Homomorphisms on the Monoid of Fuzzy Implications and the Iterative Functional Equation $I(x, I(x, y)) = I(x, y)$

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Abstract

Recently, Vemuri and Jayaram proposed a novel method of generating fuzzy implications, called the ⊛-composition, from a given pair of fuzzy implications [Representations through a Monoid on the set of Fuzzy Implications, Fuzzy Sets and Systems, 247, 51-67]. However, as with any generation process, the ⊛-composition does not always generate new fuzzy implications. In this work, we study the generative power of the ⊛-composition. Towards this end, we study some specific functional equations all of which lead to the solutions of the iterative functional equation $I(x, I(x, y)) = I(x, y)$ involving fuzzy implications which has been studied extensively for different families of fuzzy implications in this very journal, see [Information Sciences 177, 2954–2970 (2007); 180, 2487–2497 (2010); 186, 209–221 (2012)]. In this work, unlike in other existing works, we do not restrict the solutions to a particular family of fuzzy implications. Thus we take an algebraic approach towards solving these functional equations. Viewing the ⊛-composition as a binary operation ⊛ on the set $\mathbb{I}$ of all fuzzy implications one obtains a monoid structure $(\mathbb{I}, ⊛)$ on the set $\mathbb{I}$. From the Cayley's theorem for monoids, we know that any monoid is isomorphic to the set of all right translations. We determine the complete set $\mathcal{K}$ of fuzzy implications w.r.t. which the right translations also become semigroup homomorphisms on the monoid $(\mathbb{I}, ⊛)$ and show that $\mathcal{K}$ not only answers our questions regarding the generative power of the ⊛-composition but also contains many as yet unknown solutions of the iterative functional equation $I(x, I(x, y)) = I(x, y)$.

Keywords: Semigroup, monoid, homomorphism, center, idempotent element, right absorbing element, fuzzy implications, neutrality property, functional equation.
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1. Introduction

Fuzzy implications are a generalisation of classical implication from two valued logic to multivalued logic. Fuzzy implications are defined as follows:

Definition 1.1 ([4], Definition 1.1.1). A function $I \colon [0,1]^2 \longrightarrow [0,1]$ is called a *fuzzy implication* if it satisfies, for all $x, x_1, x_2, y, y_1, y_2 \in [0,1]$, the following conditions:

> if $x_1 \le x_2$, then $I(x_1, y) \ge I(x_2, y)$, i.e., $I(\,\cdot\,, y)$ is decreasing , $\qquad$ (I1)

> if $y_1 \le y_2$, then $I(x, y_1) \le I(x, y_2)$, i.e., $I(x, \,\cdot\,)$ is increasing , $\qquad$ (I2)

> $I(0,0) = 1$, $I(1,1) = 1$, $I(1,0) = 0$. $\qquad$ (I3)

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The set of all fuzzy implications will be denoted by $\mathbb{I}$. Table 1 (see also [4]) lists some examples of basic fuzzy implications.

Name	Formula
Łukasiewicz	$I_{\mathbf{LK}}(x, y) = \min(1, 1 - x + y)$
Gödel	$I_{\mathbf{GD}}(x, y) = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{if } x > y \end{cases}$
Reichenbach	$I_{\mathbf{RC}}(x, y) = 1 - x + xy$
Kleene-Dienes	$I_{\mathbf{KD}}(x, y) = \max(1 - x, y)$
Rescher	$I_{\mathbf{RS}}(x, y) = \begin{cases} 1, & \text{if } x \leq y \\ 0, & \text{if } x > y \end{cases}$
Weber	$I_{\mathbf{WB}}(x, y) = \begin{cases} 1, & \text{if } x < 1 \\ y, & \text{if } x = 1 \end{cases}$
Smallest	$I_{\mathbf{0}}(x, y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 1 \\ 0, & \text{if } x > 0 \text{ and } y < 1 \end{cases}$
Largest	$I_{\mathbf{1}}(x, y) = \begin{cases} 1, & \text{if } x < 1 \text{ or } y > 0 \\ 0, & \text{if } x = 1 \text{ and } y = 0 \end{cases}$
Most Strict	$I_{\mathbf{D}}(x, y) = \begin{cases} 1, & \text{if } x = 0 \\ y, & \text{if } x > 0 \end{cases}$

Table 1: Examples of some basic fuzzy implications

Fuzzy implications play an important role in many fields like, *fuzzy control, decision making, fuzzy image processing, data mining, approximate reasoning* etc. Due to their applicational value it is essential to generate fuzzy implications that are fit for a specific task.

Recently, in [34], the authors had proposed a novel *generative* method, called the $\circledast$-composition, which derives fuzzy implications from a given pair of fuzzy implications.

Definition 1.2 ([34], Definition 7). For any two fuzzy implications I, J, their $\circledast$-composition defined as follows is a fuzzy implication:

$$(I \circledast J)(x, y) = I(x, J(x, y)), \qquad x, y \in [0, 1]. \tag{1}$$

Note that the novelty of the proposed $\circledast$-composition arises from the following facts:

(i) It is the first composition that does not employ any other fuzzy logic connective(s) or parameters to help in the generation of fuzzy implications.
(ii) Further, the operation $\circledast$ not only leads to newer implications but also to a richer algebraic structure, namely a non-idempotent monoid, see **Theorem 2.6** below, on the set $\mathbb{I}$ of fuzzy implications.

The algebraic aspects of the monoid $(\mathbb{I}, \circledast)$ have already been explored in [36] leading up to hitherto unknown representations of the Yager's families of fuzzy implications.

1.1. Motivation for this work

As with any generation process, it is not necessary that the $\circledast$-composition always generates newer fuzzy implications different from the given pair of fuzzy implications. For instance, for a given pair $I, J \in \mathbb{I}$, their $\circledast$-composition $I \circledast J$ may be either of I or J. Thus it is essential to determine the power of the generation process. This forms the main motivation for this work.

Towards this end, specifically, the following questions regarding the generation process are explored in this work:

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Question 1: Note that, from Definition 1.2, we see that the ⊛-composition, in general, is not commutative, i.e., $I \circledast J \neq J \circledast I$ always. Looking at the ⊛-composition as a generation process, this non-commutativity is more a boon than a bane, since it is apparent that if they are equal then we obtain only half the number of new implications than we could. Hence it is important to find the set of all $J \in \mathbb{I}$ that commute with every $I \in \mathbb{I}$ to know clearly when the generative power of the ⊛-composition can diminish. Note that algebraically speaking, this is nothing but the center $\mathcal{Z}$ of the monoid $(\mathbb{I}, \circledast)$.

Question 2: Similarly, another subset of $\mathbb{I}$ that does not lead to new fuzzy implications is the following: the set of all $J \in \mathbb{I}$ that when composed with any $I \in \mathbb{I}$ subsumes it, i.e., $I \circledast J = J$. Once again, algebraically speaking, these are the right absorbing elements of the ⊛-composition on $\mathbb{I}$. Thus determining these right, and left, absorbing elements helps in understanding the generative power of the ⊛-composition.

Question 3: Finally, let $J = I \in \mathbb{I}$. Then clearly $I \circledast J = I \circledast I \in \mathbb{I}$. Note that this self-composition may or may not produce a different fuzzy implication than I. The case where it does produce different fuzzy implications, i.e., $I \circledast I \neq I$, and the limit of this repeated self-composition has already been studied in another work, see [35]. In this work, we investigate the other case, i.e., the set of all fuzzy implications I where $I \circledast I = I$. Clearly, answering this question is the algebraic equivalent of determining the set of all idempotent elements of the monoid $(\mathbb{I}, \circledast)$.

1.2. Main Contributions of this work

The main contributions of this work can be seen as two-fold in the context of both algebra and functional equations.

On the one hand, as is already alluded to above, answering the three questions listed above is equivalent to determining some special subsets of the set $\mathbb{I}$ of all fuzzy implications all of which have an algebraic connotation in the monoid $(\mathbb{I}, \circledast)$, viz., the set of all right or left absorbing elements, commuting and idempotent elements of the ⊛-composition.

On the other hand, all the above three questions can also be seen as determining the solutions of functional equations involving fuzzy implications - a topic that has been an intense area of research for two main reasons - due to their applicational value (see for instance, [8, 19, 20]) and theoretical importance, see for instance, [5, 7, 9, 11, 12, 13, 14, 21, 23, 27, 29, 30, 31, 32].

In fact, many such works have appeared in this very journal [1, 22, 33, 37, 38].

Question 1 leads us to consider the solution set $\mathcal{Z} \subseteq \mathbb{I}$ of the functional equation

$$I(x, J(x, y)) = J(x, I(x, y)), \qquad x, y \in [0, 1] , \tag{2}$$

while solving **Question 2** determines all those $J \in \mathcal{R}, I \in \mathcal{L} \subseteq \mathbb{I}$ that satisfy the functional equations

$$I(x, J(x, y)) = J(x, y), \qquad x, y \in [0, 1] , \tag{3}$$

$$I(x, J(x, y)) = I(x, y), \qquad x, y \in [0, 1] , \tag{4}$$

respectively. **Question 3** leads to the functional equation

$$I(x, I(x, y)) = I(x, y), \qquad x, y \in [0, 1] . \tag{5}$$

Let us denote the set of all fuzzy implications that satisfy the functional equation (5) as $\mathcal{I} \subsetneq \mathbb{I}$.

The iterative functional equation (5) is well-known in the literature, as also the fact that obtaining its complete solution set is a non-trivial problem. In fact, this can be seen from the many works that have appeared in this very journal [33, 37, 38]. While this functional equation has been dealt with by many authors, see for instance, [29, 31, 32], their approach has always been to determine the subset of a particular family of fuzzy implications that are solutions for the equation (5). In this work we do not restrict our study to any specific family of fuzzy implications. Further, we also obtain both characterizations and representations of the solutions of the above functional equations. It should also be pointed out that some of the functions presented as solutions of (5) in some of the works cited above are, in fact, not fuzzy

implications. In Section 8, we discuss the results obtained in this work in light of the known results from the existing works.

Our study reveals also an interesting fact. It is easy to see that, in any monoid, the right and left absorbing elements are also idempotent elements, i.e., $\mathcal{R}, \mathcal{L} \subseteq \mathcal{I}$. Clearly, if $J \in \mathcal{R}$ then $J \circledast J = J$ and hence $J \in \mathcal{I}$. However, in the monoid $(\mathbb{I}, \circledast)$ the center $\mathcal{Z} \subsetneq \mathcal{I}$, i.e., the set of all commuting elements w.r.t. the $\circledast$-composition are also its idempotent elements and hence are solutions of (5), a property that is not true, in general, in every monoid. Thus, in one sense, the entire work contained in this submission could also be seen as largely dealing with the functional equation (5).

1.3. Uniqueness of the approach in this work

Based on the discussions so far, it is clear that one could take two different approaches to answer the above questions - either a purely functional equation approach or an algebraic approach. Note that in the absence of any restrictions on $I, J \in \mathbb{I}$ employed in the above functional equations, viz., their form, properties, representations or the families they come from - as is done in the works cited above, the functional equation approach is quite hard to pursue. Hence in this work, we have taken the latter approach.

As was already shown in [34], $(\mathbb{I}, \circledast)$ becomes only a monoid but not a group. From the Cayley's theorem for monoids, we know that any monoid $(\mathbb{M}, \otimes)$ is isomorphic to the set $\mathcal{T} \subsetneq \mathbb{M}^{\mathbb{M}}$ of all right translations, where $\mathcal{T} = \{g_a : \mathbb{M} \longrightarrow \mathbb{M} | g_a(x) = x \otimes a, \text{ for a fixed } a \in \mathbb{M}\}$. In the context of $(\mathbb{I}, \circledast)$ the set $\mathcal{T}$ is given by

$$\mathcal{T} = \{g_K : \mathbb{I} \longrightarrow \mathbb{I} \mid g_K(I) = I \circledast K, \text{ for a fixed } K \in \mathbb{I}\}.$$

While it is easy to see that every g_K is a lattice homomorphism on the bounded lattice $(\mathbb{I}, \wedge, \vee, I_{\mathbf{0}}, I_{\mathbf{1}})$ (see Proposition 2.10 below for more details), the following posers still remain:

(i) Do all the right translations g_K become semigroup / monoid homomorphisms on $(\mathbb{I}, \circledast)$?
(ii) If not, what is the subset $\mathcal{K} \subsetneq \mathbb{I}$ such that for $K \in \mathcal{K}$ the corresponding g_K's become a semigroup / monoid homomorphism?
(iii) Does the set $\mathcal{K}$ have any algebraic significance?

In this work, by answering the above posers completely, we show that we solve also **Questions 1** and **2** completely. While **Question 3** is still not completely solved, we show that the above set $\mathcal{K}$ is totally contained in the set of idempotent elements, or equivalently, included in the solution set of the functional equation (5).

Further, we show that the set $\mathcal{K}$ contains fuzzy implications that are completely different from those fuzzy implications that are obtained as solutions of (5) by various authors, see, [29, 31, 32, 33, 37, 38]. Note also that our approach allows us to obtain clear and complete representations of fuzzy implications that are solutions to the functional equations (2) and (3).

