** *In a Mitsch poset $(S, \leq_{\mathcal{M}})$, if $a \in E_S$ and $x \leq_{\mathcal{M}} a$, then $x \in E_S$.*

Lemma 2.5. *Let S be a semigroup such that $(S, \leq_{\mathcal{M}})$ is bounded above poset. Then there exists an element $x_0 \in S$ such that x_0 is an identity element, i.e., S is a monoid. Moreover, the identity element is a maximum element.*

P r o o f. Since the poset $(S, \leq_{\mathcal{M}})$ is bounded above, for all $a \in S$ there exists a unique $1 \in S$ such that $a \leq_{\mathcal{M}} 1$, by reflexivity property we have $1 \leq_{\mathcal{M}} 1$, i.e.,

$$1 = x_0 1 = 1 y_0, \quad x_0 1 = 1 \qquad \text{for some } x_0, y_0 \in S. \tag{3}$$

We know that 1 is the maximum element and hence $a \leq_{\mathcal{M}} 1$ for all $a \in S$, which implies that, there exist $x, y \in S$ such that $a = x1 = 1y,\ xa = a \Longrightarrow x_0 a = x_0 1 y$. Then by (3) we have $x_0 a = 1y \Longrightarrow x_0 a = a$. Thus x_0 is a left neutral element. Similarly, we can prove that y_0 is a right neutral element.This implies that $x_0 = y_0$, i.e., S has an identity element x_0 (say).

Now, we claim that $x_0 = 1$, the maximum element of the Mitsch poset. Since $x_0 \leq_{\mathcal{M}} 1$ by the definition of order in (1), there exist $x, y \in S$ such that $x_0 = x1 = 1y,\ xx_0 = x_0$, which implies that $x_0 = x$, from whence we obtain $x_0 = 1$. □

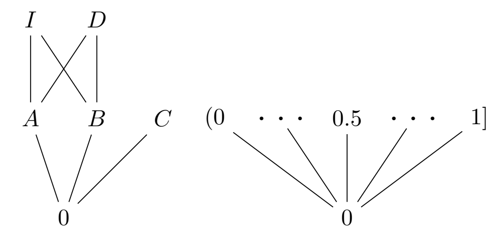
Remark 2.6. The converse of Lemma 2.5 need not be true, i.e., order obtained from a monoid need not be bounded above. For example, let S be a set of 2×2 integer matrices with the usual multiplication, where

$$S = \{0, I, A, B, C, D\} = \left\{0, I, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

It is clear that the multiplication operation forms a semigroup with the identity element I. But, it can be easily seen in the Figure 1, I is not a maximum element.

$((0,1], \times)$ is an example of an infinite monoid which is not bounded above and no two elements are comparable. On the other hand, if we consider $([0,1], \times)$ then 0 becomes the minimum element

[3] Note that by a Mitsch poset we refer to any semigroup with the order on it defined by $\leq_{\mathcal{M}}$.



$(S, \leq_{\mathcal{M}}, 0)$ $(S, \leq_{\mathcal{M}})$

FIGURE 1. The Hasse diagram for $\leq_{\mathcal{M}}$.

and so is comparable to every element in $(0, 1]$ w.r.t. $\leq_{\mathcal{M}}$, as indicated in its Hasse diagram in Figure 1.

THEOREM 2.7. *Let S be a semigroup. Then the following conditions are equivalent:*

(i) *The Mitsch poset $(S, \leq_{\mathcal{M}})$ is a bounded above.*

(ii) *The semigroup S is a band and $(E_S, \leq_{E_S})$ is bounded above.*

Proof. (i) $\Rightarrow$ (ii): From Lemma 2.5 we know that the Mitsch poset $(S, \leq_{\mathcal{M}})$ has the identity element $1 \in E_S$ as its maximum element. Since $a \leq_{\mathcal{M}} 1$ for all $a \in S$ by Lemma 2.4 we see that S is an idempotent monoid, i.e., $S = E_S$.

Since the partial order $\leq_{\mathcal{M}}$ defined in (1) coincides with the order on E_S, i.e., $\leq_{\mathcal{M}} = \leq_{E_S}$ on E_S, we have that $(E_S, \leq_{E_S})$ is bounded above.

(ii) $\Rightarrow$ (i): Trivially. $\square$

COROLLARY 2.8. *Let S be a semigroup. A poset $(S, \leq_{\mathcal{M}})$ is bounded above if and only if S is an idempotent monoid.*

Clearly, the examples given in Remark 2.6 are not idempotent semigroups. Mitsch in [15] also gave the following necessary and sufficient condition on a semigroup for a Mitsch poset defined on it to be a chain.

THEOREM 2.9 ([15: Theorem 2.3])**.** *A semigroup S is totally ordered w.r.t. $\leq_{\mathcal{M}}$ if and only if S is one of the following:*

(i) *$S = E_S$ and $(E_S, \leq_{E_S})$ is chain,*
or

(ii) *$S = E_S \cup \{a\}$ for some $a \notin E_S$ such that $ea = ae = e$ for every $e \in E_S$, and E_S is a chain with greatest element a^2.*

Note that the condition (ii) given in Theorem 2.9 is superfluous. Let $a \notin E_S$ as given and let $b \in E_S$. Clearly, $a \not\leq_{\mathcal{M}} b$, since if that were the case, by Lemma 2.4 we would have that $a \in E_S$, a contradiction to our assumption. Thus $b \leq_{\mathcal{M}} a$ for every $b \in E_S$ and hence a is the upper bound of S. However, from Theorem 2.7 we see that a is the identity element and hence $a^2 = aa = a \in E_S$, i.e., $S = E_S$.

The revised version of Theorem 2.9 is given below:

THEOREM 2.10. *A semigroup S is totally ordered w.r.t. $\leq_{\mathcal{M}}$ if and only if $S = E_S$ and $(E_S, \leq_{E_S})$ is chain.*

Note that not every band gives a total order w.r.t. $\leq_{\mathcal{M}}$. For instance, let X, Y be non-empty sets and $S = X \times Y$. Let us define the binary operation F on S by

$$F\left((x, s), (t, y)\right) = (x, y).$$

Then S is a band and the partial order w.r.t. $\leq_{\mathcal{M}}$ is trivial.

