YAGER'S CLASSES OF FUZZY IMPLICATIONS: SOME PROPERTIES AND INTERSECTIONS

Michał Baczyński and Balasubramaniam Jayaram

Recently, Yager in the article "On some new classes of implication operators and their role in approximate reasoning" [12] has introduced two new classes of fuzzy implications called the f-generated and g-generated implications. Along similar lines, one of us has proposed another class of fuzzy implications called the h-generated implications. In this article we discuss in detail some properties of the above mentioned classes of fuzzy implications and we describe their relationships amongst themselves and with the well established (S, N)-implications and R-implications. In the cases where they intersect the precise sub-families have been determined.

Keywords: fuzzy implication, f-generated implication, g-generated implication, h-generated implication, (S, N)-implication, S-implication, R-implication

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1. INTRODUCTION

Recently, Yager [12] has introduced two new families of fuzzy implications, called the f-generated and g-generated implications, respectively, and discussed their desirable properties as listed in [4] or [5]. Also, Balasubramaniam [3] has discussed f-generated implications with respect to three classical logic tautologies, viz., the distributivity, the law of importation and the contrapositive symmetry. In [2, 3] a new class of fuzzy implications, along the lines of f-generated implications, called the h-generated implications has been proposed.

In this work, we attempt to answer the following questions:

Problem 1.

(i) What are the properties of f-, g- and h-generated implications?

(ii) Do the above families of fuzzy implications intersect with the two well-known classes of fuzzy implications, viz., (S, N)-implications and R-implications?

We show that they are, in general, different from the well established (S, N)- and R-implications. In the cases where they intersect with the above families of fuzzy implications, we have determined precisely the sub-families of such intersections.

The paper is organized as follows. In Section 2 we present some preliminary definitions and results connected with (S, N)- and R-implications. In the next three sections we give the definitions of the newly proposed families of f-, g- and h-generated fuzzy implications and also prove some new results concerning them. While in Section 6 we investigate the intersections amongst the families of f-, g- and h-generated fuzzy implications, Sections 7 and 8 contain our investigations on the intersections of the families of f-, g- and h-generated fuzzy implications with (S, N)- and R-implications, respectively. Section 9 gives some concluding remarks.

2. PRELIMINARIES

We assume that the reader is familiar with the classical results concerning basic fuzzy logic connectives (fuzzy negations, t-norms, t-conorms, fuzzy implications), but we briefly recall some definitions, examples and facts that will be useful in the sequel (for more details see [5, 8] or [6]).

Definition 1. (Fodor and Roubens [5], Klement et al. [7]) A decreasing function $N\colon [0,1] \to [0,1]$ is called a fuzzy negation if $N(0) = 1$ and $N(1) = 0$. A fuzzy negation N is called

(i) strict if, in addition, it is strictly decreasing and continuous.

(ii) strong if, in addition, it is an involution, i. e., $N(N(x)) = x$ for all $x \in [0,1]$.

Example 1. Table 1 lists a few negations with the properties they satisfy. For more examples see [5] or [8].

Table 1. Examples of fuzzy negations and their properties.

Name	Formula	Properties
classical	$N_{\mathbf{C}}(x) = 1 - x$	strong
Gödel	$N_{\mathbf{G1}}(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x > 0 \end{cases}$	not continuous, smallest
dual Gödel	$N_{\mathbf{G2}}(x) = \begin{cases} 1, & \text{if } x < 1 \\ 0, & \text{if } x = 1 \end{cases}$	not continuous, greatest

In the literature, especially at the beginnings, we can find several different definitions of fuzzy implications (see [4, 5]). In this article we will use the following one, which is equivalent to the definition introduced by Fodor and Roubens (cf. [5], Definition 1.15).

Definition 2. A function $I\colon [0,1]^2 \to [0,1]$ is called a fuzzy implication if it satisfies the following conditions:

$$I \text{ is decreasing in the first variable,} \tag{I1}$$

$$I \text{ is increasing in the second variable,} \tag{I2}$$

$$I(0,0) = 1, \qquad I(1,1) = 1, \qquad I(1,0) = 0. \tag{I3}$$

Definition 3. (cf. Dubois and Prade [4], Fodor and Roubens [5], Gottwald [6]) A fuzzy implication I is said to satisfy

(i) the left neutrality property or is said to be left neutral, if

$$I(1,y) = y, \quad y \in [0,1]; \tag{NP}$$

(ii) the exchange principle, if

$$I(x, I(y,z)) = I(y, I(x,z)), \quad x,\, y,\, z \in [0,1]; \tag{EP}$$

(iii) the identity principle, if

$$I(x,x) = 1, \quad x \in [0,1]; \tag{IP}$$

(iv) the ordering property, if

$$x \le y \Longleftrightarrow I(x,y) = 1, \quad x,\, y \in [0,1]; \tag{OP}$$

(v) the contrapositive symmetry with respect to a fuzzy negation N, CP(N), if

$$I(x,y) = I(N(y), N(x)), \quad x,\, y \in [0,1]. \tag{CP}$$

Definition 4. Let I be a fuzzy implication. The function N_I defined by $N_I(x) = I(x,0)$ for all $x \in [0,1]$, is called the natural negation of I.

It can be easily shown that N_I is a fuzzy negation for every fuzzy implication I.

Definition 5. (Fodor and Roubens [5], Gottwald [6] or Baczyński and Jayaram [1]) A function $I\colon [0,1]^2 \to [0,1]$ is called an (S,N)-implication if there exist a t-conorm S and a fuzzy negation N such that

$$I(x,y) = S(N(x), y), \quad x,\, y \in [0,1].$$

If N is a strong negation, then I is called a strong implication (or S-implication).

The family of all (S,N)-implications will be denoted by $\mathbb{I}_{S,N}$.

The following characterization of (S,N)-implications is from [1], which is an extension of a result in [10] (see also [5], Theorem 1.13).

Theorem 1. (Baczyński and Jayaram [1], Theorem 5.1) For a function $I\colon [0,1]^2 \to [0,1]$ the following statements are equivalent:

(i) I is an (S,N)-implication generated from some t-conorm S and some continuous (strict, strong) fuzzy negation N.

(ii) I satisfies (I2), (EP) and N_I is a continuous (strict, strong) fuzzy negation.

Definition 6. (Dubois and Prade [4], Fodor and Roubens [5], Gottwald [6]) A function $I\colon [0,1]^2 \to [0,1]$ is called a residual implication (shortly R-implication) if there exists a left-continuous t-norm T such that

$$I(x,y) = \max\{t \in [0,1] \mid T(x,t) \le y\}, \quad x, y \in [0,1]. \tag{1}$$

The family of all R-implications will be denoted by $\mathbb{I}_T$.

In general, R-implications can be considered for all t-norms (with supremum in (1)), but this class of implications is related to a residuation concept from the intuitionistic logic and in this context this definition is proper only for left-continuous t-norms (see [6], Proposition 5.4.2 and Corollary. 5.4.1).

Theorem 2. (Fodor and Roubens [5], Theorem 1.14) For a function $I\colon [0,1]^2 \to [0,1]$ the following statements are equivalent:

(i) I is an R-implication based on some left-continuous t-norm T.
(ii) I satisfies (I2), (OP), (EP) and $I(x,\cdot)$ is right-continuous for any $x \in [0,1]$.

Example 2. The basic (S,N)- and R-implications can be found in the literature (see [5, 8] or [6]). Here we present only some examples of (S,N)- and R-implications, which will be used in the next part of this article.

(i) If S is any t-conorm and N is the Gödel negation $N_{\mathbf{G1}}$, then we always obtain the smallest (S,N)-implication

$$I_{\mathbf{G1}}(x,y) = \begin{cases} 1, & \text{if } x = 0, \\ y, & \text{if } x > 0, \end{cases} \qquad x, y \in [0,1].$$

(ii) If S is any t-conorm and N is the dual Gödel negation $N_{\mathbf{G2}}$, then we always obtain the greatest (S,N)-implication

$$I_{\mathbf{G2}}(x,y) = \begin{cases} 1, & \text{if } x < 1, \\ y, & \text{if } x = 1, \end{cases} \qquad x, y \in [0,1].$$

(iii) If S is the algebraic sum t-conorm $S_{\mathbf{P}}(x,y) = x + y - xy$ and N is the classical negation $N_{\mathbf{C}}$, then we obtain the following S-implication called the Reichenbach implication

$$I_{\mathbf{RC}}(x,y) = 1 - x + xy, \quad x, y \in [0,1].$$

(iv) The R-implication generated from the product t-norm $T_{\mathbf{P}}(x,y) = xy$ is the following Goguen implication

$$I_{\mathbf{GG}}(x,y) = \min\left(1, \frac{y}{x}\right) = \begin{cases} 1, & \text{if } x \le y, \\ \frac{y}{x}, & \text{if } x > y, \end{cases} \qquad x, y \in [0,1].$$

3. THE FAMILY OF f–GENERATED IMPLICATIONS

In this section, after giving the definition of this new family of fuzzy implications, we discuss some of their properties. Specifically, we show that the generator from which f-generated implication is obtained, is only unique up to a positive multiplicative constant. We also investigate the natural negations of the above implications.