1.4. Outline of the paper

In Section 2, we recall the notion of right translations on a monoid and the lattice ordered monoid structure $(\mathbb{I}, \circledast)$ that was proposed in [36]. Further, we introduce the right translations g_K on the monoid $(\mathbb{I}, \circledast)$ and show that not all of them become semigroup homomorphisms (denoted *s.g.h*, henceforth) on $(\mathbb{I}, \circledast)$. Hence, in Section 3, we undertake this study and obtain a few necessary conditions on $K \in \mathbb{I}$ such that g_K is an *s.g.h.*, or equivalently $K \in \mathcal{K}$, and show that one such condition on $K \in \mathbb{I}$ is that it must be idempotent. Following this, we split our study based on the range of the considered fuzzy implication $K \in \mathcal{K}$.

In Section 4, we investigate the representations of trivial range fuzzy implications $K \in \mathcal{K}$ and based on their representations show that they form the set of all right absorbing elements of the monoid $(\mathbb{I}, \circledast)$, thus obtaining the solutions of the functional equation (3). In Section 5, we show that in the case of non-trivial range fuzzy implications $K \in \mathcal{K}$, the vertical section $K(1, y)$ must be of the following forms: either, $K(1, y) = y$ for all $y \in [0, 1]$, in which case K has (NP) (see Definition 5.4) or $K(1, y) = 0$ for all $y \in [0, 1)$.

In Section 6, we characterise the set of all $K \in \mathcal{K}$ satisfying (NP) and obtain their representations, which show that they are precisely the set of all commuting elements of $(\mathbb{I}, \circledast)$, viz., the center of the monoid, thus

solving the functional equation (2). More importantly, we show that the set of all commuting elements of the monoid $(\mathbb{I}, \circledast)$ is a proper subset of the set of all idempotent elements of the same, which also shows that every solution of (2) is also a solution of (5). The representations of fuzzy implications $K \in \mathcal{K}$ satisfying $K(1, y) = 0$ for all $y \in [0, 1)$ is discussed in Section 7. In Section 8, we compare the known solutions of the functional equation (5), as presented in [33, 37, 38], with the solutions that are obtained as the fuzzy implications $K \in \mathcal{K}$. This section also highlights the as yet unknown solutions of (5) found through our investigations in this work.

2. Right Translations on the Monoid $(\mathbb{I}, \circledast)$

Translations are one of the important transformations that can be defined on a semigroup. In [15], Clifford introduced the notion of translations in the context of extension of semigroups, and later on, their role has been studied in different contexts, for more details see, for instance, [26].

2.1. Left and Right Translations on a general monoid

In this subsection we review some important concepts related to translations, inner translations and their role in the embedding of semigroups.

Definition 2.1 ([26], Chapter 10, Definition 7.2). A transformation ψ of a semigroup $(U, \cdot)$ is called a left (right) translation if for any elements x and y of U,

$$\psi(x \cdot y) = \psi(x) \cdot y \qquad (\psi(x \cdot y) = x \cdot \psi(y)) \ .$$

For every $a \in U$, the functions $\phi_a(x) = a \cdot x$ and $\psi_a(x) = x \cdot a$ are left and right translations of U, respectively, and are called the inner left and inner right translations induced by a.

Theorem 2.2 ([26], Chapter 10, Theorem 7.5)**.** *In a semigroup U, every left (right) translation is an inner right translation if and only if U has left (right) identity.*

Theorem 2.3 ([26], Chapter 10, Theorem 7.7)**.** *If a semigroup U has an identity then every translation, both left and right, is inner.*

The following is the Cayley's theorem for semigroups.

Theorem 2.4 ([24], Theorem 2.34)**.** *For every semigroup U there exists a set X and an injective map $\phi : U \to X^X$ which is a morphism of semigroups from U to X^X.*

The above result states that every semigroup U is isomorphic to a subsemigroup of the semigroup of all transformations on an appropriate set X. In fact, from the usual proofs of this result, see for instance, [26], it can be seen that U is isomorphic to the set of all right translations defined over U.

2.2. A Lattice Ordered Monoid of Fuzzy Implications: $(\mathbb{I}, \circledast, \vee, \wedge)$

As noted in the Introduction, in [34] (see Definition 7), the following generating method of fuzzy implications from fuzzy implications has been proposed.

Definition 1.2. Given $I, J \in \mathbb{I}$, define $I \circledast J : [0, 1]^2 \longrightarrow [0, 1]$ as

$$(I \circledast J)(x, y) = I(x, J(x, y)), \qquad x, y \in [0, 1] \ .$$

Theorem 2.5 ([34], Theorem 10)**.** *The function $I \circledast J$ is a fuzzy implication, i.e., $I \circledast J \in \mathbb{I}$.*

Table 2 shows some new fuzzy implications obtained from some of the basic fuzzy implications listed in Table 1 via the $\circledast$-composition defined in Definition 1.2. From Table 2 we see that the $\circledast$-composition can indeed generate newer fuzzy implications from given pair of fuzzy implications, though not always.

Theorem 2.5 shows also that $\circledast$ is indeed a binary operation on the set $\mathbb{I}$. The following result shows that $\circledast$ makes $\mathbb{I}$ a monoid. Note that this is the richest algebraic structure obtained so far on the set $\mathbb{I}$ without any assumptions.

I	J	$I \circledast J$
$I_{\mathbf{RC}}$	$I_{\mathbf{LK}}$	$\begin{cases} 1, & \text{if } x \leq y \\ 1 - x^2 + xy, & \text{if } x > y \end{cases}$
$I_{\mathbf{KD}}$	$I_{\mathbf{RS}}$	$\begin{cases} 1, & \text{if } x \leq y \\ 1 - x, & \text{if } x > y \end{cases}$
$I_{\mathbf{RC}}$	$I_{\mathbf{KD}}$	$\max(1 - x^2, 1 - x + xy)$
$I_{\mathbf{GD}}$	$I_{\mathbf{LK}}$	$\begin{cases} 1, & \text{if } x \leq \frac{1+y}{2} \\ 1 - x + y, & \text{otherwise} \end{cases}$
I	$I_{\mathbf{1}}$	$I_{\mathbf{1}}$
$I_{\mathbf{LK}}$	$I_{\mathbf{GD}}$	$I_{\mathbf{LK}}$
$I_{\mathbf{GD}}$	$I_{\mathbf{GD}}$	$I_{\mathbf{GD}}$

Table 2: Compositions of some fuzzy implications w.r.to $\circledast$.

Theorem 2.6 ([34], Theorem 11). $(\mathbb{I}, \circledast)$ *forms a monoid, whose identity element is given by*

$$I_{\mathbf{D}}(x, y) = \begin{cases} 1, & \text{if } x = 0, \\ y, & \text{if } x > 0 \,. \end{cases}$$

Bandler and Kohout [10] were the first to employ the lattice operations of *meet* and *join* to obtain fuzzy implications by taking the meet and join of fuzzy implications I and J, which are defined as follows:

$$(I \vee J)(x, y) = \max(I(x, y), J(x, y)), \qquad x, y \in [0, 1], \qquad I, J \in \mathbb{I} \,, \tag{Latt-Max}$$

$$(I \wedge J)(x, y) = \min(I(x, y), J(x, y)), \qquad x, y \in [0, 1], \qquad I, J \in \mathbb{I} \,. \tag{Latt-Min}$$

Moreover, as was proven by Baczyński and Drewniak [3], these lattice operations produce a much stronger structure on the set $\mathbb{I}$ of all fuzzy implications, as the following result illustrates.

Theorem 2.7 ([4], Theorem 6.1.1). *The set $\mathbb{I}$ is a complete, completely distributive lattice with the lattice operations join $\vee$ (Latt-Max) and meet $\wedge$ (Latt-Min).*

In fact, $(\mathbb{I}, \vee, \wedge, I_{\mathbf{0}}, I_{\mathbf{1}})$ is also a bounded lattice with $I_{\mathbf{0}}, I_{\mathbf{1}}$ being the smallest and greatest fuzzy implications. Note that from Theorem 2.6, we know that $(\mathbb{I}, \circledast)$ is a monoid. Together with all these operations we obtain the following result.

Lemma 2.8 ([36], Lemma 2.6). *The quadruple $(\mathbb{I}, \circledast, \vee, \wedge)$ is a lattice ordered monoid.*

2.3. Right Translations on the monoid $(\mathbb{I}, \circledast)$

In the following, we introduce the right translations on the monoid $(\mathbb{I}, \circledast)$ and show that they are also lattice homomorphisms.

Definition 2.9. For a fixed $K \in \mathbb{I}$, define $g_K \colon (\mathbb{I}, \circledast) \longrightarrow (\mathbb{I}, \circledast)$ by

$$g_K(I) = I \circledast K, \qquad I \in \mathbb{I}.$$

Proposition 2.10. *For every $K \in \mathbb{I}$, the map g_K is a lattice homomorphism.*

Proof. Let $K \in \mathbb{I}$. Let $I, J \in \mathbb{I}$ and $x, y \in [0, 1]$. Then,

$$\begin{aligned} g_K(I \vee J)(x, y) &= ((I \vee J) \circledast K)(x, y) \\ &= (I \vee J)(x, K(x, y)) \\ &= \max(I(x, K(x, y)), J(x, K(x, y))) \\ &= \max((I \circledast K)(x, y), (J \circledast K)(x, y)) \\ &= (g_K(I) \vee g_K(J))(x, y). \end{aligned}$$

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Similarly, for any $I, J \in \mathbb{I}$, one can prove that, $g_K(I \wedge J) = g_K(I) \wedge g_K(J)$. Thus g_K is a lattice homomorphism. $\qquad \square$

Similar to the above Proposition one can easily prove the following result.

Theorem 2.11. *The map g_K is a monoid homomorphism $\iff K = I_{\mathbf{D}}, i.e., g_K \equiv id$.*

From Theorem 2.11, it is clear that the identity function is the only monoid homomorphism on $\mathbb{I}$. Hence in this work, we study the semigroup homomorphisms of the form g_K on the monoid $(\mathbb{I}, \circledast)$.

While g_K's are lattice homomorphisms for every $K \in \mathbb{I}$, the function g_K need not be a monoid or even a semigroup homomorphism (denoted *s.g.h* hereon) on $\mathbb{I}$ for every $K \in \mathbb{I}$ as the following example illustrates.

Example 2.12. (i) On the one hand, when $K(x, y) = I_{\mathbf{LK}}(x, y) = \min(1, 1 - x + y)$, the Łukasiewicsz implication, the map g_K is not an *s.g.h.* To see this, letting $I(x, y) = I_{\mathbf{KD}}(x, y) = \max(1 - x, y)$ and $J(x, y) = I_{\mathbf{RC}}(x, y) = 1 - x + xy$ and $x = 0.4, y = 0.2$, we observe that

$$g_{I_{\mathbf{LK}}}(I \circledast J)(0.4, 0.2) = 0.92 \neq 1 = \left(g_{I_{\mathbf{LK}}}(I) \circledast g_{I_{\mathbf{LK}}}(J) \right)(0.4, 0.2) \ .$$

(ii) On the other hand, let $K \in \mathbb{I}$ be defined by

$$K(x, y) = \begin{cases} 1, & \text{if } x \leq 0.5 \ , \\ y, & \text{if } x > 0.5 \ . \end{cases}$$

For any $I, J \in \mathbb{I}$ it is easy to see that

$$(I \circledast J \circledast K)(x, y) = (I \circledast K \circledast J \circledast K)(x, y) = \begin{cases} 1, & \text{if } x \leq 0.5 \ , \\ I(x, J(x, y)), & \text{if } x > 0.5 \ , \end{cases}$$

which implies that $I \circledast J \circledast K = I \circledast K \circledast J \circledast K$, or equivalently, $g_K(I \circledast J) = g_K(I) \circledast g_K(J)$. Thus g_K becomes an *s.g.h.* However, note that $g_K(I_{\mathbf{D}}) = I_{\mathbf{D}} \circledast K = K \neq I_{\mathbf{D}}$. This implies that g_K is only an *s.g.h.* but not a monoid homomorphism.

Since g_K is not an *s.g.h* for every $K \in \mathbb{I}$, we investigate to characterise and, if possible, determine those fuzzy implications K for which g_K becomes an *s.g.h.*, or equivalently, $K \in \mathbb{I}$ such that $K \in \mathcal{K}$.

3. Necessary conditions on $K \in \mathbb{I}$ such that g_K is an *s.g.h.*

In this section, we investigate some necessary conditions on $K \in \mathbb{I}$ such that $K \in \mathcal{K}$.

Proposition 3.1. *Let $K \in \mathbb{I}$ be arbitrarily fixed. Then the following statements are equivalent:*

(i) *g_K is an s.g.h.*
(ii) *$J \circledast K = K \circledast J \circledast K$ for all $J \in \mathbb{I}$.*

Proof. (i) $\implies$ (ii) : Let $K \in \mathbb{I}$ and g_K be an *s.g.h.* Then for all $I, J \in \mathbb{I}$, $g_K(I \circledast J) = g_K(I) \circledast g_K(J)$ will imply $I \circledast J \circledast K = I \circledast K \circledast J \circledast K$. If we take $I = I_{\mathbf{D}}$, the identity in $(\mathbb{I}, \circledast)$, it follows that $J \circledast K = K \circledast J \circledast K$ for all $J \in \mathbb{I}$.

(ii) $\implies$ (i) : Let $K \in \mathbb{I}$ be such that $J \circledast K = K \circledast J \circledast K$ for all $J \in \mathbb{I}$. This directly implies that $I \circledast J \circledast K = I \circledast K \circledast J \circledast K$ for every $I \in \mathbb{I}$, since every $\circledast$ is a well-defined function on $\mathbb{I}$. Thus g_K is an *s.g.h.* $\qquad \square$

As a consequence of Proposition 3.1, we have the following result.