In the following, we present a complete characterization of the class of bands which become a total order w.r.t. $\leq_{\mathcal{M}}$.

Theorem 2.11. *A band S is totally ordered with respect to its $\leq_{E_S}$ if and only if S is commutative and one of the following holds:*

(i) *S is a left singular semigroup, (i.e., $ab = a$),*
or

(ii) *S is a right singular semigroup (i.e., $ab = b$).*

Proof. Suppose that $(E_S, \leq_{E_S})$ is totally ordered. Then for any $e, f \in E_S$, either $e \leq_{E_S} f$ or $f \leq_{E_S} e$, which implies that $e = ef = fe$ or $f = fe = ef$. Thus $ef = fe$ and $ef \in \{e, f\}$ for all $e, f \in E_S$. □

Theorem 2.12. *Let S be a semigroup. Then the following conditions are equivalent:*

(i) *$\leq_{\mathcal{M}}$ is a total order.*

(ii) *S is commutative and locally internal, i.e., $x \cdot y \in \{x, y\}$.*

On the one hand, we observe from Theorem 2.7 and Theorem 2.12 that linearity, or even boundedness of Mitsch's poset, offers a lot of information about the semigroup. On the other hand, quite quixotically, Mitsch's relation does not lead to interesting order theoretic structures from non-idempotent semigroups and thus blurring the utility of studying this relation on them.

On the contrary, as we will see in the following sections, Clifford's relation offers information at different granularities depending on the properties of the semigroups.

3. Quasi-projective semigroup

In this section, we begin by defining yet another special property of a semigroup, called the *quasi-projectivity* property which, in conjunction with (**LLI**), plays a major role in the sequel. We show that while such semigroups are different from already known special semigroups, these are still found in abundance. We conclude this section by characterising these semigroups interms of the Green's $\mathcal{L}$-relations.

Definition 3.1. Let S be a semigroup. It is said to satisfy the *Quasi-Projection* property, if for any $a, b, c \in S$,

$$abc = c \implies bc = c\ . \qquad (\mathbf{QP})$$

Some well-known semigroups that satisfy both (**LLI**) and (**QP**) are $(\mathbb{R}^{\geq 0}, +)$ and $(\mathbb{N}, \times)$. A few not-so-common examples of such semigroups are presented below. If a semigroup S satisfies (**QP**) we also term it as a **QP**-semigroup.

***Example* 3.2.** The following semigroups S_i satisfy both the (**QP**) and (**LLI**) properties on the given sets.

(i) Let $S_1 = \left\{\frac{1}{n} : n = 1, 2, 3, \dots\right\} \cup \{0\}$ and the semigroup operation be given by

$$ab = \begin{cases} 0, & \text{if } ab = 0, \\ \dfrac{1}{n+m-1}, & \text{if } a = \frac{1}{n} \text{ and } b = \frac{1}{m}. \end{cases}$$

(ii) Let $S_2 \neq \emptyset$ be s.t. $|S_2| > 2$ and let $0, 1 \in S_2$ be arbitrary but fixed and define the semigroup operation on S by

$$ab = \begin{cases} 0, & \text{if } a, b \in S_2 \setminus \{1\}, \\ a, & \text{if } b = 1, \\ b, & \text{if } a = 1. \end{cases}$$

(iii) Let us define the functions from $\{1, 2, 3\}$ to $\{1, 2, 3\}$ as follows:

$$A = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{smallmatrix}\right), \quad B = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{smallmatrix}\right), \quad C = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{smallmatrix}\right),$$

$$D = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \end{smallmatrix}\right), \quad E = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{smallmatrix}\right), \quad F = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \end{smallmatrix}\right).$$

Let $S_3 = \{A, B, C, D, E, F\}$ be a semigroup under the composition of functions, whose Cayley table we give below for ready reference:

$\circ$	A	B	C	D	E	F
A	A	A	A	E	E	E
B	A	A	A	E	E	E
C	A	B	C	D	E	F
D	A	A	A	E	E	E
E	A	A	A	E	E	E
F	A	B	C	D	E	F

(iv) $S_4 = \mathbb{Z}$ and the semigroup operation is given by $ab = |a|b$, where $|\cdot|$ is the absolute value. p S such that $xy \neq y$ for any $x, y \in S$ vacuously satisfies (**QP**).

(v) Every positive semigroup[4], semilattice or a right regular band is a **QP**-semigroup.

Remark 3.3.

(i) Not all regular, completely regular, quasi-separative or weakly-separative[5] semigroups satisfy (**QP**). Please see Example 3.4 (i).

(ii) A semigroup S does not satisfy (**QP**) if it has an identity e and if there exist $x, y \in S \setminus \{e\}$ such that $xy = e$. Clearly, a group will never satisfy (**QP**) and hence not all separative semigroups satisfy (**QP**).

(iii) Not all bands, i.e., where every element is idempotent, satisfy (**QP**) (see Example 3.4 (i)). While commutative bands are semilattices and hence satisfy (**QP**), commutativity is not necessary for a band to satisfy (**QP**) (see Example 3.4(ii)).

***Example* 3.4** ([9: Example 1]).

(i) Consider the 2×2 integer matrix semigroup

$$S = \{I, E, F\} = \left\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ -1 & 0 \end{smallmatrix}\right)\right\}$$

[4] A semigroup that has an identity element e, has no non-trivial invertible elements and satisfies the left cancellation law [17].

[5] See Drazin [9] for the definitions and relations among these special semigroups.

with multiplication. It can be easily verified that every element of S is idempotent and hence S is a band, from whence it follows that S is also regular, completely regular, separative, quasi-separative and weakly-separative.

However, S does not satisfy (**QP**), since $EFE = E$ but $FE \neq E$.

