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Yager's new class of implications J_f and some classical tautologies

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Abstract

Recently, Yager [R. Yager, On some new classes of implication operators and their role in approximate reasoning, Information Sciences 167 (2004) 193–216] has introduced a new class of fuzzy implications, denoted J_f, called the f-generated implications and has discussed some of their desirable properties, such as neutrality, exchange principle, etc. In this work, we discuss the class of J_f implications with respect to three classical logic tautologies, viz., distributivity, law of importation and contrapositive symmetry. Necessary and sufficient conditions under which J_f implications are distributive over t-norms and t-conorms and satisfy the law of importation with respect to a t-norm have been presented. Since the natural negations of J_f implications, given by $N_{J_f}(x) = J_f(x, 0)$, in general, are not strong, we give sufficient conditions under which they become strong and possess contrapositive symmetry with respect to their natural negations. When the natural negations of J_f are not strong, we discuss the contrapositivisation of J_f. Along the lines of J_f implications, a new class of implications called h-generated implications, J_h, has been proposed and the interplay between these two types of implications has been discussed. Notably, it is shown that while the natural negations of J_f are non-filling those of J_h are non-vanishing, properties which determine the compatibility of a contrapositivisation technique.

Keywords: Yager's Implications; f-Generated implications; Distributivity of fuzzy implications; Law of importation; Contrapositive symmetry; Contrapositivisation; h-Generated implications

1. Introduction

Fuzzy implication operators play an important role both in Approximate Reasoning and Fuzzy Control Theory. The most established and well-studied classes of fuzzy implications are R-, S- and QL-implications (see, for example, [15,16,19,25] for their definitions and properties). Recently, Yager [31] has introduced a new class of implications, denoted J_f, called the f-generated implications – which in general are different from

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the above categories (see [3,4]) – and discussed their desirable properties as listed in [19], such as neutrality, exchange principle, etc.

1.1. Motivation for this work

Lately there has been a spate of works that discusses and explores the validity of many classical logic tautologies in fuzzy logic, especially those that involve fuzzy implications. Three such classical logic tautologies involving fuzzy implications that have obtained maximum attention from researchers are those that deal with the distributivity of fuzzy implications over t-norms and t-conorms, the satisfaction of the law of importation with respect to a t-norm and contrapositive symmetry.

Recently there has been a lot of discussion [7,12–14,22,26] centred around a paper by Combs and Andrews [11] where they attempt to exploit the equivalence

$$(p \wedge q) \rightarrow r \equiv (p \rightarrow r) \vee (q \rightarrow r) \tag{1}$$

towards eliminating combinatorial rule explosion in fuzzy systems. (1) is only one of four such equations as listed in [17], which deals with the distributivity of implication operators with respect to t-norms and t-conorms, the rest of them being

$$(p \vee q) \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r), \tag{2}$$
$$r \rightarrow (s \wedge t) \equiv (r \rightarrow s) \wedge (r \rightarrow t), \tag{3}$$
$$r \rightarrow (s \vee t) \equiv (r \rightarrow s) \vee (r \rightarrow t). \tag{4}$$

In [26], Trillas and Alsina have investigated the conditions under which the following general form of (1), where $p, q, r \in [0, 1]$

$$J(T(p, q), r) \equiv S(J(p, r), J(q, r)) \tag{5}$$

holds for the classes of R-, S- and QL-implications, where T and S denote any t-norm and t-conorm respectively. The generalisations of Eqs. (2)–(4) are as follows:

$$J(S(p, q), r) \equiv T(J(p, r), J(q, r)), \tag{6}$$
$$J(r, T_1(s, t)) \equiv T_2(J(r, s), J(r, t)), \tag{7}$$
$$J(r, S_1(s, t)) \equiv S_2(J(r, s), J(r, t)), \tag{8}$$

where $p, q, r, s, t \in [0, 1]$.

Conditions under which Eqs. (6)–(8) hold for R- and S-implications have appeared in [7]. Except for the case when J is an R-implication obtained from a strict t-norm T and $S_1 = S_2$ is a nilpotent t-conorm in (8), in all the other cases, if J is an R- or an S-implication, the t-norm T and the t-conorm S do get fixed to $T_{\mathbf{M}}(x, y) = \min(x, y)$ and $S_{\mathbf{M}}(x, y) = \max(x, y)$, respectively. Also Eq. (7) has been discussed in [1,2] under the assumptions that $T = T_1 = T_2$ is a strict t-norm and the implication J is continuous except at $(0, 0)$.