Proposition 1. (cf. Yager [12], page 197) If $f\colon [0,1] \to [0,\infty]$ is a strictly decreasing and continuous function with $f(1) = 0$, then the function $I\colon [0,1]^2 \to [0,1]$ defined by

$$I(x,y) = f^{-1}(x \cdot f(y)), \quad x,\, y \in [0,1], \tag{2}$$

with the understanding $0 \cdot \infty = 0$, is a fuzzy implication.

P r o o f. Firstly, since for every $x,\, y \in [0,1]$ we have $x \cdot f(y) \le f(y) \le f(0)$ we see that the formula (2) is correctly defined. That I defined by (2) is a fuzzy implication can be easily shown as in [12], page 197. □

Definition 7. (Yager [12]) An f-generator $f\colon [0,1] \to [0,\infty]$ of a fuzzy implication I is a strictly decreasing and continuous function with $f(1) = 0$, such that for all $x,\, y \in [0,1]$ the function I can be represented by (2). In addition, we say that I is an f-generated implication and if I is generated from f, then we will often write I_f instead of I.

Example 3.

(i) If we take the f-generator $f(x) = -\log x$, which is a continuous additive generator of the product t-norm $T_{\mathbf{P}}$, then we obtain the Yager implication

$$I_{\mathbf{YG}}(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0, \\ y^x, & \text{otherwise,} \end{cases} \qquad x,\, y \in [0,1],$$

which is neither an (S,N)-implication nor an R-implication (see [1]).

(ii) If we take the f-generator $f(x) = 1 - x$, which is a continuous additive generator of the Łukasiewicz t-norm $T_{\mathbf{L}}(x,y) = \max(x + y - 1, 0)$, then we obtain the Reichenbach implication $I_{\mathbf{RC}}$, which is an S-implication.

For more examples of f-generated implications see Yager [12].

As can be seen from [7] Theorem 5.1 and as noted in [12] and above, the f-generators can be used as continuous additive generators of continuous Archimedean t-norms. Such generators are unique up to a positive multiplicative constant, and this is also true for the f-generators of the f-generated implications.

Theorem 3. The f-generator of an f-generated implication is uniquely determined up to a positive multiplicative constant, i.e., if f_1 is an f-generator, then f_2 is an f-generator such that $I_{f_1} = I_{f_2}$ if and only if there exists a constant $c \in (0, \infty)$ such that $f_2(x) = c \cdot f_1(x)$ for all $x \in [0, 1]$.

Proof. ($\Longrightarrow$) Let f_1, f_2 be two f-generators of an f-generated implication, i.e., $I_{f_1}(x, y) = I_{f_2}(x, y)$ for all x, $y \in [0, 1]$. Using (2) we get

$$f_1^{-1}(x \cdot f_1(y)) = f_2^{-1}(x \cdot f_2(y)), \quad x, y \in [0, 1].$$

If $f_1(0) = \infty$, then

$$I_{f_1}(x, 0) = f_1^{-1}(x \cdot f_1(0)) = f_1^{-1}(x \cdot \infty) = f_1^{-1}(\infty) = 0, \quad x \in (0, 1].$$

Hence, for all $x \in (0, 1]$, we have

$$0 = I_{f_1}(x, 0) = I_{f_2}(x, 0) = f_2^{-1}(x \cdot f_2(0)),$$

so $f_2(0) = x \cdot f_2(0)$. This implies that $f_2(0) = \infty$ or $f_2(0) = 0$. But $f_2(0) = 0$ is impossible, since f_2 is a strictly decreasing function. By changing the role of f_1 and f_2 we obtain the following equivalence:

$$f_1(0) = \infty \Longleftrightarrow f_2(0) = \infty.$$

Now, we consider the following two cases:

1. If $f_1(0) < \infty$, then $f_2(0) < \infty$ and we obtain, for every x, $y \in [0, 1]$,

$$f_1^{-1}(x \cdot f_1(y)) = f_2^{-1}(x \cdot f_2(y)) \Longleftrightarrow f_2 \circ f_1^{-1}(x \cdot f_1(y)) = x \cdot f_2(y).$$

In particular, for $y = 0$ and any $x \in [0, 1]$, we get

$$f_2 \circ f_1^{-1}(x \cdot f_1(0)) = x \cdot f_2(0) \Longleftrightarrow f_2 \circ f_1^{-1}(x \cdot f_1(0)) = x \cdot f_1(0) \cdot \frac{f_2(0)}{f_1(0)}. \quad (3)$$

Let us fix arbitrarily $x \in [0, 1]$ and consider $z = f_1(x)$. Of course $z \in [0, f_1(0)]$. Hence there exists $x_1 \in [0, 1]$ such that $z = x_1 \cdot f_1(0)$. From (3) we obtain

$$f_2 \circ f_1^{-1}(z) = f_2 \circ f_1^{-1}(x_1 \cdot f_1(0)) = x_1 \cdot f_1(0) \cdot \frac{f_2(0)}{f_1(0)} = z \cdot \frac{f_2(0)}{f_1(0)}.$$

Since f_1 is a bijection, substituting $c = \dfrac{f_2(0)}{f_1(0)} \in (0, \infty)$ we get

$$f_2(x) = f_1(x) \cdot \frac{f_2(0)}{f_1(0)} = c \cdot f_1(x).$$

But x was arbitrarily fixed, so we obtain the claim in this case.

2. If $f_1(0) = \infty$, then $f_2(0) = \infty$. First see that $f_2(0) = c \cdot f_1(0)$ and $f_2(1) = c \cdot f_1(1)$ for every $c \in (0, \infty)$. Now, for every x, $y \in [0, 1]$ we have

$$f_1^{-1}(x \cdot f_1(y)) = f_2^{-1}(x \cdot f_2(y)) \Longleftrightarrow f_2 \circ f_1^{-1}(x \cdot f_1(y)) = x \cdot f_2 \circ f_1^{-1}(f_1(y)).$$

By the substitution $h = f_2 \circ f_1^{-1}$ and $z = f_1(y)$ for $y \in [0, 1]$, we obtain the following equation

$$h(x \cdot z) = x \cdot h(z), \quad x \in [0, 1], z \in [0, \infty], \tag{4}$$

where $h\colon [0, \infty] \to [0, \infty]$ is a continuous strictly increasing bijection. Let us substitute $z = 1$ above, we get

$$h(x) = x \cdot h(1), \quad x \in [0, 1]. \tag{5}$$

Now, fix arbitrarily $x \in (0, 1)$ and consider $z = f_1(x)$. Of course $z \in (0, \infty)$. Hence there exists $x_1 \in (0, 1]$ such that $x_1 \cdot z \in (0, 1)$. From (4) and (5) we get

$$h(z) = \frac{h(x_1 \cdot z)}{x_1} = \frac{x_1 \cdot z \cdot h(1)}{x_1} = z \cdot h(1).$$

Thus, by the definition of h, we have

$$f_2 \circ f_1^{-1}(z) = z \cdot f_2 \circ f_1^{-1}(1).$$

Since f_1 is a bijection, substituting $c = f_2 \circ f_1^{-1}(1) \in (0, \infty)$ we get

$$f_2(x) = f_1(x) \cdot f_2 \circ f_1^{-1}(1) = c \cdot f_1(x).$$

But $x \in (0, 1)$ was arbitrarily fixed, so we have the proof in this direction.

($\Longleftarrow$) Let f_1 be an f-generator and $c \in (0, \infty)$. Define $f_2(x) = c \cdot f_1(x)$ for all $x \in [0, 1]$. Firstly, we note that f_2 is a well defined f-generator. Moreover, $f_2^{-1}(z) = f_1^{-1}\left(\frac{z}{c}\right)$ for every $z \in [0, f_2(0)]$. Now, for every x, $y \in [0, 1]$, we have

$$x \cdot c \cdot f_1(y) \leq c \cdot f_1(y) = f_2(y) \leq f_2(0),$$
$$\frac{x \cdot c \cdot f_1(y)}{c} = x \cdot f_1(y) \leq f_1(y) \leq f_1(0)$$

and thus

$$\begin{aligned} I_{f_2}(x, y) &= f_2^{-1}(x \cdot f_2(y)) = f_2^{-1}(x \cdot c \cdot f_1(y)) = f_1^{-1}\left(\frac{x \cdot c \cdot f_1(y)}{c}\right) \\ &= f_1^{-1}(x \cdot f_1(y)) = I_{f_1}(x, y), \end{aligned}$$

for all x, $y \in [0, 1]$. □

Remark 1. From the above result it follows, that if f is an f-generator such that $f(0) < \infty$, then the function $f_1 \colon [0,1] \to [0,1]$ defined by

$$f_1(x) = \frac{f(x)}{f(0)}, \quad x \in [0,1] \tag{6}$$

is a well defined f-generator such that $I_f = I_{f_1}$ and $f_1(0) = 1$. In other words, it is enough to consider only decreasing generators for which $f(0) = \infty$ or $f(0) = 1$.