Lemma 3.2. *Let $K \in \mathbb{I}$ be such that g_K is an s.g.h. Then $K \circledast K = K$, i.e., $K \in \mathcal{I}$.*

Proof. Let $K \in \mathbb{I}$ be such that g_K is an *s.g.h.* From Proposition 3.1, it follows that $J \circledast K = K \circledast J \circledast K$ for all $J \in \mathbb{I}$. When $J = I_{\mathbf{D}}$, the identity of $(\mathbb{I}, \circledast)$, we have that $I_{\mathbf{D}} \circledast K = K \circledast I_{\mathbf{D}} \circledast K$, or equivalently, $K = K \circledast K$. $\qquad\square$

Remark 3.3. The converse of Lemma 3.2 need not be true always, i.e., not every idempotent element K of the monoid $(\mathbb{I}, \circledast)$ makes g_K an *s.g.h.*

For example, let $K = I_{\mathbf{GD}}$. Clearly, $K \circledast K = K$, see, Theorem 11 in [33]. However, g_K is not an s.g.h. To see this, let us consider $J = I_\beta \in \mathbb{I}$ defined by

$$I_\beta(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 1 , \\ 0, & \text{if } x = 1 \text{ and } y = 0 , \\ \beta, & \text{otherwise.} \end{cases} \tag{6}$$

Now, let $\beta = 0.6, x = 0.4$ and $y = 0.2$. Then

$$(I_\beta \circledast I_{\mathbf{GD}})(0.4, 0.2) = I_\beta(0.4, I_{\mathbf{GD}}(0.4, 0.2))$$
$$= I_\beta(0.4, 0.2) = 0.6 ,$$
$$\text{while, } (I_{\mathbf{GD}} \circledast I_\beta \circledast I_{\mathbf{GD}})(0.4, 0.2) = I_{\mathbf{GD}}(0.4, I_\beta(0.4, I_{\mathbf{GD}}(0.4, 0.2)))$$
$$= I_{\mathbf{GD}}(0.4, I_\beta(0.4, 0.2)) = I_{\mathbf{GD}}(0.4, 0.6) = 1.$$

From Proposition 3.1, it follows that g_K is not an *s.g.h.*

Let $\mathcal{K}$ denote the set of all fuzzy implications whose right translations become *s.g.h.*, i.e.,

$$\mathcal{K} = \{K \in \mathbb{I} \mid g_K \text{ is an } s.g.h.\} .$$

The above two results present the necessary conditions that a $K \in \mathbb{I}$ should satisfy to belong to $\mathcal{K}$, from which we see that $\mathcal{K} \subsetneq \mathcal{I}$. Recall that $\mathcal{I}$ is the set of all fuzzy implications which satisfy the iterative functional equation (5).

In our quest for determining $\mathcal{K}$, we divide our analysis into two parts, viz., finding $K \in \mathcal{K}$ when the range of K is trivial and when the range of K is non-trivial.

4. Representations of $K \in \mathcal{K}$ with Trivial Range

In this section, we determine completely the fuzzy implications K whose range is trivial, i.e., $K(x,y) \in \{0,1\}$ for all $x, y \in [0,1]$, and for whom the map g_K is an *s.g.h.*

Towards this end, let us consider the following family of fuzzy implications, $\mathbb{K}^\delta = \{K^\delta | \delta \in [0,1]\}$ where for $\delta = 0$ we have,

$$K^0(x,y) = I_{\mathbf{1}}(x,y) = \begin{cases} 1, & \text{if } x < 1 \text{ or } y > 0, \\ 0, & \text{if } x = 1 \text{ and } y = 0 , \end{cases} \tag{7}$$

and for any $\delta \in (0, 1]$, we have

$$K^\delta(x,y) = \begin{cases} 1, & \text{if } x < 1 \text{ or } (x = 1 \text{ and } y \geq \delta) , \\ 0, & \text{if } x = 1 \text{ and } y < \delta . \end{cases}$$

Note that, w.r.t. pointwise ordering of functions, we have that $\sup \mathbb{K}^\delta = I_{\mathbf{1}}$ and $\inf \mathbb{K}^\delta = I_{\mathbf{SW}}$, given by

$$K^1(x,y) = I_{\mathbf{SW}}(x,y) = \begin{cases} 1, & \text{if } x < 1 \text{ or } y = 1, \\ 0, & \text{if } x = 1 \text{ and } y \neq 1. \end{cases} \tag{8}$$

Theorem 4.1. *The following statements are equivalent:*

(i) $K \in \mathcal{K}$ *and the range of K is trivial.*

(ii) $K = K^\delta$ *for some $\delta \in [0, 1]$.*

Proof.   (i) $\implies$ (ii). Let $K \in \mathcal{K}$, i.e., g_K is an *s.g.h.*, and the range of K be trivial.

Claim: $K(x, y) = 1$, for all $x \in [0, 1)$ and for all $y \in [0, 1]$.

Proof of the claim :

- If $x = 0$, it is trivial that $K(x, y) = 1$ for all $y \in [0, 1]$.

- Let $0 < x < 1$. Suppose that for some $y_0 \in [0, 1)$, $K(x, y_0) < 1$, i.e., $K(x, y_0) = 0$. Since g_K is an *s.g.h*, it follows that $J \circledast K = K \circledast J \circledast K$ for all $J \in \mathbb{I}$. Now,

$$(J \circledast K)(x, y_0) = J(x, K(x, y_0))$$
$$= J(x, 0),$$
$$(K \circledast J \circledast K)(x, y_0) = K(x, J(x, K(x, y_0)))$$
$$= K(x, J(x, 0)).$$

  Since the range of K is trivial, $J(x, 0) \subseteq \{0, 1\}$ for all $J \in \mathbb{I}$. This gives a contradiction if we take a $J \in \mathbb{I}$ such that $J(x, 0) \notin \{0, 1\}$.

Thus $K(x, y) = 1$, for all $x < 1$.

Now for $x = 1, y \in [0, 1]$, we have either $K(x, y) = 0$ or $K(x, y) = 1$. Let us define

$$\delta = \sup\{y \in [0, 1] | K(1, y) = 0\}.$$

Let us take $K \in \mathbb{I}$ such that $K(1, y)$ is right continuous. Then for $y \geq \delta$, $K(1, y) = 1$ and for $y < \delta$, $K(1, y) = 0$. Thus $K = K^\delta$.

(ii) $\implies$ (i). That any K^δ is of trivial range can be clearly seen from its definition, while the fact that g_{K^δ} is an *s.g.h.* can be easily verified. $\qquad\square$

Remark 4.2. Note that in the proof of Theorem 4.1 we have chosen K^δ such that it is right-continuous in the second variable when $x = 1$. However, if we choose K^δ such that it is left-continuous in the second variable at $x = 1$, i.e., $K^\delta(1, y) = 1$ when $y > \delta$ and $K^\delta(1, y) = 0$ when $y \leq \delta$, it can be easily verified that g_{K^δ} is still an *s.g.h.* This particular choice was made to conform to the tradition in the literature of requiring right-continuity in the second variable, as in the case of implications from which the deresiduum is constructed.

Interestingly, as the following result shows $\mathbb{K}^\delta$ is precisely the set of all right absorbing elements of $\mathbb{I}$ w.r.t. the $\circledast$-composition. Before doing so, recall that $I_{\mathbf{0}} \in \mathbb{I}$ is defined as follows:

$$I_{\mathbf{0}}(x, y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 1, \\ 0, & \text{if } x > 0 \text{ and } y < 1 \ . \end{cases} \tag{9}$$

Lemma 4.3. *Let $\mathcal{R} \subset \mathbb{I}$ be the set of all right absorbing elements of $\circledast$. Then $\mathcal{R} = \mathbb{K}^\delta = \{K^\delta | \delta \in [0, 1]\}$.*

Proof. On the one hand, if $K = K^\delta$ for some $\delta \in [0, 1]$, then it is easy to see that $I \circledast K = K$ for all $I \in \mathbb{I}$. Hence $\mathcal{R} \supseteq \mathbb{K}^\delta$.

On the other hand, let $K \in \mathcal{R}$, i.e., $I \circledast K = K$ for all $I \in \mathbb{I}$.

Claim: $K(x, y) = \{0, 1\}$ for all $x, y \in [0, 1]$, i.e., the range of K is trivial.

Proof of the claim: Clearly, if $x = 0$ or $y = 1$ then $K(x, y) = 1 \in \{0, 1\}$. Suppose for some $x_0 \in (0, 1], y_0 \in [0, 1)$ that $\alpha = K(x_0, y_0) \notin \{0, 1\}$. Now,

$$(I_{\mathbf{0}} \circledast K)(x_0, y_0) = I_{\mathbf{0}}(x_0, K(x_0, y_0))$$
$$= I_{\mathbf{0}}(x_0, \alpha) = 0 \neq \alpha = K(x_0, y_0),$$

contradicting $I \circledast K = K$ for all $I \in \mathbb{I}$. Thus the range of K is trivial.

Claim: $K(x, y) = 1$, for all $x \in [0, 1)$ and for all $y \in [0, 1]$.

Proof of the claim: If $x = 0$, then it is trivial. So, let $0 < x < 1$ be fixed arbitrarily. Suppose for $y_0 < 1$, that $K(x, y_0) < 1$. Since the range of K is trivial, $K(x, y_0) = 0$. Now,

$$(I_\mathbf{1} \circledast K)(x, y_0) = I_\mathbf{1}(x, K(x, y_0))$$
$$= I_\mathbf{1}(x, 0) = 1 \neq 0 = K(x, y_0),$$

contradicts the fact that $I \circledast K = K$ for all $I \in \mathbb{I}$. Now define $\delta = \sup\{t | K(1, t) = 0\}$. This implies that $K(1, y) = 0$ for all $y < \delta$ and $K(1, y) = 1$ for all $y > \delta$, because the range of K is trivial. Once again since we are interested in $K \in \mathbb{I}$ right continuous in the second variable, we take that $K(1, \delta) = 1$. Thus $K = K^\delta$ for some $\delta \in [0, 1]$ and hence $\mathcal{R} \subseteq \mathbb{K}^\delta$. $\qquad\square$

Proposition 4.4. *The monoid* $(\mathbb{I}, \circledast)$ *does not have left absorbing elements.*

Proof. Let $\mathcal{L}$ denote the set of all left-absorbing elements of the monoid $(\mathbb{I}, \circledast)$. We claim that $\mathcal{L} = \emptyset$.

On the contrary, let $I, J \in \mathcal{L}$ be two left-absorbing elements. Then $I \circledast K = I$ and $J \circledast K = J$ for all $K \in \mathbb{I}$. Now, consider a right absorbing element $K' \in \mathcal{R}$. Then it follows that $I = I \circledast K' = K'$ and $J = J \circledast K' = K'$. This shows that $I = J$ and hence $\mathcal{L}$ is utmost a singleton set. Let $L \in \mathcal{L}$.

Now, let $K_1, K_2 \in \mathcal{R}$ be two distinct right absorbing elements of $(\mathbb{I}, \circledast)$. Then we have $L = L \circledast K_1 = K_1$ and $L = L \circledast K_2 = K_2$, which leads to a contradiction, since $K_1 \neq K_2$. Thus $\mathcal{L} = \emptyset$ and $(\mathbb{I}, \circledast)$ has no left absorbing elements. $\qquad\square$

From Proposition 4.4, it follows that the functional equation (4) has no solutions.

Corollary 4.5. *The monoid* $(\mathbb{I}, \circledast)$ *has no two-sided absorbing elements.*

While $(\mathbb{I}, \circledast)$ has no two-sided absorbing elements, it is interesting to note the following. Clearly, every $K^\delta \in \mathcal{R}$ is a right absorbing element, i.e., $I \circledast K^\delta = K^\delta$, for any $I \in \mathbb{I}$. If we consider the following composition, $K^\delta \circledast I$ for any $I \in \mathbb{I}$, we obtain that $K^\delta \circledast I = K^\mu \in \mathcal{R}$ for some $\mu \in (0, 1]$. In other words, the set $\mathcal{R}$ when composed with $\mathbb{I}$ subsumes it both from the left and the right. In fact, as we show below, the set $\mathcal{R}$ forms a two-sided ideal of the monoid $(\mathbb{I}, \circledast)$.

Recall that a non-empty subset A of a semigroup U is called a two-sided ideal if $AU \subseteq A$ and $UA \subseteq A$.

Lemma 4.6. *The set* $\mathcal{R}$ *of all right absorbing elements forms a two-sided ideal of* $(\mathbb{I}, \circledast)$, *i.e.,* $\mathbb{I}\mathcal{R} = \mathcal{R} = \mathcal{R}\mathbb{I}$.

Proof. From Lemma 4.3, it follows that $\mathbb{I}\mathcal{R} = \mathcal{R}$. Now it remains to show that $\mathcal{R}\mathbb{I} = \mathcal{R}$. Before proceeding to show this, for a given $\delta \in (0, 1]$ and an $I \in \mathbb{I}$, let us define

$$\delta_I = \inf\{y \in [0, 1] | I(1, y) \geq \delta\} \ . \tag{10}$$

Note that $1 \in \{y \in [0, 1] | I(1, y) \geq \delta\}$ and hence $\delta_I \in (0, 1]$ and is well defined.