The above equations play an important role in lossless rule reduction in Fuzzy Rule Based Systems [5,6,24,29]. Thus it becomes both interesting and important to discuss the validity of these distributive equations for a given fuzzy implication in the hope of obtaining t-norms T and t-conorms S other than $T_{\mathbf{M}}$ and $S_{\mathbf{M}}$, respectively. We will see below that the family of f-generated implications has more solutions to (7) than an R- or an S-implication. On the other hand, for Yager's f-generated implication too the solution to (8) is not fully settled. Thus it is worthwhile to study the distributivity of J_f over t-norms and t-conorms.

The equation $(x \wedge y) \rightarrow z \equiv (x \rightarrow (y \rightarrow z))$, known as the law of importation, is another desirable tautology in classical logic. The general form of the above equivalence is given by

$$J(T(x, y), z) \equiv J(x, J(y, z)), \quad x, y, z \in [0, 1],$$

where T is a t-norm and J a fuzzy implication. In A-implications defined by Turksen et al. [28], the general form of the law of importation, with T as the product t-norm $T_{\mathbf{P}}(x, y) = x \cdot y$, was taken as one of the axioms. Baczyński [1] has studied the law of importation in conjunction with Eq. (7) and has given a characterisation.

Bouchon-Meunier and Kreinovich [23] have characterised fuzzy implications that have the law of importation as one of the axioms along with Eq. (7).

They have considered the minimum t-norm $T_{\mathbf{M}}$ for T and claim that Mamdani's choice of implication "min" is "not so strange after all". Ref. [8] discusses the validity of this tautology for R-, S- and QL-implications and its possible applications in Approximate Reasoning are explored.

Contrapositive symmetry is yet another classical tautology desirable for fuzzy implications. Contrapositive symmetry plays a significant role in classical logic structures wherein "proof by contradiction" is a commonly employed method to validate conjectures. Works in [9,10,20] discuss ways of imparting contrapositive symmetry with respect to any arbitrary strong negation N. Also contrapositive symmetry of fuzzy implications has been studied in a functional equation framework in [2] along with the law of importation and Eq. (7).

This work, where we discuss the class of J_f implications with respect to three classical logic tautologies, viz., distributivity over t-norms and t-conorms, law of importation and contrapositive symmetry, can be seen as part of the above efforts.

Yager in [31] has done an extensive analysis of the impact of this new class of implications in Approximate Reasoning by introducing concepts like strictness of implications and sharpness of inference, among others. This work can also be seen as a continuation of the above study on the classical tautologies satisfied by Yager's f-generated implications that have an influence in Approximate Reasoning. For more recent works on the role of fuzzy logic operators in computing with words see [27,32].

1.2. Outline of this work

Firstly, by discussing the four different general forms of distributive equations we show that the Yager's class of f-generated implications does have more solutions for one of them than that possessed by R-, S- or QL-implications. We also give necessary and sufficient conditions under which the class of J_f implications satisfies the law of importation. Following this, we give sufficient conditions under which the natural negations of J_f implications are strong and the implications J_f possess contrapositive symmetry with respect to their natural negations.

Since the natural negations of J_f implications, in general, are not strong we discuss the contrapositivisation of J_f implications using the techniques proposed in [10]. We have shown that, in general, only the upper contrapositivisation is N-compatible with J_f.

Finally, taking cue from the f-generated implications J_f, we also present a new class of implications, J_h, called h-generated implications and show that they have some very useful properties, viz., their natural negations are *non-vanishing*, and hence the lower contrapositivisation is N-compatible with J_h.

The paper is organised as follows. In Section 2, we recall the class of f-generated fuzzy implications J_f proposed by Yager in [31] and also some relevant results on t-norms and t-conorms. In Section 3, we investigate the distributivity of J_f over t-norms and t-conorms, while in Section 4, the law of importation with respect to a t-norm T is explored and in Section 5, the contrapositive symmetry of J_f implications is discussed. In Section 6, we present a new class of implications, J_h, called h-generated implications and discuss their properties vis-á-vis contrapositivisation. Section 7 gives some concluding remarks.

2. Preliminaries

To make this work self-contained, we briefly mention some of the concepts and results employed in the rest of the work.