(ii) Let $S \neq \emptyset$ and define the semigroup operation by $xy = y$ for all $x, y \in S$. Clearly, S is a non-commutative band but satisfies (**QP**).

3.1. Green's relation and (QP)

The five relations of Green [10] are known to give a peek into the structure of a semigroup by characterising the elements based on the principal ideals generated by them. In the following we show that **QP**-semigroups are precisely those for whom the Green's $\mathcal{L}$-equivalence classes are singletons.

Definition 3.5 ([10: page 164])**.** Let S be a semigroup.

(i) The principal left ideal generated by an $a \in S$ is given by $(a)_L = Sa \cup \{a\}$.

(ii) The Green's $\mathcal{L}$-relation on S is defined as follows for any $a, b \in S$:

$$a\mathcal{L}b \iff (a)_L = (b)_L. \tag{4}$$

Clearly, if S satisfies (**LLI**) or is a monoid, then $(a)_L = Sa$ for every $a \in S$. Also it is clear that the $\mathcal{L}$-relation is an equivalence relation on S and we denote the $\mathcal{L}$-equivalence class of an $a \in S$ by $\mathcal{L}_a$, i.e., $\mathcal{L}_a = \{b \in S \mid (a)_L = (b)_L\}$.

Definition 3.6. A semigroup S is called an $\mathcal{L}$-trivial semigroup if the Green's $\mathcal{L}$-equivalence classes are trivial, i.e., $\mathcal{L}_a = \mathcal{L}_b \iff a = b$.

The proof of the following lemma is straightforward:

Lemma 3.7. *Let S be a semigroup that satisfies* (**LLI**) *and let $a, b \in S$ be arbitrary. Then the following conditions are equivalent:*

(i) $Sa = Sb$.

(ii) $\mathcal{L}_a = \mathcal{L}_b$.

Theorem 3.8. *Let S be a semigroup that satisfies* (**LLI**). *Then the following conditions are equivalent:*

(i) *S is a **QP**-semigroup.*

(ii) *S is $\mathcal{L}$-trivial.*

Proof. (i) $\Rightarrow$ (ii): Suppose $\mathcal{L}_a = \mathcal{L}_b$ for some $a, b \in S$. Then clearly $a \in \mathcal{L}_b$ and $b \in \mathcal{L}_a$ which implies that $xb = a$ and $ya = b$ for some $x, y \in S$. Thus,

$$a = xb \implies ya = y(xb) = b.$$

Now, from (**QP**) we have $xb = b$ and hence $a = b$.

(ii) $\Rightarrow$ (i): Let $x, y, z \in S$ such that $xyz = z$. Then we have , $Sz = S(xyz) \subseteq S(yz) \subseteq Sz$, i.e., $S(yz) = Sz$. By Lemma 3.7, we have $\mathcal{L}_{yz} = \mathcal{L}_z$. Since S is $\mathcal{L}$-trivial we have $yz = z$. $\square$

4. Clifford's relation on a semigroup

In this section, we begin by recalling the order relation defined by Clifford in [4] and show the important role played by (**LLI**) and (**QP**) in this setting[6]. Further, we show that the order of Clifford arises naturally from the Greens's $\mathcal{L}$-relation.

Theorem 4.1. *Let S be a semigroup. The Clifford's relation $\preceq$[7] on S is defined as follows: For any $a, b \in S$,*

$$a \preceq b \iff \text{there exists } x \in S \text{ such that } xb = a. \tag{5}$$

The following are equivalent:

(i) *$(S, \preceq)$ is a poset.*

(ii) *S satisfies both* (**QP**) *and* (**LLI**).

(iii) *S is $\mathcal{L}$-trivial and satisfies* (**LLI**).

Proof. (i) ⇒ (ii): Since $(S, \preceq)$ is a poset, we have $a \preceq a$ for every $a \in S$, hence by the definition of $\preceq$, for every $a \in S$ there exists an $x \in S$ such that $xa = a$, i.e., S satisfies (**LLI**).

Towards showing that S satisfies (**QP**), for some $a, b, c \in S$ let $abc = c$. Hence, by the definition of order we have that $c \preceq bc$. However, since $bc = bc$, trivially, once again by the definition of order $bc \preceq c$, from whence by anti-symmetry we have $bc = c$.

(ii) ⇒ (i): That the reflexivity of $\preceq$ follows from (**LLI**) and its transitivity follows from associativity is easy to see. That (**QP**) implies antisymmetry can be seen as follows: Let $a, b \in S$ such that $a \preceq b$ and $b \preceq a$. Then there exist $x, y \in S$ such that $xb = a$ and $ya = b$ from whence $a = xb \Rightarrow ya = y(xb) = b$ and by (**QP**) $xb = b$ and hence $a = b$. Thus $(S, \preceq)$ is a poset.

(iii) ⇔ (ii): Follows from Theorem 3.8. □

Remark 4.2. Figure 2 gives the Hasse diagrams of the Clifford's posets presented in Example 3.2. If a Clifford's poset is bounded with $0, 1 \in S$ as the minimum and maximum elements, respectively, then we write $(S, \preceq, 0, 1)$.

(i) It is easy to see that the Clifford's posets $(S_1, \preceq, 0, 1)$ and $(S_2, \preceq, 0, 1)$ from Examples 3.2 (i) and (ii) are bounded, with the former being a chain and the latter only a lattice, while $(S_3, \preceq)$ from Example 3.2(iii) is neither bounded above nor bounded below.

(ii) The Clifford's poset $(S_4, \preceq, 0)$ from Example 3.2(iv) is bounded below but not above. 0 is the minimum element while $1, -1$ are the maximal elements. Further, it is only a meet semi-latiice in which

$$x \wedge y = \begin{cases} 0, & \text{if } (x \in \mathbb{N} \text{ and } y \in \mathbb{N}^-) \\ & \text{or } (x \in \mathbb{N}^- \text{ and } y \in \mathbb{N}), \\ \text{l.c.m.}\{x, y\}, & \text{otherwise,} \end{cases}$$

where $\mathbb{N}^-$ is the set of negative integers. Once again, note that, according to the usual ordering, $(\mathbb{Z}, \leq)$ is neither bounded below nor above.