Now, let $I \in \mathbb{I}$ and $K \in \mathcal{R}$. Since $K \in \mathcal{R}$, from Lemma 4.3 we have that $K = K^\delta$ for some $\delta \in (0, 1]$ (see, Theorem 4.1, for the definition of K^δ). Now,

$$(K \circledast I)(x, y) = K^\delta(x, I(x, y))$$

$$= \begin{cases} 1, & \text{if } x < 1 \text{ or } (x = 1 \ \& \ I(1, y) \geq \delta) \ , \\ 0, & \text{if } x = 1 \ \& \ I(1, y) < \delta \ , \end{cases}$$

$$= \begin{cases} 1, & \text{if } x < 1 \text{ or } (x = 1 \ \& \ y \geq \delta_I) \ , \\ 0, & \text{if } x = 1 \ \& \ y < \delta_I \ , \end{cases}$$

$$= K^{\delta_I}(x, y) \ ,$$

where δ_I is as defined in (10) above. Since, for every $I \in \mathbb{I}$, there exists a $\delta_I \in (0, 1]$, such that $K \circledast I = K^{\delta_I} \in \mathcal{R}$, we see that $\mathcal{R}\mathbb{I} \subseteq \mathcal{R}$. The other inclusion $\mathcal{R} \subseteq \mathcal{R}\mathbb{I}$ follows directly, as the identity $I_\mathbf{D} \in \mathbb{I}$. Thus $\mathcal{R}\mathbb{I} = \mathcal{R}$ and $\mathcal{R}$ is a two-sided ideal. $\qquad\square$

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5. Characterisations of $K \in \mathcal{K}$ with Non-Trivial Range

In Section 4, we have characterised and found the trivial range fuzzy implications K such that g_K is an *s.g.h.* Further, we have shown that these fuzzy implications form the set of all right absorbing elements of the monoid $(\mathbb{I}, \circledast)$. In this section, we determine the non-trivial range fuzzy implications K such that g_K is an *s.g.h.*

The following result shows that the range of such fuzzy implications K should be the entire $[0,1]$ interval.

Lemma 5.1. *If the range of $K \in \mathcal{K}$ is non-trivial then the range of K is equal to $[0,1]$.*

Proof. Let the range of $K \in \mathcal{K}$ be non-trivial. Since the range of K is non-trivial, there exists $\alpha \in (0,1)$ such that $K(x_0, y_0) = \alpha$ for some $x_0 \in (0,1]$ and $y_0 \in [0,1)$. Let $I_\beta \in \mathbb{I}$ be as defined in (6). Then, $(I_\beta \circledast K)(x_0, y_0) = I_\beta(x_0, K(x_0, y_0)) = I_\beta(x_0, \alpha) = \beta$. Since g_K is an *s.g.h*, $(K \circledast I_\beta \circledast K)(x_0, y_0) = \beta$, i.e., β is in the range of K. Since $\beta \in (0,1)$ is chosen arbitrarily, the range of K contains every point of $[0,1]$. Thus the range of K is $[0,1]$. $\qquad\square$

Towards characterising all such fuzzy implications, we study some specific vertical sections of $K \in \mathcal{K}$ which help us in the sequel.

5.1. *Natural Negations and Neutrality of Fuzzy Implications*

We begin with the following definitions which are more general in scope and apply to the whole of $\mathbb{I}$.

Definition 5.2 ([16]). A function $N : [0,1] \longrightarrow [0,1]$ is called a *fuzzy negation* if N is non-increasing on $[0,1]$ such that $N(0) = 1$ and $N(1) = 0$.

Definition 5.3 ([4], Definition 1.4.14). Let I be any fuzzy implication. The function $N_I : [0,1] \longrightarrow [0,1]$ defined by $N_I(x) = I(x,0)$ is a fuzzy negation and is called the *natural negation* of I.

Table 3 lists some examples of natural negations of fuzzy implications, which are also considered as the basic fuzzy negations.

Implication I	Natural Negation N_I	Remark
$I_{\mathbf{KD}}(x,y) = \max(1-x, y)$	$N_{\mathbf{C}}(x) = 1 - x$	Classical
$I_{\mathbf{D}}(x,y) = \begin{cases} 1, & \text{if } x = 0 \\ y, & \text{if } x > 0 \end{cases}$	$N_{\mathbf{D_1}}(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x > 0 \end{cases}$	Smallest
$I_{\mathbf{WB}}(x,y) = \begin{cases} 1, & \text{if } x < 1 \\ y, & \text{if } x = 1 \end{cases}$	$N_{\mathbf{D_2}}(x) = \begin{cases} 1, & \text{if } x < 1 \\ 0, & \text{if } x = 1 \end{cases}$	Greatest

Table 3: Fuzzy Implications and their natural negations, (see, also Table 1.7, [4])

Definition 5.4 ([4], Definition 1.3.1). A fuzzy implication I is said to satisfy the left neutrality property (NP) if

$$I(1, y) = y, \qquad y \in [0,1] . \tag{NP}$$

Let $\mathbb{I}_{\mathbf{NP}}$ denote the set of all fuzzy implications satisfying the property (NP).

5.2. Some Vertical Sections of $K \in \mathcal{K}$

In the following, we discuss the natural negations and the range and form of the vertical section $K(1, \cdot)$ for a $K \in \mathcal{K}$ whose range is non-trivial.

Lemma 5.5. *Let $K \in \mathcal{K}$. Then the range of $K(\cdot, 0) = \{0, 1\}$, i.e., N_K has trivial range.*

Proof. Let $K \in \mathcal{K}$. For some $x_0 \in (0, 1)$, let $K(x_0, 0) = \alpha$. Consider $I_{\mathbf{0}} \in \mathbb{I}$. Then

$$
\begin{aligned}
(K \circledast I_{\mathbf{0}} \circledast K)(x_0, 0) &= K(x_0, I_{\mathbf{0}}(x_0, K(x_0, 0))) \\
&= K(x_0, I_{\mathbf{0}}(x_0, \alpha)) = K(x_0, 0),
\end{aligned}
$$

while,
$$
\begin{aligned}
(I_{\mathbf{0}} \circledast K)(x_0, 0) &= I_{\mathbf{0}}(x_0, K(x_0, 0)) \\
&= I_{\mathbf{0}}(x_0, \alpha) = 0.
\end{aligned}
$$

Since g_K is an *s.g.h.* from Proposition 3.1, it follows that $K(x_0, 0) = 0 = \alpha$. Note that $x_0 \in (0, 1)$ is chosen arbitrarily. Hence $K(x, 0) = \{0, 1\}$ for all $x \in [0, 1]$. $\qquad\square$

Note that the above result characterises the horizontal section $K(\cdot, 0)$ and is trivially true for $K \in \mathcal{K}$ whose range is trivial. Now, before characterising the non-trivial range fuzzy implications $K \in \mathcal{K}$, we characterise the vertical section $K(1, \cdot)$ of K which helps us in getting the representations of K.

Towards this end, we propose the following definition.

Definition 5.6. Let $K \in \mathbb{I}$. Define the following two real numbers:

$$\epsilon_0 = \sup\{t \in [0, 1] | K(1, t) = 0\}, \tag{11}$$

$$\epsilon_1 = \inf\{t \in [0, 1] | K(1, t) = 1\}. \tag{12}$$

Remark 5.7. (i) Let ϵ_0, ϵ_1 be two real numbers as defined in Definition 5.6. For every $K \in \mathbb{I}$, since $K(1, 0) = 0$ and $K(1, 1) = 1$, the real numbers ϵ_0, ϵ_1 in the equations (11), (12) are well defined and exist in general.

(ii) More importantly, $0 \leq \epsilon_0 \leq \epsilon_1 \leq 1$.

(iii) Since $\epsilon_0 \leq \epsilon_1$, if $\epsilon_0 = 1$ then $\epsilon_1 = 1$.

Proposition 5.8. *Let the range of $K \in \mathcal{K}$ be non-trivial. Let $\epsilon_0, \epsilon_1 \in [0, 1]$ be defined as in Definition 5.6. Then the vertical section $K(1, .)$ has the following form:*

$$
K(1, y) = \begin{cases}
0, & \text{if } y \in [0, \epsilon_0) , \\
0 \text{ } \boldsymbol{or} \text{ } \epsilon_0, & \text{if } y = \epsilon_0 , \\
y, & \text{if } y \in (\epsilon_0, \epsilon_1) , \\
\epsilon_1 \text{ } \boldsymbol{or} \text{ } 1, & \text{if } y = \epsilon_1 , \\
1, & \text{if } y \in (\epsilon_1, 1] .
\end{cases} \tag{13}
$$

Proof. Let $K \in \mathbb{I}$ be such that the range of K is non-trivial and g_K is an *s.g.h.*

Further, since, g_K is an *s.g.h*, we see that for all $J \in \mathbb{I}$ the following equality should hold for all $y \in [0, 1]$:

$$
\begin{aligned}
(J \circledast K)(1, y) &= (K \circledast J \circledast K)(1, y) , \\
\text{i.e., } J(1, K(1, y)) &= K(1, J(1, K(1, y))) . \tag{14}
\end{aligned}
$$

(i) From the definition of ϵ_0, ϵ_1 above, it is clear that $K(1, y) = 0$ whenever $0 \leq y < \epsilon_0$ and $K(1, y) = 1$, whenever $\epsilon_1 < y \leq 1$.

(ii) Let $\epsilon_0 < y < \epsilon_1$. We claim that $K(1, y) = y$. If not, let there be a $y_0 \in [0, 1)$ such that $K(1, y_0) = y' \neq y_0$. Let us choose a $J \in \mathbb{I}$ such that $J(1, y') = y_0$. Note that such a J is always possible, for instance, $J = I_\beta$ of (6) with $\beta = y_0$. Then, we have

      LHS of (14) $= J(1, K(1, y_0)) = J(1, y') = y_0$,
      RHS of (14) $= K(1, J(1, K(1, y_0))) = K(1, J(1, y')) = K(1, y_0) = y'$,

which implies that g_K is not an *s.g.h.*, a contradiction. Thus $K(1, y) = y$ whenever $\epsilon_0 < y < \epsilon_1$.

(iii) Note that since ϵ_0, ϵ_1 are only the infimum and supremum of these sets, which are intervals due to the monotonicity of K in the second variable, they may not belong to these intervals themselves. In other words, $K(1, \epsilon_0) \geq 0$ and $K(1, \epsilon_1) \leq 1$.

  (a) Clearly, if $\epsilon_0 = \max\{t \in [0, 1] | K(1, t) = 0\}$, then $K(1, \epsilon_0) = 0$.
  (b) However, if $\epsilon_0 \notin \{t \in [0, 1] | K(1, t) = 0\}$ then clearly $0 < K(1, \epsilon_0) = \delta$. We claim that $\delta = \epsilon_0$. On the contrary, let $\delta \neq \epsilon$, then, once again, one can choose a $J \in \mathbb{I}$ such that $J(1, \delta) = \epsilon_0$. Then,

      LHS of (14) $= J(1, K(1, \epsilon_0)) = J(1, \delta) = \epsilon_0$,
      RHS of (14) $= K(1, J(1, K(1, \epsilon_0))) = K(1, J(1, \delta)) = K(1, \epsilon_0) = \delta$,

  from whence we obtain that g_K is not an *s.g.h.*, a contradiction. Thus $K(1, \epsilon_0) = \epsilon_0$.
  (c) A similar proof as above shows that if $\epsilon_1 \in \{t \in [0, 1] | K(1, t) = 1\}$ then $K(1, \epsilon_1) = 1$, while if $\epsilon_1 \notin \{t \in [0, 1] | K(1, t) = 1\}$ then $K(1, \epsilon_1) = \epsilon_1$.

$\hfill\square$

In Proposition 5.8, even though we are able to characterise the vertical sections $K(1, \cdot)$, it is not clear what values $K(1, \cdot)$ could assume. Now, we investigate all the possible values of ϵ_0, ϵ_1 in the case when the range of K is non-trivial and g_K is an *s.g.h.*

Theorem 5.9. *Let $K \in \mathcal{K}$ be such that the range of K is non-trivial and let ϵ_0, ϵ_1 be defined as in Definition 5.6. Then*

  (i) $\epsilon_1 \neq 0$.
  (ii) *If $\epsilon_0 = 0$, then $\epsilon_1 = 1$, in which case $K(1, y) = y$ for all $y \in [0, 1]$.*
  (iii) *If $0 < \epsilon_0 < 1$, then $\epsilon_0 \neq \epsilon_1$.*
  (iv) *If $\epsilon_0 > 0$, then $\epsilon_0 = 1$, in which case $K(1, y) = 0$ for all $y > 0$.*

Proof.   (i) Let $\epsilon_1 = 0$. This implies that $K(1, y) = 1$ for all $y > 0$. Again it follows from the monotonicity of I in the first variable that $K(x, y) = 1$ for all x and all $y > 0$. Now, from Lemma 5.5, it follows that the range of the negation of fuzzy implication K is trivial, i.e., $K(x, 0) \in \{0, 1\}$. So, the range of K becomes $\{0, 1\}$, a contradiction to the fact the range of K is non-trivial. Thus $\epsilon_1 \neq 0$.

(ii) Let $\epsilon_0 = 0$ and suppose that $\epsilon_1 < 1$. Then from (i), it follows that $0 < \epsilon_1 < 1$. So choose a $\delta > 0$ such that $0 < \epsilon_1 + \delta < 1$. Let $0 < y_1 < \epsilon_1$. This implies that $0 < K(1, y_1) = \alpha < 1$. Now, choose a $J \in \mathbb{I}$ such that $J(1, K(1, y_1)) = J(1, \alpha) = \epsilon_1 + \delta$. However, $K(1, J(1, K(1, y_1))) = K(1, \epsilon_1 + \delta) = 1$, which contradicts g_K being an *s.g.h.* Thus $\epsilon_1 = 1$.