2.1. Negations

Definition 1 [19, Definition 1.1, p. 3]. A negation N is a function from $[0,1]$ to $[0,1]$ such that

- $N(0) = 1$; $N(1) = 0$;
- N is non-increasing.

Definition 2. A negation N is said to be

- non-vanishing if $N(x) \neq 0$ for any $x \in [0,1)$, i.e., $N(x) = 0$ iff $x = 1$;
- non-filling if $N(x) \neq 1$ for any $x \in (0,1]$, i.e., $N(x) = 1$ iff $x = 0$.

A negation N that is not non-filling (non-vanishing) will be called filling (vanishing).

Definition 3 [19, Definition 1.2, p. 3]. A negation N is called strict if in addition N is strictly decreasing and continuous.

Note that if a negation N is strict it is both non-vanishing and non-filling, but the converse is not true.

Definition 4 [19, Definition 1.2, p. 3]. A strong negation N is a strict negation N that is also involutive, i.e., $N(N(x)) = x$, $\forall x \in [0,1]$.

2.2. T-Norms and T-conorms

Definition 5 [18, Definition 1.1, p. 4]. A function T from $[0,1]^2 \to [0,1]$ is called a triangular norm (shortly t-norm) if, for all $x, y, z \in [0,1]$,

$$T(x,y) = T(y,x), \tag{T1}$$

$$T(x,T(y,z)) = T(T(x,y),z), \tag{T2}$$

$$T(x,y) \leqslant T(x,z) \quad \text{whenever } y \leqslant z, \tag{T3}$$

$$T(x,1) = x. \tag{T4}$$

Definition 6 [18, Definition 1.13, p. 11]. A function $S:[0,1]^2 \to [0,1]$ is called a triangular conorm (shortly t-conorm) if, for all $x, y, z \in [0,1]$, it satisfies

$$S(x,y) = S(y,x), \tag{S1}$$

$$S(x,S(y,z)) = S(S(x,y),z), \tag{S2}$$

$$S(x,y) \leqslant S(x,z) \quad \text{whenever } y \leqslant z, \tag{S3}$$

$$S(x,0) = x. \tag{S4}$$

Definition 7 [18, Definition 2.9, p. 26; Definition 2.13, p. 28]. A t-norm T (t-conorm S resp.) is said to be

- Continuous if it is continuous in both the arguments.
- Idempotent if $T(x,x) = x$ ($S(x,x) = x$) for all $x \in [0,1]$.
- Archimedean if T (S resp.) is such that for every $x, y \in (0,1]$ ($x, y \in [0,1)$ resp.) there is an $n \in \mathbb{N}$ with $x_T^{(n)} < y$ ($x_S^{(n)} > y$).
- Strict if T (S resp.) is continuous and stricly monotone, i.e., $T(x,y) < T(x,z)$ ($S(x,y) < S(x,z)$) whenever $x > 0$ ($x < 1$ resp.) and $y < z$.
- Nilpotent if T (S resp.) is continuous and if each $x \in (0,1)$ is such that $x_T^{(n)} = 0$ ($x_S^{(n)} = 1$) for some $n \in \mathbb{N}$.

Table 1 lists the basic t-norms and t-conorms along with their properties.

Theorem 1 [18, Theorem 5.1, p. 122]. *T is a continuous Archimedean t-norm iff T has a continuous additive generator, i.e., there exists a continuous, strictly decreasing function $f:[0,1] \to [0,\infty]$ with $f(1) = 0$, which is uniquely determined upto a positive multiplicative constant, such that for all $x, y \in [0,1]$*

Table 1
Examples of t-norms and t-conorms and their properties

t-Norm T	t-Conorm S	Properties
$T_{\mathbf{M}}$: $\min(x,y)$	$S_{\mathbf{M}}$: $\max(x,y)$	Continuous, idempotent
$T_{\mathbf{P}}$: $x \cdot y$	$S_{\mathbf{P}}$: $x+y-x\cdot y$	Strict
$T_{\mathbf{LK}}$: $\max(x+y-1,0)$	$S_{\mathbf{LK}}$: $\min(x+y,1)$	Nilpotent
$T_{\mathbf{D}}: \begin{cases} y & \text{if } x=1 \\ x & \text{if } y=1 \\ 0 & \text{otherwise} \end{cases}$	$S_{\mathbf{D}}: \begin{cases} x & \text{if } y=0 \\ y & \text{if } x=0 \\ 1 & \text{otherwise} \end{cases}$	Archimedean, not continuous

$$T(x,y) = f^{(-1)}(f(x)+f(y)), \tag{9}$$

where $f^{(-1)}$ is the pseudo-inverse of f and is defined as

$$f^{(-1)}(x) = \begin{cases} f^{-1}(x) & \text{if } x \in [0, f(0)], \\ 0 & \text{if } x \in [f(0), \infty]. \end{cases} \tag{10}$$

Note that if $f(0) = \infty$ then T is strict and if $f(0) < \infty$ then T is nilpotent.