Now we investigate the natural negations of I_f.

Proposition 2. Let f be an f-generator of an f-generated implication I_f.

(i) If $f(0) = \infty$, then the natural negation N_{I_f} is the Gödel negation $N_{\mathbf{G1}}$, which is non-continuous.

(ii) The natural negation N_{I_f} is a strict negation if and only if $f(0) < \infty$.

(iii) The natural negation N_{I_f} is a strong negation if and only if $f(0) < \infty$ and f_1 defined by (6) is a strong negation.

Proof. Let f be an f-generator. We get

$$N_{I_f}(x) = I_f(x,0) = f^{-1}(x \cdot f(0)), \quad x \in [0,1].$$

(i) If $f(0) = \infty$, then for every $x \in [0,1]$ we have

$$\begin{aligned} N_{I_f}(x) = f^{-1}(x \cdot \infty) &= \begin{cases} f^{-1}(0), & \text{if } x = 0 \\ f^{-1}(\infty), & \text{if } x > 0 \end{cases} = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x > 0 \end{cases} \\ &= N_{\mathbf{G1}}(x). \end{aligned}$$

(ii) If $f(0) < \infty$, then N_{I_f} is a composition of real continuous functions, so it is continuous. Moreover, if $x_1 < x_2$, then $x_1 \cdot f(0) < x_2 \cdot f(0)$ and by the strictness of f^{-1} we get that N_{I_f} is a strict negation. The converse implication is a consequence of the point (i) of this proposition.

(iii) If $f(0) < \infty$, then because of Remark 1 the function f_1 defined by (6) is a well defined f-generator such that $I_f = I_{f_1}$ and $f_1(0) = 1$. In particular

$$N_{I_f}(x) = N_{I_{f_1}}(x) = f_1^{-1}(x), \quad x \in [0,1].$$

If N_{I_f} is a strong negation, then also f_1^{-1} is a strong negation, so $f_1 = f_1^{-1}$. Conversely, if f_1 is a strong negation, then $f_1^{-1} = f_1$, so N_{I_f} is also a strong negation. □

Theorem 4. (cf. Yager [12], p. 197) If f is an f-generator of an f-generated implication I_f, then

(i) I_f satisfies (NP) and (EP);

(ii) $I_f(x,x) = 1$ if and only if $x = 0$ or $x = 1$, i. e., I_f does not satisfy (IP);

(iii) $I_f(x,y) = 1$ if and only if $x = 0$ or $y = 1$, i. e., I_f does not satisfy (OP);

(iv) I_f satisfies (CP) with some fuzzy negation N if and only if $f(0) < \infty$, f_1 defined by (6) is a strong negation and $N = N_{I_f}$.

(v) I_f is continuous if and only if $f(0) < \infty$;

(vi) I_f is continuous except at the point $(0,0)$ if and only if $f(0) = \infty$.

Proof.

(i) That I_f satisfies (NP) and (EP) was shown by Yager [12], page 197.

(ii) Let $I_f(x,x) = 1$ for some $x \in [0,1]$. This implies that $f^{-1}(x \cdot f(x)) = 1$, thus $x \cdot f(x) = f(1) = 0$, hence $x = 0$ or $f(x) = 0$, which by the strictness of f means $x = 1$. The reverse implication is obvious.

(iii) Proof is similar to that for (ii).

(iv) I_f satisfies (NP) and (EP), so by Corollaries 2.3 and 2.5 from [1] it can satisfy (CP) with some fuzzy negation N if and only if $N = N_{I_f}$ is a strong negation. Therefore, if we assume that I_f satisfies CP(N_{I_f}), then the natural negation N_{I_f} is strong. Because of Proposition 2 (iii) we obtain the thesis in the first direction. Conversely, if $f(0) < \infty$ and f_1 defined by (6) is a strong negation, then again from Proposition 2 (iii) the natural negation N_{I_f} is strong, hence I_f satisfies CP(N_{I_f}).

(v) If $f(0) < \infty$, then I_f given by (2) is the composition of the real continuous functions, so it is continuous. On the other hand, if $f(0) = \infty$, then because of previous proposition, the natural negation is not continuous and therefore I_f is also non-continuous.

(vi) If $f(0) = \infty$, then I_f is continuous for every x, $y \in (0,1]$. Further, for every $y \in [0,1]$ we get $I_f(0,y) = 1$ and for every $x \in (0,1]$ we have $I_f(x,0) = 0$, so I is not continuous in the point $(0,0)$. In addition, for every fixed $y \in (0,1]$ we have $f(y) < \infty$ and

$$\lim_{x \to 0^+} I_f(x,y) = \lim_{x \to 0^+} f^{-1}(x \cdot f(y)) = f^{-1}(0) = 1 = I_f(0,y).$$

Finally, for every $x \in (0,1]$ we have

$$\lim_{y \to 0^+} I_f(x,y) = \lim_{y \to 0^+} f^{-1}(x \cdot f(y)) = f^{-1}(\infty) = 0 = I_f(x,0).$$ □

We would like to point out that by Theorem 4 (v) above, the point D-7 in [12], page 197 is untrue.

4. THE FAMILY OF g-GENERATED IMPLICATIONS

Yager [12] has also proposed another class of implications called the g-generated implications. In a similar way as in the previous section we discuss its properties.

Proposition 3. (Yager [12], page 202) If $g\colon [0,1] \to [0,\infty]$ is a strictly increasing and continuous function with $g(0) = 0$, then the function $I\colon [0,1]^2 \to [0,1]$ defined by

$$I(x,y) = g^{(-1)}\left(\frac{1}{x} \cdot g(y)\right), \quad x,\, y \in [0,1], \tag{7}$$

with the understanding $\frac{1}{0} = \infty$ and $\infty \cdot 0 = \infty$, is a fuzzy implication.

The function $g^{(-1)}$ in (7) is called the pseudo-inverse of g and is given by

$$g^{(-1)}(x) = \begin{cases} g^{-1}(x), & \text{if } x \in [0, g(1)], \\ 1, & \text{if } x \in [g(1), \infty]. \end{cases}$$

Therefore, (7) can be written in the following form

$$I(x,y) = g^{-1}\left(\min\left(\frac{1}{x} \cdot g(y), g(1)\right)\right), \quad x,\, y \in [0,1], \tag{8}$$

without explicitly using the pseudo-inverse.

Definition 8. (Yager [12]) A g-generator $g\colon [0,1] \to [0,\infty]$ of a fuzzy implication I is a strictly increasing and continuous function with $g(0) = 0$, such that for all $x,\, y \in [0,1]$ the function I can be represented by (7) (or, equivalently, by (8)). In addition, we say that I is a g-generated implication and if I is generated from g, then we will often write I_g instead of I.

Example 4.

(i) If we take the g-generator $g(x) = -\log(1-x)$, which is a continuous additive generator of the algebraic sum t-conorm $S_{\mathbf{P}}$, then we obtain the following fuzzy implication

$$I(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0, \\ 1 - (1-y)^{\frac{1}{x}}, & \text{otherwise}, \end{cases} \qquad x,\, y \in [0,1].$$

which is neither an (S,N)-implication nor an R-implication.

(ii) If we take the g-generator $g(x) = x$, which is a continuous additive generator of the Łukasiewicz t-conorm $S_{\mathbf{L}}(x,y) = \min(x+y, 1)$, then we obtain the Goguen implication $I_{\mathbf{GG}}$, which is an R-implication.

For more examples of g-generated implications see Yager [12].

The g-generators can be used as continuous additive generators of continuous Archimedean t-conorms. Such a generator is unique up to a positive multiplicative constant, and this is also true for the g-generators of the g-generated implications.

Theorem 5. The g-generator of a g-generated implication is uniquely determined up to a positive multiplicative constant, i. e., if g_1 is a g-generator, then g_2 is a g-generator such that $I_{g_1} = I_{g_2}$ if and only if there exists a constant $c \in (0, \infty)$ such that $g_2(x) = c \cdot g_1(x)$ for all $x \in [0, 1]$.