The following lemma is useful in the Remark 4.4 that follows it.

Lemma 4.3. *Let S be a semigroup that satisfies* (**LLI**)*. Then the following statements are equivalent:*

[6] A part of this result has been proven in [11] in the context where $S = [0, 1]$.

[7] To remain consistent with the notations we should have used $\leq_{\mathcal{C}}$. However, for better readability, considering the extensive use of this symbol in the sequel, we instead use $\preceq$.

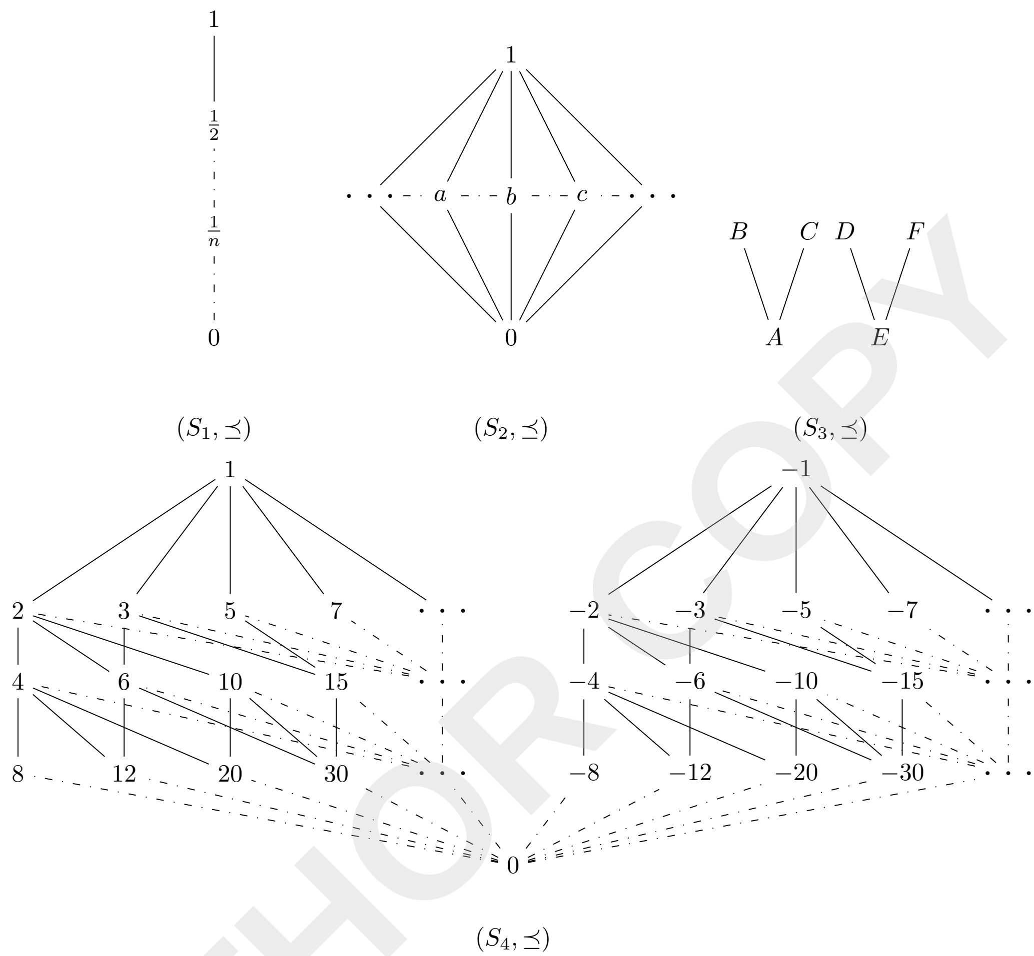


FIGURE 2. The Hasse diagrams of the Clifford's posets obtained from Examples 3.2.

(i) $Sa \subseteq Sb$,

(ii) $a \in Sb$,

(iii) $a = xb$ *for some* $x \in S$.

Remark 4.4. Consider the order relation based on Green's $\mathcal{L}$-relation given as follows. If S is a semigroup, then

$$a \leq_{\mathcal{L}} b \iff S^1 a \subseteq S^1 b \;, \tag{6}$$

where S^1 denotes the set S if S has an identity and the set S with an identity adjoined in the other case. The following two observations are noteworthy:

(i) One can derive the Clifford's order relation from the above, as shown below:

$$\begin{aligned} a \leq_{\mathcal{L}} b &\iff S^1 a \subseteq S^1 b \iff a \in S^1 b \\ &\iff a = xb \qquad \text{for some } x \in S^1 \iff a \preceq b \;. \end{aligned}$$

(ii) It is well known that $\leq_{\mathcal{L}}$ is only a pre-order on S, i.e., it is only reflexive and transitive, while it does become an order on the set of $\mathcal{L}$-equivalence classes. Theorem 4.1 shows that if S satisfies (**QP**) and (**LLI**) then $\leq_{\mathcal{L}}$ is an order on S, from whence it also follows that the corresponding $\mathcal{L}$-equivalence classes are singletons.[8]

5. Clifford's relation as a natural partial order

What constitutes a natural partial order on a semigroup has seen different interpretations - from defining it to be natural if the order is defined from the semigroup operation, see for instance, Mitsch [14] to insisting on some specific compatibility/monotonicity-type conditions, see for instance [6,12]. However, all of them, in one way or the other, seem to insist on the partial order to have the following properties w.r.t. the semigroup operation and some special subsets:

Coincidence on $(E_S, \sqsubseteq)$: Let E_S denote the set of idempotents of S. Then the *natural* partial order $\sqsubseteq$ defined on E_S is as follows:

$$e \sqsubseteq f \iff ef = fe = e, \qquad e, f \in E_S .$$

A partial order $\leq$ on S to be *natural* is expected to coincide with the order $\sqsubseteq$ when restricted to E_S.