Definition 8 [18, Definition 3.39, p. 79]. A *multiplicative generator* of a t-conorm S is a strictly decreasing function $\phi:[0,1] \to [0,1]$, which is left-continuous in 1 and satisfies $\phi(0) = 1$, such that for all $x, y \in [0,1]$ we have

$$S(x,y) = \phi^{(-1)}(\phi(x)\cdot\phi(y)),$$

where $\phi^{(-1)}$ is the pseudo-inverse of ϕ.

For more details on the pseudo-inverses of monotone functions see, for example, [18], Section 3.1.

2.3. Yager's class of implication operators

Definition 9 [19, Definition 1.15, p. 22]. A function J from $[0,1]^2$ to $[0,1]$ is called a fuzzy implication if for all $x, y, z \in [0,1]$ it has the following properties:

$$J(x,z) \geqslant J(y,z) \quad \text{if } x \leqslant y, \tag{J1}$$
$$J(x,y) \leqslant J(x,z) \quad \text{if } y \leqslant z, \tag{J2}$$
$$J(0,y) = 1, \tag{J3}$$
$$J(x,1) = 1, \tag{J4}$$
$$J(1,0) = 0. \tag{J5}$$

Definition 10 (*cf.* [25]). A fuzzy implication J is said to have

- the neutrality property or is said to be neutral if

$$J(1,y) = y, \quad y \in [0,1]; \tag{NP}$$

- the exchange property if

$$J(x, J(y,z)) = J(y, J(x,z)), \quad x, y, z \in [0,1]. \tag{EP}$$

Definition 11 [31, p. 196]. An f-generator is a function $f:[0,1] \to [0,\infty]$ that is a strictly decreasing and continuous function with $f(1) = 0$. Also we denote its pseudo-inverse by $f^{(-1)}$ given by (10).

Table 2
Examples of some J_f implications with their f-generators

Name	$f(x)$	$f(0)$	$J_f(x,y)$
Yager	$-\ln x$	∞	y^x
Frank	$-\ln\left\{\frac{s^x-1}{s-1}\right\};\ s>0,\ s\neq 1$	∞	$\log_s\{1+(s-1)^{1-x}\cdot(s^y-1)^x\}$
Trigonometric	$\cos\left(\frac{\pi}{2}\cdot x\right)$	1	$\cos^{-1}\left[x\cdot\cos\left(\frac{\pi}{2}\cdot y\right)\right]$
Yager's class	$(1-x)^{\lambda};\ \lambda>0$	1	$1-x^{\frac{1}{\lambda}}\cdot(1-y)$

Definition 12 [31, p. 197]. A function from $[0,1]^2$ to $[0,1]$ defined by an f-generator as

$$J_f(x,y) = f^{(-1)}(x\cdot f(y)),\quad x,y\in[0,1],$$

with the understanding that $0\times\infty = 0$, is called an f-generated implication.

It can easily be shown, as in [31, p. 197], that J_f is a fuzzy implication.

Remark 13. Note that since $x \leqslant 1 \Rightarrow x\cdot f(y) \leqslant f(y) \leqslant f(0)$ we have $J_f(x,y) = f^{(-1)}(x\cdot f(y)) = f^{-1}(x\cdot f(y))$ for all $x,y\in[0,1]$. Also as can be seen from Theorem 1 and as noted in [31], the f-generators can be used as additive generators for generating t-norms.

The following properties of J_f have already been discussed by Yager (see [31, p. 197]):

- *Neutrality*: $J_f(1,x) = f^{(-1)}(1\cdot f(x)) = f^{(-1)}(f(x)) = x$.
- *Exchange principle*: $J_f(x,J_f(y,z)) = f^{(-1)}(x\cdot f(J_f(y,z))) = f^{(-1)}(x\cdot f(f^{(-1)}(y\cdot f(z)))) = f^{(-1)}(x\cdot y\cdot f(z)) = J_f(y,J_f(x,z))$.