Proof. ($\Longrightarrow$) Let g_1, g_2 be two g-generators of a g-generated implication, i. e., assume that $I_{g_1}(x, y) = I_{g_2}(x, y)$ for all $x, y \in [0, 1]$. Using (7) we get

$$g_1^{(-1)}\left(\frac{1}{x} \cdot g_1(y)\right) = g_2^{(-1)}\left(\frac{1}{x} \cdot g_2(y)\right), \quad x, y \in [0, 1].$$

If $g_1(1) = \infty$, then $g_2(1) = \infty$. Indeed, let us assume that $g_2(1) < \infty$ and fix arbitrarily $y_0 \in (0, 1)$. Then there exists $x_0 \in (0, 1)$ such that $\frac{1}{x_0} \cdot g_2(y_0) > g_2(1)$, since $\lim_{x \to 0^+} \frac{1}{x} \cdot g_2(y_0) = \infty$. Hence $g_2^{(-1)}\left(\frac{1}{x_0} \cdot g_2(y_0)\right) = 1$, but $g_1^{(-1)}\left(\frac{1}{x_0} \cdot g_1(y_0)\right) = g_1^{-1}\left(\frac{1}{x_0} \cdot g_1(y_0)\right) < 1$, a contradiction to the assumption that $I_{g_1} = I_{g_2}$. By changing the role of g_1 and g_2 we obtain the following equivalence:

$$g_1(1) = \infty \Longleftrightarrow g_2(1) = \infty.$$

Now, we consider the following two cases:

1. If $g_1(1) = \infty$, then also $g_2(1) = \infty$. Firstly, note that $g_2(0) = c \cdot g_1(0)$ and $g_2(1) = c \cdot g_1(1)$ for every $c \in (0, \infty)$. Now, for every $x, y \in [0, 1]$ we have

$$\begin{aligned} I_{g_1}(x, y) = I_{g_2}(x, y) &\Longleftrightarrow g_1^{-1}\left(\frac{1}{x} \cdot g_1(y)\right) = g_2^{-1}\left(\frac{1}{x} \cdot g_2(y)\right) \\ &\Longleftrightarrow g_2 \circ g_1^{-1}\left(\frac{1}{x} \cdot g_1(y)\right) = \frac{1}{x} \cdot g_2(y) \\ &\Longleftrightarrow g_2 \circ g_1^{-1}\left(\frac{1}{x} \cdot g_1(y)\right) = \frac{1}{x} \cdot g_2 \circ g_1^{-1}(g_1(y)). \end{aligned}$$

By the substitution $h = g_2 \circ g_1^{-1}$ and $z = g_1(y)$ for $y \in [0, 1]$, we obtain the following equation

$$h\left(\frac{1}{x} \cdot z\right) = \frac{1}{x} \cdot h(z), \quad x \in [0, 1], z \in [0, \infty], \tag{9}$$

where $h\colon [0, \infty] \to [0, \infty]$ is a continuous strictly increasing bijection such that $h(0) = 0$ and $h(\infty) = \infty$. Let us substitute $z = 1$ above, we get

$$h\left(\frac{1}{x}\right) = \frac{1}{x} \cdot h(1), \quad x \in [0, 1]. \tag{10}$$

Fix arbitrarily $x \in (0, 1)$ and consider $z = g_1(x)$. Of course $z \in (0, \infty)$. Hence there exists $x_1 \in (0, 1)$ such that $x_1 \cdot \frac{1}{z} \in (0, 1)$. From (9) and (10) we get

$$h(z) = x_1 \cdot h\left(\frac{1}{x_1} \cdot z\right) = x_1 \cdot h\left(\frac{1}{\frac{x_1}{z}}\right) = x_1 \cdot \frac{1}{\frac{x_1}{z}} \cdot h(1) = z \cdot h(1).$$

Now, by the definition of h, we have

$$g_2 \circ g_1^{-1}(z) = z \cdot g_2 \circ g_1^{-1}(1),$$

thus

$$g_2(x) = g_1(x) \cdot g_2 \circ g_1^{-1}(1).$$

But $x \in (0,1)$ was arbitrarily fixed, so letting $c = g_2 \circ g_1^{-1}(1)$ we obtain the result in this case.

2. In the case $g_1(1) < \infty$ we also have $g_2(1) < \infty$. Now, for every x, $y \in [0,1]$ we have

$$\begin{aligned} & I_{g_1}(x,y) = I_{g_2}(x,y) \\ & \quad \Longleftrightarrow g_1^{-1}\left(\min\left(\frac{1}{x} \cdot g_1(y), g_1(1)\right)\right) = g_2^{-1}\left(\min\left(\frac{1}{x} \cdot g_2(y), g_2(1)\right)\right) \\ & \quad \Longleftrightarrow g_2 \circ g_1^{-1}\left(\min\left(\frac{1}{x} \cdot g_1(y), g_1(1)\right)\right) = \min\left(\frac{1}{x} \cdot g_2(y), g_2(1)\right). \end{aligned}$$

By the substitution $h = g_2 \circ g_1^{-1}$, $u = \dfrac{1}{x}$ and $v = g_1(y)$ for x, $y \in [0,1]$, we obtain the following equation

$$h\left(\min\left(u \cdot v, g_1(1)\right)\right) = \min\left(u \cdot h(v), g_2(1)\right), \quad u \in [1,\infty], v \in [0, g_1(1)],$$

where the function $h\colon [0, g_1(1)] \to [0, g_2(1)]$ is a continuous and strictly increasing function such that $h(0) = 0$ and $h(g_1(1)) = g_2(1)$. Let us fix any $x \in (0,1)$. Then $x \cdot v < g_1(1)$ for all $v \in (0, g_1(1))$. Since h is strictly increasing $h(x \cdot v) < g_2(1)$ and $h(v) < g_2(1)$ for all $v \in (0, g_1(1))$. Therefore

$$\begin{aligned} h(v) &= h\left(\frac{1}{x} \cdot x \cdot v\right) = h\left(\min\left(\frac{1}{x} \cdot x \cdot v, g_1(1)\right)\right) \\ &= \min\left(\frac{1}{x} \cdot h(x \cdot v), g_2(1)\right) = \frac{1}{x} \cdot h(x \cdot v), \end{aligned}$$

for every $v \in (0, g_1(1))$. Hence, from the continuity of h, we have

$$\begin{aligned} g_2(1) &= h(g_1(1)) = \lim_{v \to g_1(1)^-} h(v) = \lim_{v \to g_1(1)^-} \frac{1}{x} \cdot h(x \cdot v) \\ &= \frac{1}{x} \cdot h\left(x \cdot \lim_{v \to g_1(1)^-} v\right) = \frac{1}{x} \cdot h(x \cdot g_1(1)). \end{aligned}$$

Since $x \in (0,1)$ was arbitrarily fixed, we get

$$h(x \cdot g_1(1)) = x \cdot g_2(1), \quad x \in (0,1). \tag{11}$$

Now, for any fixed $v \in (0, g_1(1))$ there exists $x_1 \in (0,1)$ such that $v = x_1 \cdot g_1(1)$ and the previous equality implies

$$h(v) = h(x_1 \cdot g_1(1)) = x_1 \cdot g_2(1) = x_1 \cdot g_1(1) \cdot \frac{g_2(1)}{g_1(1)} = v \cdot \frac{g_2(1)}{g_1(1)},$$

for all $v \in (0, g_1(1))$. Note that this formula is also correct for $v = 0$ and $v = g_1(1)$. Therefore, by the definition of h we get

$$g_2 \circ g_1^{-1}(v) = v \cdot \frac{g_2(1)}{g_1(1)}, \quad v \in [0, g_1(1)],$$

thus

$$g_2(y) = g_1(y) \cdot \frac{g_2(1)}{g_1(1)}, \quad y \in [0, 1].$$

Putting $c = \dfrac{g_2(1)}{g_1(1)}$ we obtain the result.

($\Longleftarrow$) Let g_1 be a g-generator and $c \in (0, \infty)$. Define $g_2(x) = c \cdot g_1(x)$ for all $x \in [0, 1]$. Evidently g_2 is a well defined g-generator. Moreover, for any $z \in [0, \infty]$,

$$g_2^{(-1)}(z) = \begin{cases} g_1^{-1}\left(\dfrac{z}{c}\right), & \text{if } z \in [0, c \cdot g_1(1)], \\ 1, & \text{if } z \in [c \cdot g_1(1), \infty]. \end{cases}$$

This implies, that for every x, $y \in [0, 1]$ we get

$$\begin{aligned} I_{g_2}(x, y) &= g_2^{-1}\left(\min\left(\frac{1}{x} \cdot g_2(y), g_2(1)\right)\right) \\ &= g_1^{-1}\left(\frac{1}{c} \min\left(\frac{1}{x} \cdot c \cdot g_1(y), c \cdot g_1(1)\right)\right) \\ &= g_1^{-1}\left(\min\left(\frac{1}{x} \cdot g_1(y), g_1(1)\right)\right) = I_{g_1}(x, y). \end{aligned}$$ □

Remark 2. From the above result it follows, that if g is a g-generator such that $g(1) < \infty$, then the function $g_1 \colon [0, 1] \to [0, 1]$ defined by

$$g_1(x) = \frac{g(x)}{g(1)}, \quad x \in [0, 1] \tag{12}$$

is a well defined g-generator such that $I_g = I_{g_1}$ and $g_1(1) = 1$. In other words, it is enough to consider only decreasing generators for which $g(1) = \infty$ or $g(1) = 1$.