Dependence on special properties: A partial order, if applicable only to a special semigroup, should quintessentially depend on the properties of that special semigroup. For instance, the order relations defined by those of Nambooripad [16], Conrad [8] and Sussman [19] make use of the properties of regularity, weak-separativity and quasi-separativity of semigroups, respectively.

Compatibility: A partial order $\leq$ on S should be compatible with the semigroup operation, i.e., if $a \leq b$ then $ac \leq bc$ and $ca \leq cb$ for any $c \in S$.

In the following we show that the Clifford's order relation is natural in every sense of its interpretation. In fact, Remark 4.4 makes yet another argument substantiating the Clifford's order relation as *the* natural order.

5.1. Coincidence of $\preceq$ on $(E_S, \sqsubseteq)$

Lemma 5.1. *The relation $\preceq$ coincides with $\sqsubseteq$ on E_S.*

Proof. If $E_S = \emptyset$ then it is vacuously true. Let $E_S \neq \emptyset$ and $e \preceq f$ for some $e, f \in E_S$. Then there exists a $g \in E_S$ such that $e = gf$. Now,

$$e = gf \implies ef = gf^2 = gf = e.$$

Then, $gf = e \Rightarrow (gf)e = e^2 = e$, by (**QP**), we have $fe = e$. Thus $e = ef = fe$ and hence $e \sqsubseteq f$.

On the other hand, let $e \sqsubseteq f$, which by definition of $\sqsubseteq$ implies $e = ef$ and hence, clearly, $e \preceq f$. □

[8] Note that if S satisfies (**LLI**), the set S^1 in the definition of $\leq_{\mathcal{L}}$ in (6) can be replaced by S itself.

5.2. Compatibility of $\preceq$ on S

$$a = xb \implies ac = (xb)c = x(bc) \implies ac \preceq bc.$$

However, $\preceq$ need not be compatible from the left. While commutativity of S is sufficient, it is not necessary, for instance, see Example 3.2 (iii).

In the following, we give a necessary and sufficient condition for this.

Proposition 5.2. *Let S be a semigroup and let $\preceq$ as defined in (5) be a partial order on S. The following statements are equivalent:*

(i) *$\preceq$ is compatible with the semigroup operation.*

(ii) *For all $c, y, b \in S$ there exists an element $z \in S$ such that $cyb = zcb$.*

Proof. (i) $\Rightarrow$ (ii): Let $c, y, b \in S$ be arbitrary. Let $a = yb \in S$. This implies by (5) that $a \preceq b$ and hence by the compatibility of $\preceq$ we have that

$$ca \preceq cb \Longrightarrow cyb \preceq cb \Longrightarrow cyb = zcb$$

for some $z \in S$, once again by the definition of (5).

(ii) $\Rightarrow$ (i): Let $a \preceq b$. It is sufficient to prove that $ca \preceq cb$ for any $c \in S$. Now, there exist a $y \in S$ such that $a = yb$. By the hypothesis, for each $c, y, b \in S$ there exist an element $z \in S$ such that $cyb = zcb$. Hence, $ca = cyb = zcb$, which implies that $ca \preceq cb$. □

6. Clifford's order and some special posets

In the very same work [4] where Clifford had introduced his order relation (5), he has also proven his now famous results on the structure of semigroups that give rise to a total order. We give these results below for ready reference and also to highlight its importance.[9] .

Lemma 6.1 (cf. [4: Lemma 1.1]). *Let S be a commutative semigroup that is totally ordered w.r.t. the Clifford's relation (5). If $P \neq \emptyset$ is a proper, absorbent prime ideal[10] of S then $P, S \setminus P$ are subsemigroups that are totally ordered w.r.t. (5) and S is the ordinal sum of $S \setminus P$ and P in that order.*

Theorem 6.2 (cf. [4: Theorem 1]). *Every commutative semigroup that is totally ordered w.r.t. the Clifford's relation (5) is uniquely expressible as the ordinal sum of a totally ordered set of ordinally irreducible such semigroups.*

However, so far, there has been no work discussing the conditions under which the Clifford's relation gives rise to a total order. In this section, we not only present an answer to this poser but also discuss the context in which we may obtain further special order theoretic structures, viz., a discrete or a bounded poset.

Theorem 6.3. *Let S be a semigroup that satisfies* (**LLI**). *Then the following are equivalent:*

(i) *S is $\mathcal{L}$-trivial and the principal left ideals form a chain with respect to inclusion.*

(ii) *$(S, \preceq)$ is a totally ordered poset.*

---

[9] In Section 7.1 we show that even when the Clifford's order is not total the results may still hold.

[10] Please see Section 7.2 for the definitions of these terms.

P r o o f. (i) $\Rightarrow$ (ii): Since S is an $\mathcal{L}$-trivial semigroup that satisfies (**LLI**), by Theorem 4.1, $(S, \preceq)$ is a poset. Since all principal left ideals form a chain with respect to inclusion and by the $\mathcal{L}$-triviality of S we have the following equivalences for any $a, b \in S$:

$$\begin{aligned} Sa \subseteq Sb \text{ or } Sb \subseteq Sa &\iff a \in Sb \text{ or } b \in Sa \\ &\iff a = xb \text{ or } b = ya \qquad \text{for some } x, y \in S \\ &\iff a \preceq b \text{ or } b \preceq a. \end{aligned}$$

i.e., $(S, \preceq)$ is a totally ordered poset.

(ii) $\Rightarrow$ (i): Let $(S, \preceq)$ be a totally ordered poset. Then for any $a, b \in S$ either $a \preceq b$ or $b \preceq a$, which implies that $xb = a$ or $ya = b$ for some $x, y \in S$ and hence, the principal left ideals form a chain with respect to inclusion. □

An alternate form of Theorem 6.2 purely in terms of the algebraic properties of a semigroup is given below:

Theorem 6.4. *Let S be a commutative **QP**-semigroup in which the principal left ideals form a chain with respect to inclusion. Then S is uniquely expressible as the ordinal sum of a totally ordered set of ordinally irreducible such semigroups.*

A partial order on S is said to be trivial if all elements of S are incomparable.