Table 2 gives a few examples from the above class J_f (see [31, pp. 198–200]).

3. On the distributivity of *f*-generated implications J_f over *t*-norms and *t*-conorms

In this section, we study the distributivity of the f-generated implications J_f over t-norms and t-conorms, by studying the conditions under which J_f implications satisfy Eqs. (5)–(8).

3.1. On Eqs. (5) and (6)

Theorem 2 (cf. [7, Theorems 5 and 6]). *Any neutral fuzzy implication J that is one-to-one in the first variable, when the second variable is in* $(0,1)$, *reduces*

(i) (5) *to* (1) *and satisfies* (1);
(ii) (6) *to* (2) *and satisfies* (2).

Proposition 14 [7, Propositions 3 and 6]. *Let J be a binary operator on* $[0,1]$. *Then the following are equivalent*:

- *J is non-increasing in the first variable*;
- *J satisfies* (1), *i.e.*, (5) *with $T = T_{\mathbf{M}}$ and $S = S_{\mathbf{M}}$*;
- *J satisfies* (2), *i.e.*, (6) *with $T = T_{\mathbf{M}}$ and $S = S_{\mathbf{M}}$.*

Lemma 1. *Let J_f be an f-generated implication. J_f is one-to-one in the first variable, while the second variable lies in* $(0,1)$.

Proof. Let $y\in(0,1)$ be fixed and let $x_1,x_2\in[0,1]$ such that $J_f(x_1,y) = J_f(x_2,y)$. Now since $f(y)\in(0,\infty)$ we have

$$\begin{aligned} J_f(x_1, y) = J_f(x_2, y) &\Rightarrow f^{-1}(x_1 \cdot f(y)) = f^{-1}(x_2 \cdot f(y)) \\ &\Rightarrow x_1 \cdot f(y) = x_2 \cdot f(y) \\ &\Rightarrow x_1 = x_2. \qquad \square \end{aligned}$$

Theorem 3. *Let J_f be an f-generated implication. Then J_f satisfies (5) if and only if $S = S_{\mathbf{M}}$ and $T = T_{\mathbf{M}}$.*

Proof. ($\Rightarrow$:) Let J_f satisfy (5). Since J_f is neutral and one-to-one in the first variable with the second variable in $(0,1)$, by Theorem 2(i) we have that $S = S_{\mathbf{M}}$ and $T = T_{\mathbf{M}}$.

($\Leftarrow$:) On the other hand, since J_f is a fuzzy implication it has (J1) and thus by Proposition 14 J satisfies (5) with $S = S_{\mathbf{M}}$ and $T = T_{\mathbf{M}}$. $\square$

Theorem 4. *Let J_f be an f-generated implication. Then J_f satisfies (6) if and only if $S = S_{\mathbf{M}}$ and $T = T_{\mathbf{M}}$.*

Proof. Again by the one-to-oneness of J_f in the first variable and Theorem 2(ii). $\square$

3.2. On Eqs. (7) and (8)

To discuss Eqs. (7) and (8) w.r.to J_f we consider two cases under each of them, viz., when $f(0) = \infty$ and $f(0) < \infty$.

First we note that, since J_f is neutral, i.e., $J_f(1, y) = y$, $\forall y \in [0,1]$, we have that $T_1 = T_2 = T$ in (7) and $S_1 = S_2 = S$ in (8). Hence when J_f satisfies (7) and (8) they reduce to (11) and (12), respectively

$$J_f(r, T(s,t)) \equiv T[J_f(r,s), J_f(r,t)], \tag{11}$$
$$J_f(r, S(s,t)) \equiv S[J_f(r,s), J_f(r,t)]. \tag{12}$$

Proposition 15 [7, Propositions 9 and 12]. *Let J be a binary operator on $[0,1]$. Then the following are equivalent*:

- *J is non-decreasing in the second variable*;
- *J satisfies (3), i.e., (11) with $T = T_{\mathbf{M}}$*;
- *J satisfies (4), i.e., (12) with $S = S_{\mathbf{M}}$.*

3.2.1. On Eqs. (7) and (8) when $f(0) = \infty$

In the following result we show that J_f implications obtained from f-generators, such that $f(0) = \infty$, satisfy (11) for t-norms other than min.