Proposition 4. Let g be a g-generator. The natural negation of I_g is the Gödel negation $N_{\mathbf{G1}}$, which is not continuous.

Proof. Let g be a g-generator For every $x \in [0, 1]$ we get

$$\begin{aligned} N_{I_g}(x) &= I_g(x, 0) = g^{(-1)}\left(\frac{1}{x} \cdot g(0)\right) = g^{(-1)}\left(\frac{1}{x} \cdot 0\right) \\ &= \begin{cases} g^{(-1)}(\infty), & \text{if } x = 0 \\ g^{(-1)}(0), & \text{if } x > 0 \end{cases} = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x > 0 \end{cases} = N_{\mathbf{G1}}(x). \end{aligned}$$ □

Theorem 6. (cf. Yager [12], page 201) If g is a g-generator of a g-generated implication I_g, then

(i) I_g satisfies (NP) and (EP);

(ii) I_g satisfies (IP) if and only if $g(1) < \infty$ and $x \leq g_1(x)$ for every $x \in [0,1]$, where g_1 is defined by (12);

(iii) if $g(1) = \infty$, then $I_g(x,y) = 1$ if and only if $x = 0$ or $y = 1$, i. e., I_g does not satisfy (OP) when $g(1) = \infty$;

(iv) I_g does not satisfy the contrapositive symmetry (CP) with any fuzzy negation;

(v) I_g is continuous except at the point $(0,0)$.

Proof.

(i) That I_g defined by (7) satisfies (NP) and (EP) was shown by Yager [12], page 201.

(ii) Let us assume firstly that $g(1) = \infty$. This implies that $g^{(-1)} = g^{-1}$. Let $I_g(x,x) = 1$ for some $x \in [0,1]$. This implies that $g^{-1}\left(\frac{1}{x} \cdot g(x)\right) = 1$, thus $\frac{1}{x} \cdot g(x) = g(1) = \infty$, hence $x = 0$ or $g(x) = \infty$, which by the strictness of g means $x = 1$. Therefore I_g does not satisfy (IP) when $g(1) = \infty$. Let us assume now, that I_g satisfies the identity property (IP). Therefore it should be $g(1) < \infty$. By Theorem 5 the function g_1 defined by (12) is a well defined g-generator such that $I_g = I_{g_1}$ and $g_1(1) = 1$. Now (IP) implies, that for every $x \in (0,1]$ we get

$$\begin{aligned} I_g(x,x) = 1 &\iff I_{g_1}(x,x) = 1 \iff g_1^{(-1)}\left(\frac{1}{x} \cdot g_1(x)\right) = 1 \\ &\iff g_1^{-1}\left(\min\left(\frac{1}{x} \cdot g_1(x), g_1(1)\right)\right) = 1 \\ &\iff \frac{1}{x} \cdot g_1(x) \geq g_1(1) \iff \frac{1}{x} \cdot g_1(x) \geq 1 \\ &\iff x \leq g_1(x). \end{aligned}$$

The converse implication is a direct consequence of the above equivalences.

(iii) Let us assume that $g(1) = \infty$. This implies that $g^{(-1)} = g^{-1}$. Let $I_g(x,y) = 1$ for some x, $y \in [0,1]$. This implies that $g^{-1}(\frac{1}{x} \cdot g(y)) = 1$, thus $\frac{1}{x} \cdot g(y) = g(1) = \infty$, hence $x = 0$ or $g(y) = \infty$, which by the strictness of g means $y = 1$. The reverse implication is obvious.

(iv) By the point (i) above the g-generated implication I_g satisfies (NP) and (EP), so again it can satisfy the contrapositive symmetry only with N_{I_g} which should be a strong negation. But from Proposition 4 we see that the natural negation N_{I_g} is not strong.

(v) By the formula (8), the implication I_g is continuous for every $x, y \in (0,1]$. Further, for every $y \in [0,1]$ we get $I_g(0,y) = 1$ and for every $x \in (0,1]$ we have $I_g(x,0) = 0$, so I_g is not continuous in the point $(0,0)$. In addition, for every fixed $y \in (0,1]$ we have $g(y) > 0$ and consequently

$$\lim_{x\to 0^+} I_g(x,y) = \lim_{x\to 0^+} g^{-1}\left(\min\left(\frac{1}{x}\cdot g(y), g(1)\right)\right) = g^{-1}(g(1)) = 1 = I_g(0,y).$$

Finally, for every $x \in (0,1]$ we have $\frac{1}{x} < \infty$, thus

$$\lim_{y\to 0^+} I_f(x,y) = \lim_{y\to 0^+} g^{-1}\left(\min\left(\frac{1}{x}\cdot g(y), g(1)\right)\right) = g^{-1}(0) = 0 = I_g(x,0). \qquad \square$$

We would like to point out that by Theorem 6 (v) above, the point D-7 in [12], page 202 is untrue.

In the last theorem in this section, we will show that I_g satisfies the ordering property only for a rather special class of g-generators.

Theorem 7. If g is a g-generator, then the following statements are equivalent:

(i) I_g satisfies (OP).

(ii) $g(1) < \infty$ and there exists a constant $c \in (0,\infty)$ such that $g(x) = c\cdot x$ for all $x \in [0,1]$.

(iii) I_g is the Goguen implication $I_{\mathbf{GG}}$.

Proof. (i) $\Longrightarrow$ (ii) Let us assume that I_g satisfies the ordering property (OP). From Theorem 6 (iii) we have $g(1) < \infty$. By Remark 2 the function g_1 defined by (12) is a well defined g-generator such that $I_g = I_{g_1}$ and $g_1(1) = 1$. Now (OP) implies, that for every $x, y \in (0,1]$ we get

$$\begin{aligned} x \le y &\Longleftrightarrow I_g(x,y) = 1 \Longleftrightarrow I_{g_1}(x,y) = 1 \Longleftrightarrow g_1^{(-1)}\left(\frac{1}{x}\cdot g_1(y)\right) = 1 \\ &\Longleftrightarrow g_1^{-1}\left(\min\left(\frac{1}{x}\cdot g_1(y), g_1(1)\right)\right) = 1 \Longleftrightarrow \frac{1}{x}\cdot g_1(y) \ge g_1(1) \\ &\Longleftrightarrow \frac{1}{x}\cdot g_1(y) \ge 1 \\ &\Longleftrightarrow x \le g_1(y). \end{aligned} \tag{13}$$

This equivalence can be also written in the following form

$$x > y \Longleftrightarrow x > g_1(y), \quad x, y \in (0,1]. \tag{14}$$

We show that $g_1(x) = x$ for all $x \in (0,1]$. Suppose that this does not hold, i.e., there exists $x_0 \in (0,1)$ such that $g_1(x_0) \neq x_0$. If $x_0 < g_1(x_0)$, then by the continuity and strict monotonicity of the generator g_1 there exists $y_0 \in (0,1)$ such that

$$x_0 < g_1(y_0) < g_1(x_0). \tag{15}$$

Because of (13) we get $x_0 \leq y_0$. Since g_1 is strictly increasing $g_1(x_0) \leq g_1(y_0)$, a contradiction to (15).
If $0 < g_1(x_0) < x_0$, then by the continuity and strict monotonicity of the generator g_1 there exists $y_0 \in (0,1)$ such that

$$g_1(x_0) < g_1(y_0) < x_0. \tag{16}$$

Because of (14) we get $y_0 < x_0$. Since g_1 is strictly increasing $g_1(y_0) < g_1(x_0)$, a contradiction to (16).

We showed, that $g_1(x) = x$ for all $x \in (0,1)$, but also $g_1(0) = 0$ and $g_1(1) = 1$. By virtue of (12) we get that $g(x) = g(1) \cdot x$.

(ii) $\Longrightarrow$ (iii) If $g(1) < \infty$ and $g(x) = c \cdot x$ for all $x \in [0,1]$, with some $c \in (0,\infty)$, then $g(1) = c$ and g-generator given by (12) is equal to $g_1(x) = x$. From Example 4 (ii) we conclude, that $I_{g_1} = I_g$ is the Goguen implication.

(ii) $\Longrightarrow$ (iii) If I_g is the Goguen implication, then it is a well known result that it satisfies the ordering property (OP). □

5. THE FAMILY OF h-GENERATED IMPLICATIONS

As noted earlier the f- and g-generators can be seen as the continuous additive generators of t-norms and t-conorms, respectively. Taking cue from this, a new family of fuzzy implications called the h-generated implications has been proposed by Balasubramaniam [2], where h can be seen as a multiplicative generator of a continuous Archimedean t-conorm. In this section we give its definitions and discuss a few of its properties.

Proposition 5. (Balasubramaniam [2]) If $h\colon [0,1] \to [0,1]$ is a strictly decreasing and continuous function with $h(0) = 1$, then the function $I\colon [0,1]^2 \to [0,1]$ defined by

$$I(x,y) = h^{(-1)}\left(x \cdot h(y)\right), \quad x, y \in [0,1], \tag{17}$$

is a fuzzy implication.