Proposition 6.5. *Let S be a semigroup. The following are equivalent:*

(i) *The Clifford's poset $(S, \preceq)$ is trivial.*
(ii) *The semigroup S is right singular, i.e., $xy = y$ for all $x, y \in S$.*

P r o o f. (i) $\Rightarrow$ (ii): Suppose that the semigroup operation is not a right projection map. Then there exist $x, y \in S$ and $t \neq y \in S$ such that $xy = t$. Hence, by definition of order in (5), we have $t \preceq y$, which is a contradiction.

(ii) $\Rightarrow$ (i): Follows trivially. □

Proposition 6.6. *Let $(S, \preceq)$ be a poset. The following are equivalent:*

(i) *S is bounded above.*
(ii) *There exits a unique $1 \in S$ such that $S1 = S$.*

P r o o f. (i) $\Rightarrow$ (ii): Since the poset $(S, \preceq)$ is bounded above, there exists a unique $1 \in S$ such that $a \preceq 1$ for all $a \in S$. By definition of order in (5), for every $a \in S$ there exists $x \in S$ such that $x1 = a$, which implies that $S \subseteq S1$. Clearly, $S1 \subseteq S$, hence $S1 = S$.

(ii) $\Rightarrow$ (i): Since $S1 = S$, for every $a \in S$ there exists $x \in S$ such that $x1 = a$. Hence, by (5), we have $a \preceq 1$ for all $a \in S$. □

Lemma 6.7. *Let S be a commutative semigroup such that the Clifford's poset $(S, \preceq)$ is bounded above. Then S is an integral ordered monoid.*[11]

P r o o f. Let 1 be the upper bound of S. We claim that 1 is the identity of S.

From Proposition 6.6 we know that there exists a $w \in S$ s.t. $w1 = 1$. Let $a \in S$ be arbitrary. Since $a \preceq 1$, we have that there exists an $x \in S$ s.t. $x1 = a$. Now, $x1 = x(w1) = w(x1) = wa = a$, i.e., w is the left identity and by commutativity of S also its right and the unique identity.

Finally, since $w1 = 1w = 1$ we have that $1 \preceq w$, from whence we obtain that $w = 1$ and that S is an integral monoid. □

[11] A monoid where the operation is compatible w.r.t. the order and the identity is also the top element.

The results of this section show that, unlike in the case of the order from Mitsch, the Clifford's relation allows a large enough class of semigroups to exhibit various rich order theoretic structures.

7. Clifford's relation on an ordered semigroup

As has already been mentioned, an order theoretic exploration of an algebraic object allows one to glean hitherto unknown insights. In the following we consider a particular situation which offers yet novel perspectives.

It is often the case that a semigroup S may already have been endowed with an order, consider for instance, the case of $S = (\mathbb{N}, +)$ with the usual order on $\mathbb{N}$. Note that this order may be different from the one we may obtain employing the many (ordering) relations proposed in the literature, based on the type of semigroup S is. Clearly, one always gets an order from the relation defined by Mitsch [14].

In this context, we present two results in this section, the first of which enables us to gain some alternate perspectives on the semigroup and the second allows us to construct examples showing that the assumption of total order in Lemma 6.1 and Theorem 6.2 are only sufficient and not necessary.

The results of this section show that Clifford's order not only leverages existing underlying orders but also offers sufficient structural information about a semigroup even when the obtained poset is not grandiose.

7.1. Semigroups and orders: An alternate perspective

In this section, we present a result which is quite interesting in its own right. Firstly, it shows that if the semigroup operation satisfies a particular type of boundedness w.r.t. any order on S, then it satisfies (5) and hence may lead to, arguably, a richer kind of order-theoretic structure. Secondly, it also shows that w.r.t. the new order, the semigroup itself may augment its algebraic characteristics.

Proposition 7.1. *Let $(S, \leq)$ be a semigroup with a partial order $\leq$. If the semigroup operation satisfies one of the following inequalities:*

$$xy \leq y \qquad \text{for every } x, y \in S, \tag{7}$$

$$xy \geq y \qquad \text{for every } x, y \in S, \tag{8}$$

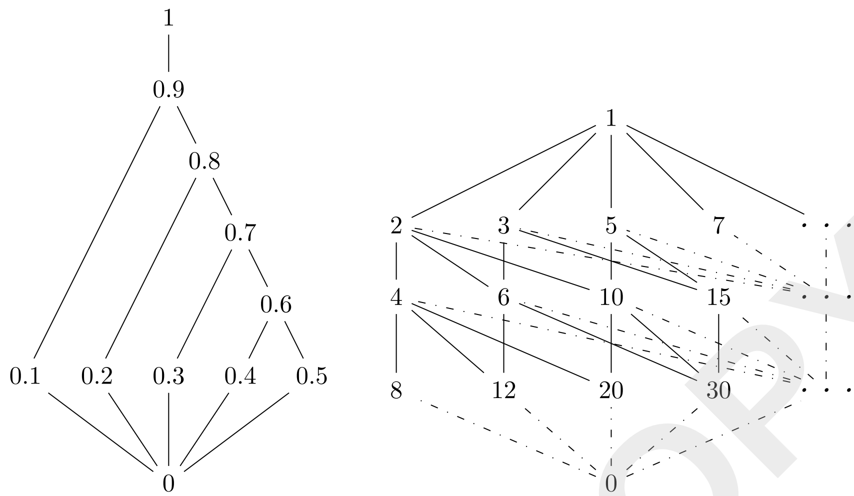
then S satisfies (**QP**).