(iii) Let $0 < \epsilon_0 < 1$. Suppose that $\epsilon_0 = \epsilon_1$. Then $K(1, \cdot)$ will be of the form

$$K(1, y) = \begin{cases} 1, & \text{if } y \geq \epsilon_0, \\ 0, & \text{if } y < \epsilon_0. \end{cases} \tag{15}$$

This implies that $K(x, y) = 1$ for all $x \in [0, 1], y \geq \epsilon_0$. Now we prove that $K(x, y) = 1$ for all $x \in [0, 1), y \in [0, \epsilon_0)$. On the contrary suppose that $\alpha = K(x_0, y_0) < 1$ for some $x_0 \in (0, 1)$, $y_0 \in [0, \epsilon_0)$. Since $0 < \epsilon_0 < 1$, choose a $\delta > 0$ such that $0 < \epsilon_0 + \delta < 1$. Now choose a $J \in \mathbb{I}$ such that $J(x_0, K(x_0, y_0)) = J(x_0, \alpha) = \epsilon_0 + \delta \neq 1$. Now, $K(x_0, J(x_0, K(x_0, y_0))) = K(x_0, J(x_0, \alpha)) = K(x_0, \epsilon_0 + \delta) = 1$ a contradiction to the fact that g_K is an *s.g.h.* Thus $K(x_0, y_0) = 1$ for all $x_0 \in [0, 1)$ and $y_0 \in [0, \epsilon_0)$ and $K(x, y) = 1$ for all $x < 1$. Finally from the Eq.(15) it follows that the range of K is trivial, a contradicition. Thus $\epsilon_0 \neq \epsilon_1$.

(iv) Let $\epsilon_0 > 0$. Suppose $\epsilon_0 < 1$, i.e., $0 < \epsilon_0 < 1$. Now from (iii) it follows that $\epsilon_0 \neq \epsilon_1$. Let $y_1 \in (\epsilon_0, \epsilon_1)$. Then the Eq.(13) implies that $K(1, y_1) = y_1$. Choose a $J \in \mathbb{I}$ be such that $J(1, y_1) = \frac{\epsilon_0}{2}$. Then

$$(J \circledast K)(1, y_1) = J(1, K(1, y_1)) = J(1, y_1) = \frac{\epsilon_0}{2},$$

$$\text{and } (K \circledast J \circledast K)(1, y_1) = K(1, J(1, K(1, y_1)))$$

$$= K(1, J(1, y_1)) = K\left(1, \frac{\epsilon_0}{2}\right) = 0 \;,$$

a contradiction to the fact that g_K is an *s.g.h.* Thus $\epsilon_0 = 1$. $\qquad\square$

From Proposition 5.8 and Remark 5.7, Corollary 5.10 gives the possible values of ϵ_0, ϵ_1 of non-trivial $K \in \mathcal{K}$.

Corollary 5.10. *Let $K \in \mathcal{K}$ be such that the range of K is non-trivial and let ϵ_0 and ϵ_1 be defined as in* (11) *and* (12), *respectively. Then*

(i) *If $\epsilon_0 = 0$ then $\epsilon_1 = 1$.*
(ii) *If $\epsilon_0 > 0$ then $\epsilon_0 = 1$.*

Corollary 5.11. *Let the range of $K \in \mathcal{K}$ be non-trivial. Then one of the following conditions holds:*

(i) $K(1, y) = y$, *for all $y \in [0, 1]$.*
(ii) $K(1, y) = 0$, *for all $y \in [0, 1)$.*

From the above results, it is clear that if K is a non-trivial range implication such that g_K is an *s.g.h* then K has either (NP) or $K(1, y) = 0$ for all $y \in [0, 1)$. We analyse each of these two cases in Sections 6 and 7.

6. Representations of $K \in \mathcal{K}$ such that K satisfies (NP), i.e., $K \in \mathcal{K} \cap \mathbb{I}_{\mathbf{NP}}$

To get the representation of $K \in \mathcal{K}$ satisfying (NP), we take the help of an important subset of the monoid $(\mathbb{I}, \circledast)$, namely, the center. Let us recall that the center of the monoid $(\mathbb{I}, \circledast)$ is defined as follows:

$$\mathcal{Z} = \{I \in \mathbb{I} | I \circledast J = J \circledast I, \forall J \in \mathbb{I}\} \;.$$

It is well-known that the center is a commutative submonoid of any monoid. However, the relation between the center and the idempotent elements of a monoid is not clear. It is interesting to note that in the monoid $(\mathbb{I}, \circledast)$, as the following lemma illustrates, we have an inclusion relation between $\mathcal{Z}$ and $\mathcal{I}$. As will be seen later, this result plays an important role when dealing with $K \in \mathcal{K}$ satisfying (NP).

Lemma 6.1. *The center $\mathcal{Z}$ of the monoid $(\mathbb{I}, \circledast)$ is contained in the set $\mathcal{I}$, i.e., $\mathcal{Z} \subset \mathcal{I}$.*

Proof. Let $K \in \mathcal{Z}$. We need to show that $K \circledast K = K$,

$$\text{i.e., } K(x, K(x, y)) = K(x, y), \qquad x, y \in [0, 1].$$

Suppose for some $x_0 \in (0, 1], y_0 \in [0, 1)$ that

$$\alpha = K(x_0, K(x_0, y_0)) \neq K(x_0, y_0) = \beta.$$

Thus $K(x_0, \beta) = \alpha$.
Claim: $\beta \notin \{0, 1\}$.
Proof of the claim: Let $\beta = 0$, i.e., $K(x_0, y_0) = 0$ and hence $K(x_0, K(x_0, y_0)) = K(x_0, 0) = \alpha \neq \beta = 0$. Then let us consider $I_{\mathbf{0}} \in \mathbb{I}$ as given in (9). Now,

$$(I_{\mathbf{0}} \circledast K)(x_0, y_0) = I_{\mathbf{0}}(x_0, K(x_0, y_0)) = I_{\mathbf{0}}(x_0, 0) = 0,$$

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$$(K \circledast I_{\mathbf{0}})(x_0, y_0) = K(x_0, I_{\mathbf{0}}(x_0, y_0)) = K(x_0, 0) = \alpha \neq 0.$$

Thus $I_{\mathbf{0}} \circledast K \neq K \circledast I_{\mathbf{0}}$, contradicting the fact $K \in \mathcal{Z}$. Thus $\beta \neq 0$.

Let $\beta = 1$, i.e., $K(x_0, y_0) = 1$. Then it implies that $K(x_0, K(x_0, y_0)) = 1$, contradicting our assumption $K(x_0, y_0) \neq K(x_0, K(x_0, y_0))$. Thus $\beta \neq 1$.

Claim: $\alpha \neq 1$.

Proof of the claim: Let $\alpha = 1$, i.e., $K(x_0, K(x_0, y_0)) = \alpha = 1$. We have already proven that $\beta \neq 0, 1$. Now define $I_\beta \in \mathbb{I}$ as in (6).

Now, $I_\beta(x_0, K(x_0, y_0)) = \beta$ and $K(x_0, I_\beta(x_0, y_0)) = K(x_0, \beta) = \alpha = 1$. Thus

$$I_\beta(x_0, K(x_0, y_0)) \neq K(x_0, I_\beta(x_0, y_0)),$$

a contradiction to the fact that $K \in \mathcal{Z}$. Thus $\alpha \neq 1$.

Now, $I_\beta(x_0, K(x_0, \beta)) = I_\beta(x_0, \alpha) = \beta$ and $K(x_0, I_\beta(x_0, \beta)) = K(x_0, \beta) = \alpha$. Thus

$$I_\beta(x_0, K(x_0, \beta)) \neq K(x_0, I_\beta(x_0, \beta)),$$

a contradiction to the fact that $K \in \mathcal{Z}$. Thus $K \in \mathcal{I}$ and hence $\mathcal{Z} \subset \mathcal{I}$. □

Remark 6.2. In Lemma 6.1, the inclusion is strict. To see this, let us consider $I_{\mathbf{1}} \in \mathbb{I}$ as given in (7). Then it is strightforward to see that $I_{\mathbf{1}} \circledast I_{\mathbf{1}} = I_{\mathbf{1}}$, i.e., $I_{\mathbf{1}} \in \mathcal{I}$. However, at $x = 1, y = 0.4$ we observe that

$$(I_{\mathbf{1}} \circledast I_{\mathbf{0}})(1, 0.4) = I_{\mathbf{1}}(1, I_{\mathbf{0}}(1, 0.4)) = I_{\mathbf{1}}(1, 0) = 0,$$

$$(I_{\mathbf{0}} \circledast I_{\mathbf{1}})(1, 0.4) = I_{\mathbf{0}}(1, I_{\mathbf{1}}(1, 0.4)) = I_{\mathbf{0}}(1, 1) = 1.$$

which implies that $I_{\mathbf{1}} \circledast I_{\mathbf{0}} \neq I_{\mathbf{1}} \circledast I_{\mathbf{0}}$ and consequently $I_{\mathbf{1}} \notin \mathcal{Z}$. Similarly, one can observe that $I_{\mathbf{GD}}, I_{\mathbf{0}} \in \mathcal{I}$ but $I_{\mathbf{GD}}, I_{\mathbf{0}} \notin \mathcal{Z}$.

Based on Lemma 6.1, we have a first partial characterisation of $K \in \mathcal{K} \cap \mathbb{I}_{\mathbf{NP}}$.

Lemma 6.3. *If $K \in \mathcal{Z}$ then g_K is an s.g.h., i.e., $\mathcal{Z} \subsetneq \mathcal{K}$.*

Proof. Let $K \in \mathcal{Z}$. Then, from Lemma 6.1, it follows that $K \in \mathcal{I}$. Let $I, J \in \mathbb{I}$. Now,

$$\begin{aligned}
g_K(I) \circledast g_K(J) &= (I \circledast K) \circledast (J \circledast K) \\
&= (I \circledast K) \circledast (K \circledast J) \\
&= I \circledast (K \circledast K) \circledast J \\
&= I \circledast (K \circledast J) = I \circledast (J \circledast K) \\
&= (I \circledast J) \circledast K = g_K(I \circledast J).
\end{aligned}$$

Thus g_K is an *s.g.h.* □

In fact, as we show in the following the converse of Lemma 6.3 is also true, i.e., any $K \in \mathbb{I}_{\mathbf{NP}}$ for which g_K is an *s.g.h.* also belongs to the center $\mathcal{Z}$, i.e., $\mathcal{K} \cap \mathbb{I}_{\mathbf{NP}} \subseteq \mathcal{Z}$. Before proving this fact, we need the following result which gives a complete characterisation of $\mathcal{K} \cap \mathbb{I}_{\mathbf{NP}}$.

Lemma 6.4. *If $K \in \mathcal{Z}$, then the range of K is non-trivial.*

Proof. Let $K \in \mathcal{Z}$. Suppose that the range of K is trivial. Since $K \in \mathcal{Z}$, from Lemma 6.3 it follows that g_K is an *s.g.h.* Again from Theorem 4.1, it follows that $K = K^\delta$ for some $\delta \in (0, 1]$. Here we claim that $\delta \neq 1$. If $\delta = 1$, then $K = I_{\mathbf{SW}}$ given by (8), which is recalled here for convenience:

$$K^1(x, y) = I_{\mathbf{SW}}(x, y) = \begin{cases} 1, & \text{if } x < 1 \text{ or } y = 1, \\ 0, & \text{if } x = 1 \text{ and } y \neq 1. \end{cases}$$

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Now it is easy to see that $(I_1 \circledast K)(1, 0.2) = 0$ where as $(K \circledast I_1)(1, 0.2) = 1$, proving that $K \notin \mathcal{Z}$, a contradiction to the fact $K \in \mathcal{Z}$. Thus $\delta \neq 1$. Now, it is easy to find two real numbers $\delta', \delta'' \in (0, 1]$ such that $\delta'' < \delta < \delta'$. Let $I = I_\beta$ as defined in (6) with $\beta = \delta''$. Then

$$(I \circledast K^\delta)(1, \delta') = I(1, K^\delta(1, \delta')) = 1,$$
$$\text{while, } (K^\delta \circledast I)(1, \delta') = K^\delta(1, I(1, \delta')) = K^\delta(1, \delta'') = 0,$$

contradicting that $K \in \mathbb{Z}$. Thus the range of K is non-trivial. $\qquad\square$

Proposition 6.5. *If $K \in \mathcal{Z}$, then K satisfies* (NP).

Proof. Let $K \in \mathcal{Z}$. From Lemma 6.3, it follows that g_K is an *s.g.h* and also from Lemma 6.4, it follows that range of K is non-trivial. To prove that K has (NP), from Proposition 5.8 it suffices to show that $K(1, y) \neq 0$ or 1 for any $y \in (0, 1)$.

On the contrary, let $K(1, y_0) = 0$ for some $y_0 \in (0, 1)$. Then, on the one hand,

$$(I_1 \circledast K)(1, y_0) = I_1(1, K(1, y_0)) = I_1(1, 0) = 0 \ ,$$

and on the other hand,

$$(K \circledast I_1)(1, y_0) = K(1, I_1(1, y_0)) = K(1, 1) = 1 \ ,$$

which contradicts the fact that $K \in \mathcal{Z}$. Thus for any $y_0 \in (0, 1)$, $K(1, y_0) \neq 0$.