The function $h^{(-1)}\colon [0,1] \to [0,1]$ in the above formula is again the pseudo-inverse of h and is given by

$$h^{(-1)}(x) = \begin{cases} h^{-1}(x), & \text{if } x \in [h(1),1], \\ 1, & \text{if } x \in [0,h(1)]. \end{cases}$$

Therefore (17) can be written in the following form

$$I(x,y) = h^{-1}\left(\max\left(x \cdot h(y), h(1)\right)\right), \quad x, y \in [0,1], \tag{18}$$

without explicitly using the pseudo-inverse.

Definition 9. (Balasubramaniam [2]) An h-generator $h\colon [0,1] \to [0,1]$ of a fuzzy implication I is a strictly decreasing and continuous function with $h(0) = 1$, such that for all $x, y \in [0,1]$ the function I can be represented by (17) (or, equivalently, by (18)). In addition, we say that I is an h-generated implication and if I is generated from h, then we will often write I_h instead of I.

Example 5.

(i) If we take $h(x) = 1 - x$, which is a continuous multiplicative generator of the algebraic sum t-conorm $S_{\mathbf{P}}$, then we obtain the Reichenbach implication $I_{\mathbf{RC}}$, which is an S-implication.

(ii) If we consider the family of h-generators $h_n(x) = 1 - \dfrac{x^n}{n}$, $n \in \mathbb{N}$, then we obtain the following fuzzy implications

$$I_n(x,y) = \min\left((n - n\cdot x + x\cdot y^n)^{\frac{1}{n}}, 1\right), \quad x,\, y \in [0,1],$$

which are (S,N)-implications.

For more examples of h-generated implications see Balasubramaniam [2] or [3].

Firstly we prove the following result.

Theorem 8. The h-generator of an h-generated implication is uniquely determined, i.e., h_1, h_2 are h-generators such that $I_{h_1} = I_{h_2}$ if and only if $h_1 = h_2$.

Proof. Let h_1, h_2 be two h-generators of h-generated implication, i.e., $I_{h_1}(x,y) = I_{h_2}(x,\, y)$ for all $x,\, y \in [0,1]$. Using (18) we have, for all $x,\, y \in [0,1]$

$$\begin{aligned} I_{h_1}(x,y) &= I_{h_2}(x,y) \\ &\iff {h_1}^{-1}\left(\max\left(x\cdot h_1(y), h_1(1)\right)\right) = {h_2}^{-1}\left(\max\left(x\cdot h_2(y), h_2(1)\right)\right) \\ &\iff h_2 \circ {h_1}^{-1}\left(\max\left(x\cdot h_1(y), h_1(1)\right)\right) = \max\left(x\cdot h_2(y), h_2(1)\right). \qquad (19) \end{aligned}$$

Now letting $g = h_2 \circ {h_1}^{-1}$, $h_2(y) = u$ and $h_1(y) = v$ we get $h_2(y) = h_2 \circ {h_1}^{-1} \circ h_1(y) = g \circ h_1(y) = g(v)$. Also $g : [h_1(1), 1] \to [h_2(1), 1]$ is a continuous and strictly increasing function such that $g(h_1(1)) = h_2(1)$ and $g(1) = 1$. Substituting the above in (19) we obtain

$$g(\max\left(x\cdot v, h_1(1)\right)) = \max\left(x\cdot g(v), g(h_1(1))\right), \quad x \in [0,1], v \in [h_1(1), 1].$$

Let us take any $x \in (h_1(1), 1]$ and put $v = 1$ above. We get

$$g(x) = g(x\cdot 1) = g(\max\left(x\cdot 1, h_1(1)\right)) = \max\left(x\cdot g(1), h_2(1)\right) = \max\left(x, h_2(1)\right).$$

Since the function g is strictly increasing we get $g(x) = x$ for all $x \in (h_1(1), 1]$. From the continuity this is also true for $x = h_1(1)$. Substituting for g we get $h_2 \circ {h_1}^{-1}(v) = v$ for all $v \in [h_1(1), 1]$ or that $h_1(x) = h_2(x)$ for all $x \in [0,1]$.

The reverse implication is obvious. □

Proposition 6. Let h be an h-generator of I_h.

(i) The natural negation N_{I_h} is a continuous fuzzy negation.

(ii) The natural negation N_{I_h} is a strict negation if and only if $h(1) = 0$.

(iii) The natural negation N_{I_h} is strong negation if and only if $h = h^{-1}$.

Proof. Since for every $x \in [0, 1]$ we get

$$N_{I_h}(x) = I_h(x, 0) = h^{(-1)}(x \cdot h(0)) = h^{(-1)}(x) = h^{-1}\left(\max\left(x, h(1)\right)\right),$$

it is obvious, that N_{I_h} is a continuous fuzzy negation. The other points are the consequence of the definitions of strict (strong) negations and h-generators. □

Theorem 9. If h is an h-generator of an h-generated implication I_h, then

(i) I_h satisfies (NP) and (EP);

(ii) I_h satisfies (IP) if and only if $h(1) > 0$ and $x \cdot h(x) \leq h(1)$ for every $x \in [0, 1]$;

(iii) I_h does not satisfy (OP);

(iv) I_h satisfies (CP) with some fuzzy negation N if and only if $h = h^{-1}$ and $N = N_{I_h}$;

(v) I_h is continuous.

Proof.

(i) For every h-generator h and $y \in [0, 1]$ we have

$$I_h(1, y) = h^{(-1)}\left(1 \cdot h(y)\right) = y$$

and for all x, $y, z \in [0, 1]$ we get

$$\begin{aligned} I_h(x, I_h(y, z)) &= h^{(-1)}\left(x \cdot h(I_h(y, z))\right) \\ &= h^{-1}\left(\max\left(x \cdot h(h^{-1}\left(\max\left(y \cdot h(z), h(1)\right)\right)), h(1)\right)\right) \\ &= h^{-1}\left(\max\left(x \cdot \max\left(y \cdot h(z), h(1)\right), h(1)\right)\right) \\ &= h^{-1}\left(\max\left(x \cdot y \cdot h(z), x \cdot h(1), h(1)\right)\right) \\ &= h^{-1}\left(\max\left(x \cdot y \cdot h(z), h(1)\right)\right), \end{aligned}$$

since $x \cdot h(1) \leq h(1)$. Similarly we get that

$$I_h(y, I_h(x, z)) = h^{-1}\left(\max\left(y \cdot x \cdot h(z), h(1)\right)\right).$$

Thus I_h satisfies the neutral property and the exchange principle.

(ii) Firstly, note that if $h(1) = 0$, then it can be seen as the f-generator, and by virtue of Theorem 4 (ii) it does not satisfy (IP). Let us assume that I_h satisfies the identity property (IP). Therefore it should be $h(1) > 0$. Now, for every $x \in [0,1]$ we get

$$I_h(x,x) = 1 \Longleftrightarrow h^{(-1)}(x \cdot h(x)) = 1 \Longleftrightarrow h^{-1}(\max(x \cdot h(x), h(1))) = 1 \\ \Longleftrightarrow x \cdot h(x) \le h(1).$$

The converse implication is a direct consequence of the above equivalences.

(iii) If $h(1) = 0$, then h can be seen as the f-generator, and because of Theorem 4 (iii) it does not satisfy (OP). Let us assume that there exists an h-generator h_0 with $h_0(1) > 0$, such that I_{h_0} satisfies the ordering property. Thus

$$x \le y \Longleftrightarrow I_{h_0}(x,y) = 1 \Longleftrightarrow h_0^{(-1)}(x \cdot h_0(y)) = 1 \\ \Longleftrightarrow h_0^{-1}(\max(x \cdot h_0(y), h_0(1))) = 1 \Longleftrightarrow x \cdot h_0(y) \le h_0(1),$$

but there exist $x_0, y_0 \in (0,1)$ such that $0 < y_0 < x_0 < h(1)$ and $x_0 \cdot h_0(y_0) < x_0 < h_0(1)$, i.e., $I_{h_0}(x_0, y_0) = 1$, a contradiction to the assumed ordering property.

(iv) This is a direct consequence of Proposition 6 and Corollaries 2.3, 2.5 from [1].

(v) For every h-generator the function given by (18) is a composition of continuous functions, so it is a continuous function. □

Example 6. As an example of h-generated implication which satisfies (IP) consider the h-generator $h(x) = 1 - \frac{x}{2}$. By easy calculations we get

$$I_h(x,y) = \min(2 - 2x + xy, 1), \quad x, y \in [0,1].$$

Firstly see that $x \cdot h(x) \le h(1)$ for every $x \in [0,1]$. Therefore, by above theorem, I_h satisfies (IP). Indeed, $I_h(x,x) = \min\left(2 - 2x + x^2, 1\right) = 1$ for all $x \in [0,1]$.