Theorem 5. *Let J_f be obtained from an f-generator where $f(0) = \infty$. Then J_f satisfies the distributive law (11) if*

(i) *$T = T_{\mathbf{M}}$, or*
(ii) *T is the t-norm obtained using f as the additive generator, i.e., $T(x,y) = f^{(-1)}(f(x) + f(y))$.*

Proof

(i) Since J_f is an implication operator, and thus has (J2), from Proposition 15 we see that J_f satisfies (11) when $T = T_{\mathbf{M}}$.
(ii) Let T be the t-norm obtained using f as the additive generator, i.e., $T(x,y) = f^{(-1)}(f(x) + f(y))$. Since $f(0) = \infty$ we know that $f^{(-1)} = f^{-1}$, $f \circ f^{-1} = id$ and we have

$$
\begin{aligned}
J_f(r, T(s,t)) &= f^{-1}(r \cdot f(T(s,t))) \\
&= f^{-1}(r \cdot f \circ f^{-1}(f(s) + f(t))) \\
&= f^{-1}(r \cdot (f(s) + f(t))) \qquad \because f \circ f^{-1} = id \\
&= f^{-1}(r \cdot f(s) + r \cdot f(t)) \\
&= f^{-1}(f \circ f^{-1}(r \cdot f(s)) + f \circ f^{-1}(r \cdot f(t))) \\
&= f^{-1}(f(J_f(r,s)) + f(J_f(r,t))) \\
&= T[J_f(r,s), J_f(r,t)]. \qquad \square
\end{aligned}
$$

In [7] it was shown that when J is an R- or an S-implication (7) holds if and only if $T_1 = T_2 = T_{\mathbf{M}}$. Along similar lines the same can be proven when J is a QL-implication too. As noted earlier, since (7) has an important role to play in rule reduction, from Theorem 5, we note that when J is an f-generated implication, there exist other choices for T than min, unlike in the case of R-, S- and QL-implications.

Example 1 shows that if the t-norm T in (11) is such that its generator is different from the f-generator used to obtain J_f then (11) may not be satisfied.

Example 1. Let $f(x) = -\ln x$, then $f(0) = \infty$, $J_f(x,y) = J_{\mathbf{Y}}(x,y) = y^x$, the Yager's implication [30]. Considering f as an additive generator we get the product t-norm $T_{\mathbf{P}}(x,y) = x \cdot y$. Now, let T be the Łukasiewicz t-norm $T_{\mathbf{LK}}(x,y) = \max(0, x+y-1)$.

Now, letting $r = s = t = 0.4$, we have $J_{\mathbf{Y}}(r,s) = J_{\mathbf{Y}}(r,t) = 0.4^{0.4} = 0.693$ while $T_{\mathbf{LK}}(s,t) = \max(0, 0.4 + 0.4 - 1) = 0$. Hence $J_{\mathbf{Y}}(r, T_{\mathbf{LK}}(s,t)) = 0^{0.4} = 0$ while $T_{\mathbf{LK}}[J_{\mathbf{Y}}(r,s), J_{\mathbf{Y}}(r,t)] = T_{\mathbf{LK}}(0.693, 0.693) = 0.386$, i.e $J_{\mathbf{Y}}(r, T_{\mathbf{LK}}(s,t)) \neq T_{\mathbf{LK}}[J_{\mathbf{Y}}(r,s), J_{\mathbf{Y}}(r,t)]$, when $r = s = t = 0.4$.

Again by (J2) and Proposition 15 we see that J_f satisfies (12) when $S = S_{\mathbf{M}}$. From Example 2 it is reasonable to surmise that a result similar to Theorem 5 may not be possible.

Example 2. Let $f(x) = -\ln x$, then $f(0) = \infty$, $J_f(x,y) = J_{\mathbf{Y}}(x,y) = y^x$, the Yager's implication [30].

- Considering f as an additive generator we get the product t-norm $T_{\mathbf{P}}$ whose dual t-conorm with respect to $1 - x$ is the *strict* Algebraic Sum t-conorm $S_{\mathbf{P}}(x,y) = x + y - x \cdot y$.
- Let us define an increasing continuous function $\phi : [0,1] \to [0, \infty)$ from $f(x) = -\ln x$ as follows: $\phi(x) = \exp\{-f(x)\} = \exp\{\ln x\} = x$. Now, considering ϕ as an additive generator of a t-conorm, we get the *nilpotent* Łukasiewicz t-conorm $S_{\mathbf{LK}}(x,y) = \min(x+y, 1)$.