Similarly, by taking I_0 instead of I_1, above we can show that for any $y_0 \in (0, 1)$, $K(1, y_0) \neq 1$. From Proposition 5.8, we see that this is equivalent to stating $\epsilon_0 = 0$ and $\epsilon_1 = 1$ and hence it follows that K must have (NP). $\qquad\square$

We define below a special class of fuzzy implications satisfying (NP).

Definition 6.6. For $\epsilon \in [0, 1)$ define

$$K_\epsilon(x, y) = \begin{cases} 1, & \text{if } x \leq \epsilon \ , \\ y, & \text{if } x > \epsilon, \end{cases} \tag{16}$$

and for $\epsilon = 1$, $K_\epsilon = I_{\mathbf{WB}}$ where

$$I_{\mathbf{WB}}(x, y) = \begin{cases} 1, & \text{if } x < 1, \\ y, & \text{if } x = 1. \end{cases} \tag{17}$$

Note that $K_\epsilon \in \mathbb{I}$, for all $\epsilon \in [0, 1]$. For notational convenience, we denote the set of all such K_ϵ fuzzy implications by

$$\mathbb{K}_\epsilon = \{I \in \mathbb{I} | I = K_\epsilon \text{ for some } \epsilon \in [0, 1]\}.$$

Clearly, $\sup \mathbb{K}_\epsilon = K_1 = I_{\mathbf{WB}}$ and $\inf \mathbb{K}_\epsilon = K_0 = I_{\mathbf{D}}$.

The following results list a few properties of fuzzy implications from the set $\mathbb{K}_\epsilon$, whose proofs are straightforward and hence are omitted.

Proposition 6.7. *The following properties hold true.*

(i) $\epsilon_1 < \epsilon_2 \implies K_{\epsilon_1} \leq K_{\epsilon_2}$
(ii) $K_{\epsilon_1} \circledast K_{\epsilon_2} = K_{\max(\epsilon_1, \epsilon_2)} = K_{\epsilon_2} \circledast K_{\epsilon_1}$
(iii) $\epsilon_1 < \epsilon_2 \implies g_{K_{\epsilon_1}}(K_{\epsilon_2}) = g_{K_{\epsilon_2}}(K_{\epsilon_1}) = K_{\epsilon_2}$
(iv) $g_{K_\epsilon}(I) = g_I(K_\epsilon), \ \text{for all } I \in \mathbb{I}$
(v) $\epsilon_1 < \epsilon_2 \implies g_{K_{\epsilon_2}}(\mathbb{I}) \subset g_{K_{\epsilon_1}}(\mathbb{I}).$

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Proposition 6.8. $(\mathbb{K}_\epsilon, \circledast)$ *is a commutative submonoid of* $(\mathbb{I}, \circledast)$.

The following result presents the inclusions among the sets $\mathbb{K}_\epsilon, \mathcal{Z}, \mathcal{K}$ and $\mathcal{I}$.

Lemma 6.9. $\mathbb{K}_\epsilon \subseteq \mathcal{Z} \subsetneq \mathcal{K} \subsetneq \mathcal{I}$.

Proof. In Lemma 6.3, we proved that $\mathcal{Z} \subset \mathcal{K}$. Again from Lemma 3.2 it follows that $\mathcal{K} \subset \mathcal{I}$ and hence we have $\mathcal{Z} \subsetneq \mathcal{K} \subsetneq \mathcal{I}$. So here it is enough to show that $\mathbb{K}_\epsilon \subseteq \mathcal{Z}$. Let $I \in \mathbb{K}_\epsilon$. For any $J \in \mathbb{I}$ and $I = K_\epsilon \in \mathbb{K}_\epsilon$ for some $\epsilon \in [0,1)$, we have

$$(I \circledast J)(x,y) = (J \circledast I)(x,y) = \begin{cases} 1, & \text{if } x \leq \epsilon, \\ J(x,y), & \text{if } x > \epsilon, \end{cases}$$

showing that $I \in \mathcal{Z}$. For $\epsilon = 1$, $I = K_\epsilon = I_{\mathbf{WB}}$, then

$$(I \circledast J)(x,y) = (J \circledast I)(x,y) = \begin{cases} 1, & \text{if } x < 1, \\ J(1,y), & \text{if } x = 1, \end{cases}$$

for all $J \in \mathbb{I}$. Thus $I_{\mathbf{WB}} \in \mathcal{Z}$. $\qquad\qquad\square$

In fact, the opposite inclusion, i.e., $\mathcal{Z} \subseteq \mathbb{K}_\epsilon$, is also true, a fact that we prove in Lemma 6.11. Now, we are ready to give a complete characterisation and representation of $K \in \mathbb{I}$ satisfying (NP) for which g_K will be an *s.g.h.*

Theorem 6.10. *The following statements are equivalent:*

 (i) $K \in \mathcal{K}$ *and satisfies* (NP).
 (ii) $K \in \mathbb{K}_\epsilon$.

Proof. (i) $\implies$ (ii): Let $K \in \mathcal{K}$ satisfy (NP). Since K has (NP) the range of K is $[0,1]$. Let $\alpha < 1$ be chosen arbitrarily. Then there exists some $x_0 \in (0,1], y_0 \in [0,1)$, such that $K(x_0, y_0) = \alpha < 1$. We keep K fixed, vary J and investigate the equivalence $J \circledast K = K \circledast J \circledast K$.
When $J = I_{\mathbf{0}}$, we have

$$\begin{aligned} (J \circledast K)(x_0, y_0) &= I_{\mathbf{0}}(x_0, K(x_0, y_0)) \\ &= I_{\mathbf{0}}(x_0, \alpha) = 0, \\ (K \circledast J \circledast K)(x_0, y_0) &= K(x_0, I_{\mathbf{0}}(x_0, K(x_0, y_0))) \\ &= K(x_0, 0). \end{aligned}$$

Since g_K is an *s.g.h.*, $K(x_0, 0) = 0$. Hence, if $K(x_0, y_0) = \alpha < 1$, then $K(x_0, 0) = 0$. Now,

$$\begin{aligned} (J \circledast K)(x_0, 0) &= J(x_0, K(x_0, 0)) \\ &= J(x_0, 0), \\ \text{and } (K \circledast J \circledast K)(x_0, 0) &= K(x_0, J(x_0, K(x_0, 0))) \\ &= K(x_0, J(x_0, 0)). \end{aligned}$$

Now let us, once again, choose J as in (6) with $\beta = y_0$. Thus we have $J(x_0, 0) = y_0$ and hence

$$\begin{aligned} y_0 = J(x_0, 0) &= K(x_0, J(x_0, 0)) \\ &= K(x_0, y_0) = \alpha. \\ \implies \alpha &= y_0. \end{aligned}$$

Since α is chosen arbitrarily, we have

$$K(x_0, y) = y, \qquad y \in [0,1]. \tag{18}$$

Let $x^* = \inf\{x|K(x,y) = y,\ \text{for all}\ \ y\} \geq 0$. Note that the infimum exists because K has (NP), i.e., $1 \in \{x|K(x,y) = y,\ \text{for all}\ \ y\} \neq \emptyset$.

Claim: $K(s,y) = 1$, for any $s \in [0,x^*)$ and for all $y \in [0,1]$.

Proof of the claim: On the contrary, let us suppose that $1 > K(s,y_0) = y_1 > y_0$ for some $y_0, y_1 \in [0,1)$ and $s \in [0,x^*)$. Now,

$$J(s, K(s,y_0)) = J(s,y_1)\ ,$$
$$\text{and,}\ K(s, J(s, K(s,y_0))) = K(s, J(s,y_1))\ .$$

Once again, choosing a J as in (6) with $\beta = y_0$, we have

$$J(s,y_1) = y_0\ \text{and}\ K(s, J(s,y_1)) = K(s,y_0) = y_1\ ,$$
$$\implies J(s, K(s,y_0)) \neq K(s, J(s, K(s,y_0))),$$

i.e., g_K is not an *s.g.h.*, a contradiction. Thus $K(s,y) = 1$, for all $s \in [0,x^*)$.
Now the question is what value should one assign to $K(x^*,y)$. Since it is customary to assume left-continuity of fuzzy implications in the first variable, we let $K(x^*,y) = 1$. Note that letting $K(x^*,y) = y$ also gives a K such that g_K is an *s.g.h.*
From the above claim and (18) we see that every K is of the form (16) for some $\epsilon \in [0,1)$ or $K = I_{\mathbf{WB}}$.

(ii) $\implies$ (i) : That $\mathbb{K}_\epsilon \subsetneq \mathcal{K}$ follows from Lemma 6.9, while that every $K \in \mathbb{K}_\epsilon$ satisfies (NP) follows from Proposition 6.5. $\qquad\square$

Lemma 6.11. *Let $K \in \mathcal{Z}$. Then $K \in \mathbb{K}_\epsilon$, i.e., $\mathcal{Z} \subseteq \mathbb{K}_\epsilon$.*

Proof. Let $K \in \mathcal{Z}$. From Lemma 6.3 it follows that g_K is an *s.g.h* and also from Lemma 6.4 it follows that range of K is non-trivial. Further, from Proposition 6.5 we know that K has (NP). Again from Theorem 6.10 it follows that $K \in \mathbb{K}_\epsilon$. $\qquad\square$

Corollary 6.12. $\mathcal{Z} = \mathbb{K}_\epsilon$.

Proof. In Lemma 6.9, we proved that $\mathcal{Z} \supseteq \mathbb{K}_\epsilon$. From Lemma 6.11 it follows that $\mathcal{Z} \subseteq \mathbb{K}_\epsilon$. $\qquad\square$

Remark 6.13. From the results leading up to Corollary 6.12, one can clearly see the important role played by the right translation semigroup homomorphisms g_K in determining the center $\mathcal{Z}$ of the monoid $\mathbb{I}$. Further, note that, not only do we have characterised the center but also have clear representations of the fuzzy implications K satisfying the functional equation $I \circledast K = K \circledast I$ for all $I \in \mathbb{I}$.

7. Representations of $K \in \mathcal{K}$ such that $K(1,y) = 0$ for all $y \in [0,1)$

Recall from Corollary 5.11 that if the range of $K \in \mathcal{K}$ is non-trivial then either, $K(1,y) = y$ for all $y \in [0,1]$ or $K(1,y) = 0$ for all $y \in [0,1)$. In Section 6, we have characterised and found representations of fuzzy implications K such that g_K is an *s.g.h.* in the case of $K(1,y) = y$ for all $y \in [0,1]$, i.e., K has (NP). Now it remains to characterise the non-trivial range non-neutral implications $K \in \mathcal{K}$ give their presentations. We take up this task in this section.

We begin this section by defining the following class of fuzzy implications.

Definition 7.1. For $\epsilon \in (0,1]$, define

$$K^\epsilon(x,y) = \begin{cases} 1, & \text{if } x < \epsilon\ , \\ y, & \text{if } \epsilon \leq x < 1\ , \\ 0, & \text{if } x = 1\ \&\ y \neq 1 \end{cases} \tag{19}$$

and for $\epsilon = 0$, define $K^\epsilon(x,y) = I_{\mathbf{IW}}(x,y) = \begin{cases} 1, & \text{if } x = 0\ , \\ y, & \text{if } x < 1\ , \\ 0, & \text{if } x = 1\ \&\ y \neq 1\ . \end{cases}$

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For notational convenience, we denote the set of all such K^ϵ fuzzy implications by

$$\mathcal{W} = \mathbb{K}^\epsilon = \{I \in \mathbb{I} | I = K^\epsilon \text{ for some } \epsilon \in [0,1]\}.$$

Note once again that $\sup \mathcal{W} = I_{\mathbf{SW}}$ (see (8)) and $\inf \mathcal{W} = I_{\mathbf{IW}}$.

Theorem 7.2. *The following statements are equivalent:*

(i) $K \in \mathcal{K}$ *and* $K(1,y) = 0$ *for all* $y \neq 1$.

(ii) $K \in \mathbb{K}^\epsilon$.

Proof. (i) $\implies$ (ii): Let $K \in \mathcal{K}$ be such that $K(1,y) = 0$ for all $y \neq 1$. Since the range of K is non-trivial, from Lemma 5.1, the range of K is whole of $[0,1]$. Let $0 < \alpha < 1$ be chosen arbitrarily. Then there exist some $x_0 \in (0,1), y_0 \in [0,1)$, such that $0 < K(x_0, y_0) = \alpha < 1$. We keep K fixed, vary $J \in \mathbb{I}$ and investigate the equivalence $J \circledast K = K \circledast J \circledast K$.

When $J = I_{\mathbf{0}}$, we have

$$(J \circledast K)(x_0, y_0) = I_{\mathbf{0}}(x_0, K(x_0, y_0)) = I_{\mathbf{0}}(x_0, \alpha) = 0 \ ,$$
$$(K \circledast J \circledast K)(x_0, y_0) = K(x_0, I_{\mathbf{0}}(x_0, K(x_0, y_0))) = K(x_0, 0).$$

Since g_K is an *s.g.h.*, $K(x_0, 0) = 0$. Hence, if $K(x_0, y_0) = \alpha < 1$, then $K(x_0, 0) = 0$. Now, for any $J \in \mathbb{I}$, we have

$$(J \circledast K)(x_0, 0) = J(x_0, K(x_0, 0)) = J(x_0, 0),$$
$$\text{and } (K \circledast J \circledast K)(x_0, 0) = K(x_0, J(x_0, K(x_0, 0))) = K(x_0, J(x_0, 0)).$$

Now let us, once again, choose $J \in \mathbb{I}$ such that $J(x_0, 0) = y_0$. Then

$$y_0 = J(x_0, 0) = K(x_0, J(x_0, 0)) = K(x_0, y_0) = \alpha,$$

which implies that $\alpha = y_0$. Since α is chosen arbitrarily, we have

$$K(x_0, y) = y, \qquad y \in [0,1]. \tag{20}$$

Let $x^* = \inf\{x | K(x,y) = y, \text{ for all } y\} \geq 0$. Note that the infimum exists because x_0 satisfies (20).