6. THE INTERSECTIONS BETWEEN $\mathbb{I}_F, \mathbb{I}_G$ AND $\mathbb{I}_H$

Let us denote the following families of fuzzy implication:

- $\mathbb{I}_{F,\infty}$ – f-generated implications such that $f(0) = \infty$,
- $\mathbb{I}_{F,\aleph}$ – f-generated implications such that $f(0) < \infty$,
- $\mathbb{I}_F = \mathbb{I}_{F,\infty} \cup \mathbb{I}_{F,\aleph}$,
- $\mathbb{I}_{G,\infty}$ – g-generated implications such that $g(1) = \infty$,
- $\mathbb{I}_{G,\aleph}$ – g-generated implications such that $g(1) < \infty$,

- $\mathbb{I}_G = \mathbb{I}_{G,\infty} \cup \mathbb{I}_{G,\aleph}$,
- $\mathbb{I}_{H,O}$ – h-generated implications I_h obtained from h such that $h(1) = 0$,
- $\mathbb{I}_{H,E}$ – h-generated implications I_h obtained from h such that $h(1) > 0$,
- $\mathbb{I}_H = \mathbb{I}_{H,O} \cup \mathbb{I}_{H,E}$.

Proposition 7. The following equalities are true:

$$\mathbb{I}_{F,\aleph} \cap \mathbb{I}_G = \emptyset, \tag{20}$$
$$\mathbb{I}_F \cap \mathbb{I}_{G,\aleph} = \emptyset, \tag{21}$$
$$\mathbb{I}_{F,\infty} = \mathbb{I}_{G,\infty}, \tag{22}$$
$$\mathbb{I}_{F,\aleph} = \mathbb{I}_{H,O}, \tag{23}$$
$$\mathbb{I}_{F,\infty} \cap \mathbb{I}_{H,O} = \emptyset, \tag{24}$$
$$\mathbb{I}_F \cap \mathbb{I}_{H,E} = \emptyset, \tag{25}$$
$$\mathbb{I}_G \cap \mathbb{I}_H = \emptyset. \tag{26}$$

Proof.

(i) The equation (20) is the consequence of Proposition 2 (ii) and Proposition 4.

(ii) Let $I \in \mathbb{I}_F$. Because of Theorem 4 (ii) we know that $I(x, x) = 1$ if and only if $x = 0$ or $x = 1$. On the other side, if we assume that $I \in \mathbb{I}_{G,\aleph}$, then there exists a g-generator such that I has the form (7) and $g(1) < \infty$. Thus, for every $x \in (0, 1)$ we get

$$I\left(\frac{g(x)}{g(1)}, x\right) = g^{(-1)}\left(\frac{g(1)}{g(x)} \cdot g(x)\right) = g^{(-1)}(g(1)) = 1.$$

Therefore we obtain (21).

(iii) Let us assume that $I \in \mathbb{I}_{F,\infty}$, i. e., there exists an f-generator f with $f(0) = \infty$ such that I has the form (2). Let us define the function $g\colon [0, 1] \to [0, \infty]$ by

$$g(x) = \frac{1}{f(x)}, \quad x \in [0, 1],$$

with the assumptions, that $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$. We see that g is a g-generator with $g(1) = \infty$. Moreover $g^{(-1)}(x) = g^{-1}(x) = f^{-1}(\frac{1}{x})$. Hence, for every x, $y \in [0, 1]$ we have

$$I_g(x, y) = g^{-1}\left(\frac{1}{x} \cdot g(y)\right) = g^{-1}\left(\frac{1}{x} \cdot \frac{1}{f(y)}\right) = f^{-1}(x \cdot f(y)) = I(x, y).$$

Conversely, if $I \in \mathbb{I}_{G,\infty}$, then there exists a g-generator g with $g(1) = \infty$ such that I has the form (7). Defining the function $f\colon [0,1] \to [0,\infty]$ by

$$f(x) = \frac{1}{g(x)}, \quad x \in [0,1],$$

we get that f is an f-generator such that $I_f = I$.

(iv) Let us assume that $I \in \mathbb{I}_{F,\aleph}$, i. e., there exists an f-generator f with $f(0) < \infty$ such that I has the form (2). Because of Remark 1 the function f_1 defined by (6) is an f generated implication such that $I_f = I_{f_1}$ and $f_1(0) = 1$. Therefore $h = f_1$ can be seen as a h-generator, with $h(1) = 0$, such that $I_f = I_h$.

Conversely, if $I \in \mathbb{I}_{H,O}$, i. e., there exists an h-generator h with $h(1) = 0$ such that I has the form (17), then $h^{(-1)} = h^{-1}$ and it can be seen as an f-generator with $f(0) < \infty$.

(v) The equation (24) is a consequence of (23).

(vi) If $I \in \mathbb{I}_F$, then by Theorem 4 (ii) we get $I(x,x) = 1$ if and only if $x = 0$ or $x = 1$. On the other hand, if $I \in \mathbb{I}_{H,E}$, then there exists an h-generator h such that $h(1) > 0$. Therefore, by Theorem 9 (iii)

$$I(x,x) = 1 \Longleftrightarrow x \cdot h(x) \le h(1),$$

which is always true for all $x \in [0, h(1)]$. Therefore we get (25).

(vii) From (22), (24) and (25) we see that $\mathbb{I}_H \cap \mathbb{I}_{G,\infty} = \emptyset$. Similarly, from (20), (21) and (23) we see that $\mathbb{I}_H \cap \mathbb{I}_{G,\aleph} = \emptyset$. □

7. THE INTERSECTIONS OF $\mathbb{I}_F, \mathbb{I}_G, \mathbb{I}_H$ WITH $\mathbb{I}_{S,N}$

In this section we investigate whether any of the families $\mathbb{I}_F, \mathbb{I}_G, \mathbb{I}_H$ intersect with $\mathbb{I}_{S,N}$. In other words, if and when any of the members of the above families can be written as an (S,N)-implication of an appropriate t-conorm – fuzzy negation pair. It is obvious from Proposition 4.1 in [1] that for a fuzzy implication I to be an (S,N)-implication it should satisfy

- the left neutrality property (NP), and
- the exchange principle (EP).

We already know that the families $\mathbb{I}_F, \mathbb{I}_G, \mathbb{I}_H$ all have (NP) and (EP). Hence, because of the characterizations of some subclasses of (S,N)-implications presented in Section 2, we need to check their natural negations. These was done in previous sections and we have the following results.

Theorem 10. If f is an f-generator, then the following statements are equivalent:

(i) I_f is an (S, N)-implication.

(ii) $f(0) < \infty$.

Proof. (i) $\Longrightarrow$ (ii) Let f be an f-generator such that $f(0) = \infty$ and assume that I_f is an (S, N)-implication generated from a t-conorm S and a fuzzy negation N. From Proposition 4.1 in [1] we get that $N_{I_f} = N$, but Proposition 2 (i) gives, that N_{I_f} is the Gödel negation $N_{\mathbf{G1}}$. Hence, from Example 2 (i), it follows that $I_f = I_{\mathbf{G1}}$. Thus $f^{-1}(x \cdot f(y)) = y$ for all $x \in (0, 1], y \in [0, 1]$, which implies $x \cdot f(y) = f(y)$ for all $x \in (0, 1], y \in [0, 1]$, a contradiction.

(ii) $\Longrightarrow$ (i) Let f be an f-generator such that $f(0) < \infty$. From Proposition 2 (ii) the natural negation N_{I_f} is a strict negation. Theorem 1 implies that I_f is an (S, N)-implication generated from some t-conorm and some strict negation. □

It is important, that in this case we can fully describe t-conorms and strict negations from which the f-generated implications are obtained. Let us denote by Φ the family of all increasing bijections $\varphi\colon [0, 1] \to [0, 1]$. We say that functions $F, G\colon [0, 1]^2 \to [0, 1]$ are Φ-conjugate, if there exists a $\varphi \in \Phi$ such that $G = F_\varphi$, where $F_\varphi(x, y) := \varphi^{-1}(F(\varphi(x), \varphi(y)))$, for all $x, y \in [0, 1]$.

Corollary 1. If $f(0) < \infty$, then the function $S\colon [0, 1]^2 \to [0, 1]$ defined by

$$S(x, y) = I_f(N_{I_f}^{-1}(x), y), \quad x,\, y \in [0, 1]$$

is a strict t-conorm, i. e., it is Φ-conjugate with the algebraic sum t-conorm $S_{\mathbf{P}}$.