From Table 3 it is clear that (12) does not hold when J is $J_{\mathbf{Y}}$ and S is either the Łukasiewicz t-conorm $S_{\mathbf{LK}}$ or the Algebraic Sum t-conorm $S_{\mathbf{P}}$ with $r = 0.3$, $s = t = 0.1$.

Hence, determining t-conorms S such that (12) holds when $J = J_f$ is worthy of exploration in view of the importance of (12) in the field of rule reduction.

3.2.2. On Eqs. (7) and (8) when $f(0) < \infty$

Again by the neutrality of J_f it is enough to consider (11) and (12). Towards investigating (11) when $f(0) < \infty$ we need the following modified versions of Theorems 7 and 8 in [7] and the lemma given below.

Theorem 6 (cf. [7, Theorems 7 and 8]). *Any neutral fuzzy implication J such that $J(\cdot, 0)$ is onto reduces*

- *(7) to (3) and satisfies (3), i.e., J satisfies (7) only if $T_1 = T_2 = T_{\mathbf{M}}$;*
- *(8) to (4) and satisfies (4), i.e., J satisfies (8) only if $S_1 = S_2 = S_{\mathbf{M}}$.*

Table 3
Examples of t-conorms (both nilpotent and strict) which do not satisfy (12) with $J_{\mathbf{Y}}$

S	$S(s,t)$	$J(r,s) = J(r,t)$	LHS (12)	RHS (12)
$S_{\mathbf{LK}}$	0.2	0.50011	0.6170	1
$S_{\mathbf{P}}$	0.19	0.50011	0.607612	0.751186

Lemma 2. *Let $f(0) < \infty$. Then $J_f(\cdot, 0)$ is onto.*

Proof. To show that $J_f(\cdot, 0)$ is onto we need to show that for every $y \in [0, 1]$ there exists $x \in [0, 1]$ such that $J_f(x, 0) = y$. Let $y \in [0, 1]$ be arbitrary. Then

$$\begin{aligned} J_f(x,0) = y &\Rightarrow f^{-1}(x \cdot f(0)) = y \\ &\Rightarrow x \cdot f(0) = f(y) \\ &\Rightarrow x = \frac{f(y)}{f(0)}. \end{aligned}$$

Now $1 \geqslant y \geqslant 0 \Rightarrow f(1) \leqslant f(y) \leqslant f(0) < \infty \Rightarrow 1 \geqslant x \geqslant 0$. Thus for any $y \in [0, 1]$ there exists $x = \frac{f(y)}{f(0)}$ such that $J_f(x,0) = y$ and so $J_f(\cdot, 0)$ is onto. □

From Theorem 6 and Lemma 2 we have the following:

Theorem 7. *Let J_f be an f-generated implication with $f(0) < \infty$. Then J_f satisfies (7) if and only if $T_1 = T_2 = T_{\mathbf{M}}$.*

Proof. (⇒:) Let J_f satisfy (7). Then $T_1 = T_2$ by the neutrality of J_f. Then by the ontoness of $J_f(\cdot, 0)$ and Theorem 6 we have that $T_1 = T_2 = T_{\mathbf{M}}$.

(⇐:) On the other hand, if $T_1 = T_2 = T_{\mathbf{M}}$ in (7), since J_f is a fuzzy implication and so has (J2) we have by Proposition 15 that J_f satisfies (11). □

Theorem 8. *Let J_f be an f-generated implication with $f(0) < \infty$. Then J_f satisfies (8) if and only if $S_1 = S_2 = S_{\mathbf{M}}$.*

A summary of the above results is given in Table 5 in Section 7.

4. On the law of importation $J(T(x, y), z) = J(x, J(y, z))$

In this section we consider the following general form of law of importation:

$$J(T(x,y),z) = J(x,J(y,z)), \quad x,y,z \in [0,1], \tag{LI}$$

where T is a t-norm and J is a fuzzy implication.