Proof. Let $a, b, c \in S$. If possible, let $a(bc) = c$ but $bc = d \neq c$. If $xy \geq y$ for every $x, y \in S$, then we have that $bc = d > c$, from whence we obtain $a(bc) = ad \geq d > c$, a contradiction. Thus $bc = c$ and S satisfies (**QP**). □

In the rest of the section, we present some illustrative examples that highlight different aspects of the above result.

***Example* 7.2.** Note that in the above, even if S satisfies (**LLI**), $\preceq$ need not coincide with $\leq$. For instance, consider the monoid $S = \{0, .1, \ldots, .9, 1\}$ with the operation $*$ given as:

$$x * y = \begin{cases} 0, & \text{if } x + y < 1, \\ \min(x, y), & \text{otherwise.} \end{cases} \tag{9}$$



$(S, \preceq, 0, 1)$ $\quad$ $(\mathbb{N} \cup \{0\}, \preceq_\times)$

FIGURE 3. The Clifford's posets from Examples 7.2 and 7.3.

We clearly see that with the usual ordering on S, which is a chain, $x * y \leq y$ and hence the monoid $(S, *)$ satisfies (**QP**). Further, since $(S, *)$ satisfies (**LLI**) the Clifford's relation $\preceq$ gives rise to a partial order on S. However, the Clifford's order can be seen to be different as given in the Hasse diagram in Figure 3.

On the other hand, if we consider $(\mathbb{N} \cup 0, +)$ we see, that $x + y \geq y$ and hence it satisfies (**QP**) and the obtained order $\preceq$ is the dual of the natural order.

***Example* 7.3.** Consider the semigroup $(\mathbb{N} \cup \{0\}, \times)$ with the usual order $\leq$ on $\mathbb{N}$. It is easy to see that it does not satisfy either (8) or (7). For instance, let $y > 0$. If $x = 0$ then $xy \not\geq y$, while if $x > 0$ then $xy \not\leq y$. However, $(\mathbb{N} \cup \{0\}, \times)$ is a **QP**-semigroup.

While $(\mathbb{N}, \leq)$ is only a bounded below chain, the poset $\mathbb{P} = (\mathbb{N} \cup \{0\}, \preceq_\times, 0, 1)$ (see Figure 3) is bounded (both above and below) with 1 and 0 as the maximum and minimum elements. In fact, it is the bounded lattice with the meet and join operations given as the g.c.d. and l.c.m. of any two numbers.

The following example plays a dual role - not only does it show that neither of the conditions (8) and (7) is necessary, but allows us to illustrate another perspective.

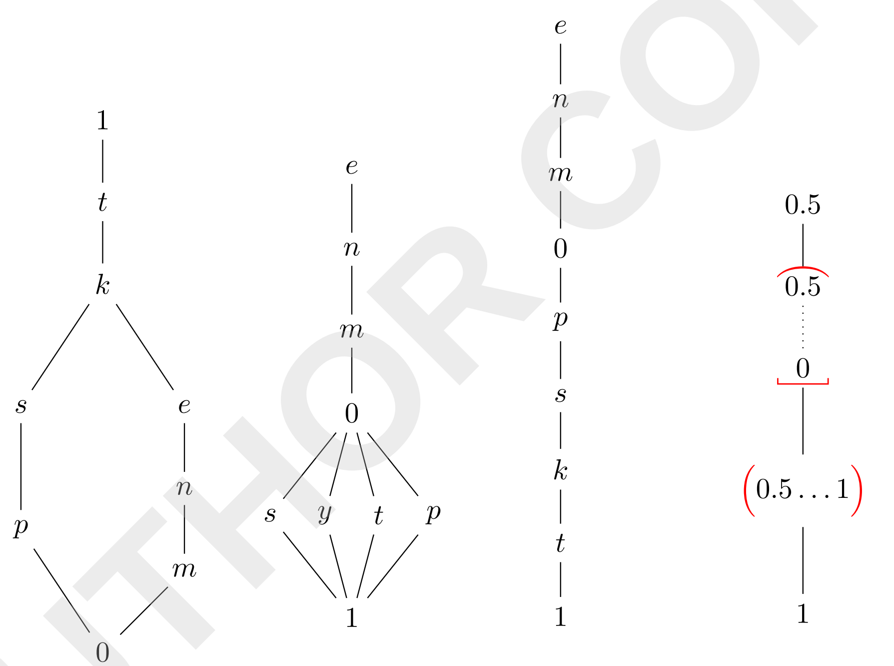
***Example* 7.4.** Consider the lattice $(S = \{0, m, n, e, p, s, k, t, 1\}, \leq, 0, 1)$ whose Hasse diagram is given in Figure 4 and the operations U and U_E defined on S_1 in Table 1. Note that while the original lattice $(S, \leq)$ is only bounded it is neither modular (consider the sublattice $\{0, m, e, s, k\}$) nor a chain, whereas the obtained Clifford's posets $(S, \preceq_U)$ and $(S, \preceq_{U_E})$ are modular and a chain, respectively.

Example 7.4 also allows to make an interesting observation. Note that while U_1 and U_2 are ordered monoids w.r.t. $(S, \leq)$, they become integral ordered monoids w.r.t. the Clifford's poset $(S, \preceq)$ obtained.

TABLE 1. The Functions U and U_E give order semigroups on $(S, \leq)$.

U	0	m	n	e	p	s	k	t	1
0	0	0	0	0	p	s	k	t	1
m	0	m	m	m	p	s	k	t	1
n	0	m	n	n	p	s	k	t	1
e	0	m	n	e	p	s	k	t	1
p	p	p	p	p	1	1	1	1	1
s	s	s	s	s	1	1	1	1	1
k	k	k	k	k	1	1	1	1	1
t	t	t	t	t	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1

U_E	0	m	n	e	p	s	k	t	1
0	0	0	0	0	p	s	k	t	1
m	0	m	m	m	p	s	k	t	1
n	0	m	n	n	p	s	k	t	1
e	0	m	n	e	p	s	k	t	1
p	p	p	p	p	p	s	k	t	1
s	s	s	s	s	s	s	k	t	1
k	k	k	k	k	k	k	k	t	1
t	t	t	t	t	t	t	t	t	1
1	1	1	1	1	1	1	1	1	1



(a) $(S_1, \leq, 0, 1)$ *(b)* $(S, \preceq_U, 1, e)$ *(c)* $(S, \preceq_{U_E}, 1, e)$ *(d)* $([0,1], \preceq_*, 1, 0.5)$

FIGURE 4. (a)–(c) The lattice $(S_1, \leq, 0, 1)$ and Clifford's order obtained from the operations given in Table 1. (d) Clifford's poset obtained on $[0,1]$ by the operation (10).