Claim: $K(s,y) = 1$, for any $s \in [0, x^*)$ and for all $y \in [0,1]$.

Proof of the claim: On the contrary, let us suppose that $1 > K(s, y_0) = y_1 > y_0$ for some y_0, y_1. Now,

$$J(s, K(s, y_0)) = J(s, y_1) \ ,$$
$$K(s, J(s, K(s, y_0))) = K(s, J(s, y_1)) \ .$$

Once again, choosing a $J \in \mathbb{I}$ such that $J(s, y_1) = y_0$, we get

$$J(s, y_1) = y_0 \text{ and } K(s, J(s, y_1)) = K(s, y_0) = y_1 \ ,$$
$$\implies J(s, K(s, y_0)) \neq K(s, J(s, K(s, y_0))),$$

i.e., g_K is not an *s.g.h.*, a contradiction. Thus $K(s,y) = 1$, for all $s \in [0, x^*)$.
Now the question is what value should one assign to $K(x^*, y)$. Since it is customary to assume left-continuity of fuzzy implications in the first variable, we let $K(x^*, y) = 1$. Note that letting $K(x^*, y) = y$ also gives a K such that g_K is a homomorphism.
From the above claim and (20) we see that every K is of the form (19) for some $\epsilon \in [0,1)$.

(ii) $\implies$ (i): That for every $K \in \mathbb{K}^\epsilon$ the g_K is an *s.g.h.* can be easily verified and hence $\mathbb{K}^\epsilon \subsetneq \mathcal{K}$.

Further, from the definition of K^ϵ, we see that $K^\epsilon(1,y) = 0$ for all $y \in [0,1)$. $\square$

Corollary 7.3. *Let* $K \in \mathbb{I}$. *Then following statements are equivalent:*

(i) g_K *is an s.g.h.*

(ii) $K \in \mathcal{R} \cup \mathcal{Z} \cup \mathcal{W}$.

Corollary 7.4. $\mathcal{K} = \mathcal{R} \cup \mathcal{Z} \cup \mathcal{W}$.

8. The subset $\mathcal{K}$ and known solutions of (5)

In this section, we compare the solution set $\mathcal{K} = \mathcal{R} \cup \mathcal{Z} \cup \mathcal{W}$ with the known solutions of (5) as presented in [33, 37, 38]. As already noted, in the above works, which are the major works dealing with the iterative functional equation (5) and all of which have appeared in this very journal, the authors have discussed the solutions of (5) from different families of fuzzy implications. In particular,

- Shi *et al.* [33] have discussed the fuzzy implications from the families of (S, N)-, R- and QL-implications that satisfy the iterative functional equation (5).

- In [38], Xie and Qin discuss the solutions to (5) from the family of continuous D-implications, while Xie *et al.* [37] discuss the same from three families of fuzzy implications obtained from uninorms, viz., the RU-, (U, N)- and QLU-implications.

8.1. Scope of this comparative study

From Lemma 5.5, note that for any $K \in \mathcal{K}$ its natural negation N_K has trivial range. In other words, due to the non-increasingness of a fuzzy negation, we see that N_K belongs to the following family $\mathcal{N}_\tau = \{N_{\{t\}} \,|\, t \in [0, 1]\}$ of threshold negations, where for any $t \in [0, 1)$

$$N_{\{t\}}(x) = \begin{cases} 1, & \text{if } x \leq t \text{ ,} \\ 0, & \text{if } x > t \text{ ,} \end{cases}$$

and for $t = 1$ we write

$$N_{\{1\}}(x) = \begin{cases} 1, & \text{if } x < 1 \text{ ,} \\ 0, & \text{if } x = 1 \text{ .} \end{cases}$$

Note that $N_{\{0\}}, N_{\{1\}}$ are often denoted as $N_{\mathbf{D1}}, N_{\mathbf{D2}}$, respectively (see **Table 3**) and are also the smallest and largest fuzzy negations.

Thus we only compare the solutions obtained here with fuzzy implications I from the different families such that they are both solutions of (5) and whose natural negations N_I are of trivial range. For the definition of different families listed above, we refer the readers to the above cited works [33, 37, 38], or the research monographs [4] or [2]. Similarly, for definitions of t-norms, t-conorms and their properties, please refer to the excellent monograph [25].

8.2. The subset $\mathcal{K}$ and known solutions from (S, N)-implications

Let $\mathbb{I}_{\mathbb{S}, \mathcal{N}_\tau}$ denote the set of (S, N)-implications obtained from any t-conorm S and let the fuzzy negation $N \in \mathcal{N}_\tau$. Clearly, if $I \in \mathbb{I}_{\mathbb{S}, \mathcal{N}_\tau}$, then

$$I(x, y) = S(N_{\{t\}}(x), y) = \begin{cases} 1, & \text{if } x \leq t \text{ ,} \\ y, & \text{if } x > t \text{ ,} \end{cases} \quad \text{or } I = I_{\mathbf{WB}}, \text{ when } N = N_{\{1\}} = N_{\mathbf{D2}}. \quad (21)$$

From **Theorem 10** of Shi *et al.* [33], we see that every (S, N)-implication whose natural negation is of type $N_{\{t\}}$ for some $t \in [0, 1)$ is a solution of (5), i.e., if $I \in \mathbb{I}_{\mathbb{S}, \mathcal{N}_\tau}$, then I satisfies (5). In fact, these are exactly the set $\mathcal{Z}$, which is the center of the monoid $(\mathbb{I}, \circledast)$. To see this, it suffices to put $t = \epsilon$ and the formulae (16), (17) and (21) are identical, i.e., $\mathbb{I}_{\mathbb{S}, \mathcal{N}_\tau} = \mathbb{K}_\epsilon = \mathcal{Z}$.

Thus $\mathcal{Z}$ gives all such (S, N)-implications that are both solutions of (5) and whose natural negations are of trivial range. Note that while neither does $\mathcal{Z}$ enlarge the solution set of (5) nor is the representation of $\mathbb{I}_{\mathbb{S}, \mathcal{N}_\tau}$ implications difficult to obtain, our study does show that any I from this subfamily of (S, N)-implications also satisfies the following functional equation with every $J \in \mathbb{I}$:

$$I(x, J(x, y)) = J(x, I(x, y)), \qquad x, y \in [0, 1] \text{ .} \quad (2)$$

8.3. The subset $\mathcal{K}$ and known solutions from R-implications

Shi *et al.* [33] have only considered R-implications obtained from left-continuous t-norms and hence have obtained the following result, which is paraphrased suitably to our context:

Theorem 8.1 (cf. [33], Corollary 3). *An R-implication I obtained from a left-continuous t-norm satisfies (5) if and only if $I = I_{\mathbf{GD}}$, the Gödel implication.*

Our study has shown that the Weber implication $I_{\mathbf{WB}} = \sup \mathcal{Z}$ is also a solution of (5), which is an R-implication obtained from the non-left-continuous drastic t-norm $T_{\mathbf{D}}$ given as below:

$$T_{\mathbf{D}}(x, y) = \begin{cases} \min(x, y), & \text{if } \max(x, y) = 1 \;, \\ 0, & \text{otherwise} \;. \end{cases}$$

Note, however, that the following question still remains open.

Problem 8.2. (i) Are $I_{\mathbf{WB}}$ and $I_{\mathbf{GD}}$ the only R-implications that satisfy (5)?
  (ii) If not, characterise all R-implications obtained from non-left-continuous (in fact, non-border-continuous) t-norms that satisfy (5).

8.4. The subset $\mathcal{K}$ and known solutions from QL-implications

Once again, Shi *et al.* [33] have only considered QL-implications obtained from strong negations and hence it is immediately clear that the solutions obtained by them do not cover our context.

As the following result shows, the only QL-implication whose natural negation is of the threshold type, is the Weber implication $I_{\mathbf{WB}}$.

Theorem 8.3. *Let I be a QL-operation whose $N_I \in \mathcal{N}_\tau$. Then the following statements are equivalent:*

  (i) *I is a fuzzy implication.*
  (ii) *$I = I_{\mathbf{WB}}$.*

Proof. Let I be a QL-operation given by

$$I(x, y) = S(N(x), T(x, y)) \;, \tag{22}$$

for some t-conorm S, a t-norm T and a fuzzy negation N. Further, let the natural negation $N_I \in \mathcal{N}_\tau$. Then from Proposition 2.6.2 in [4], it follows that $N_I = N \in \mathcal{N}_\tau$. With this N, the QL-operation in (22) becomes

$$I(x, y) = \begin{cases} 1, & \text{if } x \le t \;, \\ T(x, y), & \text{if } x > t \;, \end{cases} \tag{23}$$

when $N = N_{\{t\}}$ for some $t \in [0, 1)$, or

$$I(x, y) = \begin{cases} 1, & \text{if } x < 1 \;, \\ y = T(1, y), & \text{if } x = 1 \;, \end{cases} \tag{24}$$

when $t = 1$, in which case $I = I_{\mathbf{WB}}$. Further, note that a QL-operation I is called a QL-implication if and only if I is a fuzzy implication. Now we show that I is a QL-implication if and only if $t = 1$.

(i) $\Longrightarrow$ (ii): Let I be a QL-implication. Suppose $t < 1$. Then choose $x_1, x_2 \in (t, 1]$ such that $x_1 < x_2$. Then from (23), it follows that $T(x_1, 1) = x_1 < x_2 = T(x_2, y)$, which implies that $I(x_1, y) < I(x_2, y)$ and hence I does not satisfy (I1), a contradiction to the fact that I is a QL-implication. Hence $t = 1$.

(ii) $\Longrightarrow$ (i): Obvious. $\qquad\square$

Thus the only QL-implication which satisfies (5) and whose natural negation is of trivial range is the Weber implication $I_{\mathbf{WB}}$.

8.5. The subset $\mathcal{K}$ and known solutions from D-implications

Once again, we have the following result, whose proof can be obtained along similar lines as that of **Theorem 8.3**.

Theorem 8.4. *Let I be a D-operation whose $N_I \in \mathcal{N}_\tau$. Then the following statements are equivalent:*

(i) *I is a fuzzy implication.*
(ii) *$I = I_{\mathbf{D}}$.*

Thus the only D-implication which satisfies (5) and whose natural negation is of trivial range is $I_{\mathbf{D}}$.

8.6. The subset $\mathcal{K}$ and known solutions from implications obtained from Uninorms

Xie *et al.* [37] discuss the solutions of (5) from the families of fuzzy implications obtained from uninorms, viz., (i) RU-implications obtained as the residuals of uninorms, (ii) (U, N)-implications obtained as a generalisation of the material implication with uninorms representing the disjunction and the QLU-implications which are the quantum logic implications suitably generalised by using conjunctive and disjunctive uninorms. Further, they have considered only the uninorms coming from the following sub-classes, viz., (i) those that are continuous on $(0, 1)^2$, which also contain the representable uninorms, (ii) the idempotent uninorms and (iii) the pseudo-continuous uninorms, i.e., uninorms verifying that both functions $U(x, 0)$ and $U(x, 1)$ are continuous except at their point of neutrality $e \in (0, 1)$.

Once again, we refer the readers to [5, 28, 37] for the definitions of RU-, (U, N)- and QLU-implications and to the following works [6, 17, 18, 39] for the definitions and further details of the different classes of listed uninorms.

In fact, their study shows that many of the considered families do not satisfy (5). For instance, in the case of RU-implications, it is clear that those that are obtained from representable uninorms or uninorms continuous on $(0, 1)^2$ do not satisfy (5), see **Theorem 2** and **Corollary 1** in [37]. Similarly, in the case of QLU-implications, the sub-families considered in [37] do not contain any solutions of (5). Hence we limit the scope of this section to only consider the overlap between the set $\mathcal{K}$ and subsets of the above three families which are known to contain solutions of the iterative functional equation (5).

8.6.1. The subset $\mathcal{K}$ and known solutions from RU-implications

Case 1: RU-implications obtained from idempotent uninorms

In the case of RU-implications obtained from idempotent uninorms, from **Theorem 13** in [37] we see that the only solutions are those that have as their natural negation the least negation $N_{\mathbf{D1}}$. Further, all of these fuzzy implications are not of trivial range. Now, the only fuzzy implications $K \in \mathcal{K}$ with non-trivial range and whose natural negation is $N_{\mathbf{D1}}$ are $I_{\mathbf{D}} \in \mathcal{Z}$ and $I_{\mathbf{SW}} \in \mathcal{W}$, which clearly do not belong to the solution set obtained in **Theorem 13** of [37].

Thus, if $\mathbb{I}_{\mathbb{U}}^{\mathbf{G}}$ denotes the set of all RU-implications obtained from idempotent uninorms, then $\mathbb{I}_{\mathbb{U}}^{\mathbf{G}} \cap \mathcal{K} = \emptyset$.