Proof. Let us assume that f is a decreasing generator such that $f(0) < \infty$. Then the function f_1 defined by the formula (6) is a strict negation. We know also that $I_f = I_{f_1}$, so we get

$$\begin{aligned} S(x, y) &= I_f(N_{I_f}^{-1}(x), y) = I_{f_1}(N_{I_{f_1}}^{-1}(x), y) = I_{f_1}((I_{f_1}(x, 0))^{-1}, y) \\ &= I_{f_1}((f_1^{-1})^{-1}(x), y) = I_{f_1}(f_1(x), y) \\ &= f_1^{-1}(f_1(x) \cdot f_1(y)), \end{aligned}$$

for all $x,\, y \in [0, 1]$. Let us define the function $\varphi\colon [0, 1] \to [0, 1]$ by $\varphi(x) = 1 - f_1(x)$ for all $x \in [0, 1]$. Evidently, φ is an increasing bijection. Moreover $f_1^{-1}(x) = \varphi^{-1}(1 - x)$ for all $x \in [0, 1]$. This implies that

$$\begin{aligned} S(x, y) &= f_1^{-1}(f_1(x) \cdot f_1(y)) = f_1^{-1}((1 - \varphi(x)) \cdot (1 - \varphi(y))) \\ &= \varphi^{-1}(\varphi(x) + \varphi(y) - \varphi(x) \cdot \varphi(y)) \end{aligned}$$

for all $x,\, y \in [0, 1]$, i. e., S is Φ-conjugate with the algebraic sum t-conorm $S_{\mathbf{P}}$. Therefore, by virtue of Theorem 1.9 in [5], S is a strict t-conorm. □

This means, that for $f(0) < \infty$ we have $I_f(x,y) = S(N_{I_f}(x), y)$ for x, $y \in [0,1]$, where S is Φ-conjugate with the algebraic sum t-conorm $S_{\mathbf{P}}$. But

$$N_{I_f}(x) = N_{I_{f_1}}(x) = f_1^{-1}(x) = \varphi^{-1}(1-x), \quad x \in [0,1].$$

Hence, if $f(0) < \infty$, then we do not obtain any new implication but only (S,N)-implication generated from Φ-conjugate algebraic sum t-conorm for

$$\varphi(x) = 1 - \frac{f(x)}{f(0)}, \quad x \in [0,1],$$

and the strict negation $N(x) = \varphi^{-1}(1-x)$ for all $x \in [0,1]$.

Now, under the following restricted situations we can obtain S-implications.

Theorem 11. Let f be an f-generator. The function I_f is an S-implication if and only if $f(0) < \infty$ and the function f_1 defined by (6) is a strong negation.

Theorem 12. If g is a g-generator, then I_g is not an (S,N)-implication.

Proof. Assume, that there exists a g-generator g such that I_g is an (S,N)-implication generated from a t-conorm S and a fuzzy negation N. We get that $N_{I_g} = N$, but Proposition 4 gives, that N_{I_g} is the Gödel negation $N_{\mathbf{G1}}$. Hence, from Example 2 (i), it follows that $I_g = I_{\mathbf{G1}}$. Thus, for all $x \in (0,1], y \in [0,1]$,

$$g^{(-1)}\left(\frac{1}{x} \cdot g(y)\right) = y,$$

which implies

$$g^{-1}\left(\min\left(\frac{1}{x} \cdot g(y), g(1)\right)\right) = y.$$

Let us take any x, $y \in (0,1)$, we get $\frac{1}{x} \cdot g(y) = g(y)$, a contradiction. □

Theorem 13. If h is an h-generator, then I_h is an (S,N)-implication generated from some t-conorm S and continuous negation N.

Proof. Let h be an h-generator. By Theorem 6 the natural negation N_{I_h} is a continuous negation. By virtue of Theorem 1 we get that I_h is an (S,N)-implication generated from some t-conorm S and continuous negation N. □

Corollary 2. Let h be an h-generator. I_h is an (S,N)-implication generated from some t-conorm and some strict negation if and only if $h(1) = 0$.

Corollary 3. Let h be an h-generator. I_h is an S-implication generated from some t-conorm and some strong negation if and only if $h = h^{-1}$.

8. THE INTERSECTIONS OF $\mathbb{I}_F, \mathbb{I}_G, \mathbb{I}_H$ WITH $\mathbb{I}_T$

In this section we investigate whether any of the families $\mathbb{I}_F, \mathbb{I}_G, \mathbb{I}_H$ intersect with $\mathbb{I}_T$. In other words, if and when any of the members of the above families can be written as an R-implication of an appropriate left-continuous t-norm. It is obvious from the characterization of R-implications (see Theorem 2) that for a fuzzy implication I to be an R-implication it should

- be increasing in the second variable (I2),
- satisfy the ordering property (OP),
- satisfy the exchange principle (EP), and
- $I(x, .)$ should be right-continuous for any $x \in [0, 1]$.

We already know that the families $\mathbb{I}_F, \mathbb{I}_G, \mathbb{I}_H$ all have (I2) and (EP). Hence we need to check for (OP) and the right-continuity of their members in the second variable. This was done again in previous sections and we have the following results.

Because of Theorem 4 (iii) we get

Theorem 14. If f is an f-generator, then I_f is not an R-implication.

The next fact follows from Theorem 7.

Theorem 15. If g is a g-generator of I_g, then the following statements are equivalent:

(i) I_g is an R-implication.

(ii) There exists a constant $c \in (0, \infty)$ such that $g(x) = c \cdot x$ for all $x \in [0, 1]$.

(iii) I_g is the Goguen implication $I_{\mathbf{GG}}$.

Finally, by Theorem 9 (iv), we have

Theorem 16. If h is an h-generator, then I_h is not an R-implication.

9. CONCLUSION

In this work, firstly we discussed some properties of the newly proposed families of fuzzy implications, viz., f-, g- and h-generated implications. In the light of the properties that these classes possess, we investigated the intersections that exist amongst these classes of fuzzy implications, following which these investigations were extended to study the intersections that exist among the above classes and two of the well-established classes of fuzzy implications, viz., (S, N)- and R-implications. Table 2 gives a summary of the results in this work, while Figure gives a diagrammatic representation of the intersections.

A few interesting observations can be made with the help of the above table.

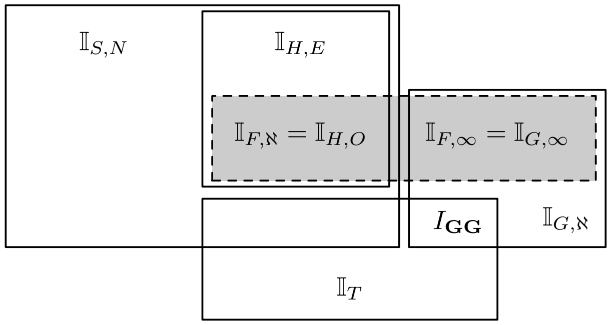

Fig. Intersections between f-,g- and h-generated implications with (S, N)- and R-implications.

Table 2. Intersections between families of fuzzy implications.

$\cap$	$\mathbb{I}_{S,N}$	$\mathbb{I}_T$
$\mathbb{I}_{F,\infty} = \mathbb{I}_{G,\infty}$	$\emptyset$	$\emptyset$
$\mathbb{I}_{F,\aleph} = \mathbb{I}_{H,O}$	$\mathbb{I}_{F,\aleph}$	$\emptyset$
$\mathbb{I}_{G,\aleph}$	$\emptyset$	$I_{\mathbf{GG}}$
$\mathbb{I}_{H,E}$	$\mathbb{I}_{H,E}$	$\emptyset$

- If the f-generator is such that $f(0) = \infty$, or equivalently a g-generator is such that $g(1) = \infty$, then we get totally new families of fuzzy implications.
- On the other hand, if the f-generator is such that $f(0) < \infty$, or equivalently the h-generator is such that $h(1) = 0$, then the f-generated, or equivalently the h-generated implication, becomes an (S, N)-implication for an appropriate t-conorm S and a continuous fuzzy negation N, but never is an R-implication. On the other hand, if the g-generator is such that $g(1) < \infty$, then the g-generated implication does not become an (S, N)-implication and the only g-generated implication that is an R-implication is the one obtained from the g-generator $g(x) = x$.
- If h is an h-generator, then $f(x) = -\ln h(1 - x)$ is an f-generator. Now, let the h-generator be such that $h(1) = 0$, then $f(0) = \infty$ and $f(1) = 0$. Interestingly, the f-generated implication I_f is neither an R-implication nor an (S, N)-implication, whereas the h-generated implication I_h is an (S, N)-implication.
- Also because of the characterization of (S, N)-implications we see that h-generated implications are only another representation of a subclass of (S, N)-implications.

The full characterization of f-, g- and h-generated fuzzy implications is as yet unknown and is significant enough to merit attention. Also the intersections of f-, g- and h-generated implications with the other classes of fuzzy implication, like QL-, A- and R_n-implications (see [9, 11]) may turn out to be interesting. Our future endeavors will be along these lines.

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Michał Baczyński, Institute of Mathematics, University of Silesia, ul. Bankowa 14, 40-007 Katowice. Poland.
e-mail: michal.baczynski@us.edu.pl

Balasubramaniam Jayaram, Department of Mathematics and Computer Sciences, Sri Sathya Sai University, Prasanthi Nilayam, A.P-515134. India.
e-mail: jbala@ieee.org