7.2. On the necessity of a total order from Clifford's relation

We begin this subsection with a result which characterises some special ordered monoids in terms of its constitutent **QP**-subsemigroups. It also paves the way to construct examples that show that some of the assumptions in Lemma 6.1 and Theorem 6.2 are not necessary.

THEOREM 7.5. *Let $(S, \cdot, \leq)$ be an ordered monoid with identity e. If e is comparable with all elements of S w.r.t. $\leq$, then S can be written as the disjoint union of two* ***QP****-subsemigroups.*

Proof. Let $(S, \cdot, \leq)$ be an ordered monoid with identity e. Let us denote by S^+ the set of all elements greater than e, i.e., $S^+ = \{x \in S : e \leq x\}$. Let $S^- = S \setminus S^+ = \{x \in S\ : x < e\}$.

Let $x, y \in S^+$, i.e., $e \leq x$ and $e \leq y$. By the compatibility of $\cdot$ w.r.t. $\leq$ we have that $e \leq x = xe \leq xy$ and therefore $xy \in S^+$. Clearly, S^+ inherits the associative property from S so $(S^+, \cdot)$ is a semigroup.

From the above it can be seen that for any $x, y \in S^+$ we have, that $y = ey \leq xy$ and by Proposition 7.1, we see that $\cdot$ satisfies (**QP**) on S^+.

Similarly, one can show that $(S^-, \cdot)$ is also a **QP**-semigroup. Thus $S = S^+ \cup S^-$ is a disjoint union of its **QP**-subsemigroups. □

Corollary 7.6. *Every totally ordered monoid is a disjoint union of* ***QP****-semigroups.*

Corollary 7.7. *Let S be an ordered semigroup with identity e. If e is either the top or the bottom element of S then S is a* ***QP****-semigroup.*

***Example* 7.8.**

(i) The additive groups of $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{Q}$ are all ordered groups under the usual ordering and by Theorem 7.5 can be written as disjoint union of two **QP**-semigroups, viz., of the negative and non-negative elements.

(ii) Consider the ordered monoid $S =([0,1], *, \leq, e = 0.5)$ where $\leq$ is the usual order and $*$ is given as follows:

$$x * y = \begin{cases} 0, & \text{if } (x, y) \in \{(0,1),(1,0)\} \\ \dfrac{xy}{xy + (1-x)(1-y)} & \text{otherwise} \end{cases} .$$

That $*$ does not satisfy (**QP**) on $[0,1]$ is clear since $0.1 * (0.9 * 0.9) = 0.9$ but $0.9 * 0.9 = \frac{81}{82} = 0.987 \neq 0.9$. However, the above result shows that $([0, 0.5), *)$ and $([0.5, 1], *)$ are both **QP**-semigroups.

In fact, it can be seen that $([0, 0.5], *)$ and $([0.5, 1], *)$ are both **QP**-monoids with 0.5 as the identity.

It can be surmised from [4] that the author considered only semigroups that were totally ordered, w.r.t. Clifford's relation (5), due to the terse structural reduction obtained in terms of irreducible subsemigroups of the same kind. However, based on the above results, we can construct examples that show that the assumption of total order w.r.t. (5) in Lemma 6.1 and Theorem 6.2 is not necessary.

We recall some of the definitions that will be used in the immediate sequel. An ideal P of S is prime if $S \setminus P$ is a subsemigroup of S. If $a, b \in S$ are such that $ab = a$, a is said to absorb b. An ideal P of S will be called absorbent if $a \in P$ and $b \notin P$ imply $ab = a$.

Now, let us consider the following ordered monoid $S = ([0,1], *, \leq, e = 0.5)$ where

$$x * y = \begin{cases} 1, & \text{if } x, y \in [0.5, 1], \\ \min(x, y), & \text{if } x, y \in [0, 0.5], \\ \max(x, y), & \text{otherwise.} \end{cases} \tag{10}$$

Let $S^+ = [0.5, 1]$ and $S^- = [0, 0.5)$. It is clear from Theorem 7.5 that S^+ is a **QP**-subsemigroup. In fact, it can be seen that $SS^+ = S^+S \subseteq S^+$ and hence is an ideal. Further, $S \setminus S^+ = S^-$ is a subsemigroup of S and hence S^+ is, in fact, a prime ideal and by the definition of $*$ in (10) S^+ is also absorbent. It is easy to see now that S is, in fact, an ordinal sum of $S \setminus S^+$ and S^+ as given in Lemma 6.1.

Now, S itself satisfies both (**LLI**) and (**QP**) and hence gives rise to a poset w.r.t. the Clifford's relation (5). However, this order is not a total order as can be seen from the Hasse diagram given in Figure 4 (d). Note that no two elements $x, y \in (0.5, 1)$ are comparable between themselves.

8. Some concluding remarks

The study contained in this paper can be seen as an exploration of semigroups that are linear w.r.t. some known and useful order relations, namely those of Mitsch and Clifford. As the results in this work show, mere boundedness w.r.t. Mitsch's relation seems to be very stringent on the semigroup and does not lead us to any further insights into the structure of the semigroup. In the case of Clifford's relation, even when it does not make the semigroup linear, we seem to be able to extract a lot of structural information.

In conjunction with the interesting connections between Clifford's relation and Green's $\mathcal{L}$-equivalences that have been shown, the current study nudges the Clifford's relation towards preeminence among all the orders defined on semigroups.

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