Case 2: RU-implications obtained from pseudo-continuous uninorms

Let $\mathbb{I}_{\mathbb{U}}^{\mathbf{E}}$ denote the set of all RU-implications obtained from pseudo-continuous uninorms.

From **Theorem 14** in [37] we see that, in the case of RU-implications obtained from pseudo-continuous uninorms, only the set of fuzzy implications $\mathbb{J}_e = \{I_e \in \mathbb{I} \mid e \in (0, 1)\}$ satisfies (5), where I_e is given as follows:

$$I_e(x, y) = \begin{cases} y, & \text{if } x \geq y \text{ or } x \leq y \ \& \ (x, y) \in [0, e]^2 \ , \\ 1, & \text{otherwise} \ . \end{cases}$$

Once again, it is clear to see that the natural negation of each of the fuzzy implications I_e is $N_{I_e} = N_{\mathbf{D1}}$, but neither $I_{\mathbf{D}} \in \mathbb{J}_e$ nor $I_{\mathbf{SW}} \in \mathbb{J}_e$. Thus $\mathbb{I}_{\mathbb{U}}^{\mathbf{E}} \cap \mathcal{K} = \emptyset$.

8.6.2. The subset $\mathcal{K}$ and known solutions from (U, N)-implications

In the case of (U, N)-implications, Xie *et al.* [37] have shown that every (U, N)-implication obtained from a disjunctive idempotent uninorm satisfies (5) ([37], **Theorem 17**), while in the case of (U, N)-implication obtained from disjunctive pseudo-continuous uninorms they have found the conditions under which they satisfy (5), see ([37], **Theorem 15**). Hence we discuss the overlaps that may exist between $\mathcal{K}$ and the solutions obtained in [37].

Case 1: (U, N)-implications obtained from disjunctive pseudo-continuous uninorms

Let $\mathbb{I}_{\mathbb{U,N}}^{\mathbf{E}}$ denote the set of all (U, N)-implications obtained from disjunctive pseudo-continuous uninorms.

From the formula for disjunctive pseudo-continuous uninorms whose neutral element $e \in (0, 1)$, we see that the (U, N)-implication obtained from them is given as follows:

$$I_{U,N}(x, y) = U(N(x), y) = \begin{cases} eT(\frac{N(x)}{e}, \frac{y}{e}), & \text{if } N(x), y \in [0, e], \\ e + (1 - e)S(\frac{N(x)-e}{1-e}, \frac{y-e}{1-e}), & \text{if } N(x), y \in [e, 1], \\ \max(N(x), y), & \text{otherwise} . \end{cases}$$

Note that the natural negation of any such $I_{U,N}$ is given by

$$N_{I_{U,N}}(x) = I_{U,N}(x, 0) = \begin{cases} 0, & \text{if } N(x) \in [0, e] , \\ N(x), & \text{if } N(x) \in [e, 1] . \end{cases} \tag{25}$$

Clearly, an $I \in \mathbb{I}_{\mathbb{U,N}}^{\mathbf{E}} \cap \mathcal{K}$ if and only if $N_I \in \mathcal{N}_\tau$. From (25) above we see that $I \in \mathbb{I}_{\mathbb{U,N}}^{\mathbf{E}} \cap \mathcal{K}$ if and only if the negation N used to obtain the I is a threshold negation, i.e., $N_I = N = N_t \in \mathcal{N}_\tau$ for some $t \in [0, 1]$.

From **Theorem 15** in [37] we see that any $I_t^e \in \mathbb{I}_{\mathbb{U,N}}^{\mathbf{E}}$ with $N \in \mathcal{N}_\tau$ does satisfy (5) and are given as follows:

$$I_t^e(x, y) = U(N_t(x), y) = \begin{cases} 1, & \text{if } x \in [0, t], y \in [0, 1] , \\ y, & \text{if } x \in (t, 1], y \in (e, 1] , \\ 0, & \text{otherwise} . \end{cases} \tag{26}$$

$\underline{\mathbb{I}_{\mathbb{U,N}}^{\mathbf{E}} \cap \mathcal{R}}$: Note, firstly, that for each $e, t \in (0, 1)$, the fuzzy implications I_t^e are of non-trivial range and hence $\mathbb{I}_{\mathbb{U,N}}^{\mathbf{E}} \cap \mathcal{R} = \emptyset$.

$\underline{\mathbb{I}_{\mathbb{U,N}}^{\mathbf{E}} \cap \mathcal{Z}}$: From (26), we see that $I_t^e(1, y) = U(N_t(1), y) = \begin{cases} 0, & \text{if } y \in [0, e] , \\ y, & \text{if } y \in (e, 1] . \end{cases}$

Thus I_t^e does not satisfy (NP) for any $e \in (0, 1)$. Since every $I \in \mathcal{Z}$ satisfies (NP), we see that $I \in \mathbb{I}_{\mathbb{U,N}}^{\mathbf{E}} \cap \mathcal{Z} = \emptyset$.

$\underline{\mathbb{I}_{\mathbb{U,N}}^{\mathbf{E}} \cap \mathcal{W}}$: Finally, since every $I \in \mathcal{W}$ is such that $I(1, y) = 0$ for all $y \in [0, 1)$, we see that $\mathbb{I}_{\mathbb{U,N}}^{\mathbf{E}} \cap \mathcal{W} = \emptyset$.

Case 2: (U, N)-implications obtained from disjunctive idempotent uninorms

Let $\mathbb{I}_{\mathbb{U,N}}^{\mathbf{G}}$ denote the set of all (U, N)-implications obtained from disjunctive idempotent uninorms. Let $g(0) = \alpha \in (0, 1]$.

From **Theorem 17** in [37] we see that any $I_{U,N}^g \in \mathbb{I}_{\mathbb{U,N}}^{\mathbf{G}}$ satisfies (5) and is given by:

$$I_{U,N}^g(x, y) = U(N(x), y) = \begin{cases} \min(N(x), y), & y < g(N(x)) \text{ or } (y = g(N(x))) \text{ and } N(x) < g(g(N(x))) , \\ \max(N(x), y), & y > g(N(x)) \text{ or } (y = g(N(x))) \text{ and } N(x) > g(g(N(x))) , \\ N(x) \text{ or } y, & \text{otherwise} . \end{cases}$$

Note that the natural negation of any such $I_{U,N}$ is given by

$$N_{I_{U,N}^g}(x) = I_{U,N}^g(x,0) = \begin{cases} 0, & \text{if } 0 < g(N(x)) \text{ or } \big(0 = g(N(x)) \ \& \ 0 < g(g(N(x)))\big) , \\ N(x), & \text{if } \big(0 = g(N(x)) \ \& \ 0 < g(g(N(x)))\big) . \end{cases} \quad (27)$$

Once again, it can be easily shown that an $I \in \mathbb{I}_{\mathbb{U},\mathbb{N}}^{\mathbf{G}} \cap \mathcal{K}$ if and only if $N_I = N \in \mathcal{N}_\tau$.

Further, any $I^g \in \mathbb{I}_{\mathbb{U},\mathbb{N}}^{\mathbf{G}}$ obtained from a threshold negation $N = N_t \in \mathcal{N}_\tau$ has the following form:

$$I^g(x,y) = U(N_t(x),y) = \begin{cases} 1, & \text{if } x \in [0,t], y \in [0,1] , \\ y, & \text{if } x \in (t,1] \text{ and } \{y > g(0) \text{ or } \big(y = \alpha \ \& \ 0 = g(\alpha)\big)\} , \\ 0, & \text{if } x \in (t,1] \text{ and } \{y < g(0) \text{ or } \big(y = \alpha \ \& \ 0 < g(\alpha)\big)\} . \end{cases} \quad (28)$$

It is immediately clear from (28) that

$$I^g(1,y) = \begin{cases} 0, & \text{if } y < \alpha \text{ or } \big(y = \alpha \ \& \ 0 < g(\alpha)\big) , \\ y, & \text{if } y > \alpha \text{ or } \big(y = \alpha \ \& \ 0 = g(\alpha)\big) . \end{cases} \quad (29)$$

To discuss the intersection of $\mathbb{I}_{\mathbb{U},\mathbb{N}}^{\mathbf{G}}$ with $\mathcal{K}$, it is sufficient to investigate the different values $\alpha = g(0)$ can assume.

Case 1: $g(0) = \alpha = 1$: From (29) above, it is clear that I^g satisfies (NP) only if $\alpha = 0 = g(0)$, which is a contradiction to the fact that g is the associated function of U. Since every $I \in \mathcal{Z}$ satisfies (NP), we see that $I \in \mathbb{I}_{\mathbb{U},\mathbb{N}}^{\mathbf{G}} \cap \mathcal{Z} = \emptyset$.

Case 2: $g(0) = \alpha \in (0,1)$: In the case $g(0) = \alpha \in (0,1)$, we see that the range of $I^g(1,y)$ is non-trivial and hence $\mathbb{I}_{\mathbb{U},\mathbb{N}}^{\mathbf{G}} \cap \mathcal{R} = \emptyset$. Further, for any $\beta > \alpha$, we have that $I^g(1,\beta) = \beta \neq 0$ and hence $I^g \notin \mathcal{W}$, i.e., $\mathbb{I}_{\mathbb{U},\mathbb{N}}^{\mathbf{G}} \cap \mathcal{W} = \emptyset$.

Case 3: $g(0) = \alpha = 1$: Finally, let us consider the case $g(0) = \alpha = 1$, in which case from (28) we obtain

$$I^g(x,y) = \begin{cases} 1, & \text{if } x \leq t \text{ or } y = 1 , \\ 0, & \text{otherwise} . \end{cases}$$

Clearly, the only $I^g \in \mathbb{I}_{\mathbb{U},\mathbb{N}}^{\mathbf{G}} \cap \mathcal{K}$ is the fuzzy implication generated from $N_{\{1\}} = N_{\mathbf{D2}}$, i.e., $I^g = I_{\mathbf{SW}} \in \mathcal{W} \cap \mathcal{R}$.

8.7. New solutions of (5) - The subsets $\mathcal{R}, \mathcal{W}$

Let us denote the families of (S,N)-, R-, QL-, and D-implications as $\mathbb{I}_{\mathbb{S},\mathbb{N}}, \mathbb{I}_{\mathbb{T}}, \mathbb{I}_{\mathbb{QL}}$ and $\mathbb{I}_{\mathbb{D}}$, respectively. Then we can summarise the above discussions as follows:

$$\begin{aligned} \mathcal{K} \cap \mathbb{I}_{\mathbb{S},\mathbb{N}} &= \mathbb{I}_{\mathbb{S},\mathcal{N}_\tau} = \mathcal{Z} , \\ \mathcal{K} \cap \mathbb{I}_{\mathbb{T}} &= I_{\mathbf{WB}} = \sup \mathcal{Z} , \\ \mathcal{K} \cap \mathbb{I}_{\mathbb{QL}} &= I_{\mathbf{WB}} = \sup \mathcal{Z} , \\ \mathcal{K} \cap \mathbb{I}_{\mathbb{D}} &= I_{\mathbf{D}} = \inf \mathcal{Z} , \\ \mathcal{K} \cap \mathbb{I}_{\mathbb{U}}^{\mathbf{E}} &= \emptyset , \\ \mathcal{K} \cap \mathbb{I}_{\mathbb{U}}^{\mathbf{G}} &= \emptyset , \\ \mathcal{K} \cap \mathbb{I}_{\mathbb{U},\mathbb{N}}^{\mathbf{E}} &= \emptyset , \\ \mathcal{K} \cap \mathbb{I}_{\mathbb{U},\mathbb{N}}^{\mathbf{G}} &= I_{\mathbf{SW}} = \sup \mathcal{W} = \inf \mathcal{R} . \end{aligned}$$

Thus we see that our study does provide new solutions of the iterative functional equation (5) in the form of fuzzy implications contained in the subsets $\mathcal{R}, \mathcal{W}$ of $\mathbb{I}$.

It should, however, be emphasised that the solution set $\mathcal{K}$ does not characterise all fuzzy implications which are both solutions of (5) and whose natural negation is of trivial range. While the Gödel implication $I_{\mathbf{GD}}$ is one such example, see Remark 3.3, there exist many such fuzzy implications.

9. Concluding Remarks

In this work, we have investigated the generative power of the $\circledast$-composition proposed in [34]. Specifically, we set out to find the set of fuzzy implications $\mathcal{K} \subset \mathbb{I}$ that did not give rise to new fuzzy implications either on self-composition or when composed with other fuzzy implications and those that were $\circledast$-commutative with every fuzzy implication in $\mathbb{I}$. This led us to study three functional equations involving fuzzy implications.

Since we do not make any assumptions on $I, J \in \mathbb{I}$ employed in the above functional equations, viz., their form, properties, representations or the families they come from - as is done in many of the works, we pursued an algebraic approach towards determining $\mathcal{K}$. Note that considering the $\circledast$-composition as a binary operation on the set of all fuzzy implications $\mathbb{I}$, one obtains a monoid structure on $(\mathbb{I}, \circledast)$.

In particular, by determining the set of all fuzzy implications K whose right translations g_K were also semigroup homomorphisms on the monoid $(\mathbb{I}, \circledast)$ we have obtained clear and complete characterisations and representations of $\mathcal{K}$. The obtained solutions not only answer the question of the generative power of the $\circledast$-composition but also offer new and as yet unknown solutions to the well-studied iterative functional equation involving fuzzy implications, viz., $I(x, I(x, y)) = I(x, y)$.

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