Theorem 9. *J_f satisfies the law of importation (LI) if and only if $T = T_{\mathbf{P}}$, the product t-norm.*

Proof. (⇐:) Let T be the product t-norm. Then,

$$\begin{aligned} \text{RHS (LI)} &= J_f(x, J_f(y,z)) \\ &= f^{-1}[x \cdot f(J_f(y,z))] \\ &= f^{-1}[x \cdot f \circ f^{-1}(y \cdot f(z))] \\ &= f^{-1}[x \cdot (y \cdot f(z))] \\ &= J_f(x \cdot y, z) = \text{LHS (LI)}. \end{aligned}$$

(⇒:) Let J_f obey the law of importation (LI). Let $z \in (0, 1)$ then $f(z) \in (0, \infty)$. Now for any $x, y \in [0, 1]$, we have

$$\begin{aligned} & J_f(T(x,y),z) = J_f(x, J_f(y,z)) \\ \Rightarrow \quad & f^{-1}[T(x,y) \cdot f(z)] = f^{-1}[x \cdot f \circ f^{-1}(y \cdot f(z))] \\ \Rightarrow \quad & f \circ f^{-1}[T(x,y) \cdot f(z)] = f \circ f^{-1}[x \cdot y \cdot f(z)] \\ & \text{i.e., } [T(x,y) \cdot f(z)] = x \cdot y \cdot f(z) \\ & \text{i.e., } T(x,y) = x \cdot y. \qquad \square \end{aligned}$$

By the commutativity of a t-norm T it is obvious that if a fuzzy implication J has (LI) then it has (EP).

5. Contrapositive symmetry of *f*-generated implications

In the framework of classical two-valued logic, contrapositiviy of a binary implication operator is a tautology, i.e., $\alpha \Rightarrow \beta \equiv \neg\beta \Rightarrow \neg\alpha$. In fuzzy logic, contrapositive symmetry of a fuzzy implication J with respect to strong negation N – CP(N) – plays an important role in the applications of fuzzy implications, viz., Approximate Reasoning, Deductive Systems, Decision Support Systems, Formal Methods of Proof, etc. (see also [20,21]).

Definition 16. A fuzzy implication J is said to have contrapositive symmetry with respect to a strong negation N, denoted CP(N), if

$$J(x,y) = J(N(y),N(x)), \quad x,y \in [0,1]. \tag{CP}$$

Definition 17. Let J be any fuzzy implication. The natural negation of J, denoted by N_J, is given by $J(x,0) = N_J(x)$, $\forall x \in [0,1]$. Clearly $N_J(0) = 1$ and $N_J(1) = 0$.

Usually, the contrapositive symmetry of a fuzzy implication J is studied with respect to its natural negation, denoted CP(N_J), provided N_J is strong. Also in the setting of fuzzy logic, contrapositive symmetry is the characterising property of strong fuzzy implications obtained from a t-conorm and a strong negation, which are defined as follows:

Definition 18 [19, Definition 1.16, p. 24]. An S-implication $J_{S,N}$ is obtained from a t-conorm S and a strong negation N as follows:

$$J_{S,N}(x,y) = S(N(x),y), \quad x,y \in [0,1]. \tag{13}$$

The following theorem characterises S-implications:

Theorem 10 [19, Theorem 1.13, p. 24]. *A fuzzy implication J is an S-implication for an appropriate t-conorm S_J and a strong negation N if and only if J has* CP(N), *the exchange property* (EP) *and is neutral* (NP), *where $S_J(x,y) = J(N(x),y)$.*

In general, the natural negation N_J of J need not be strong. Even if N_J is strong J still may not have CP(N_J). For example, consider the fuzzy implication $J_{\mathbf{K}}(x,y) = [1 - x + x\cdot y^2]^{\frac{1}{2}}$. The natural negation of $J_{\mathbf{K}}$ is $N_{J_{\mathbf{K}}}(x) = J_{\mathbf{K}}(x,0) = [1-x]^{\frac{1}{2}}$ which is not a strong negation and hence $J_{\mathbf{K}}$ does not have CP($N_{J_{\mathbf{K}}}$). On the other hand, though the natural negation of the implication $J_{\mathbf{GG}}(x,y) = \min\left\{1, \frac{1-x}{1-y}\right\}$, $N_{J_{\mathbf{GG}}}(x) = 1 - x$, is a strong negation $J_{\mathbf{GG}}$ does not have CP($1-x$).

In this section, we analyse the nature of the natural negations of J_f, N_{J_f}, under different boundary conditions on the underlying generator f and give a sufficient condition under which N_{J_f} is strong and J_f has CP(N_{J_f}).

5.1. The family of J_f implications and CP(N)

The natural negation of J_f, given by $N_{J_f}(x) = J_f(x,0) = f^{(-1)}(x\cdot f(0))$, is quite evidently a negation. To discuss the nature of N_{J_f} we consider the following two cases:

Case I: $f(0) < \infty$

If $f(0) < \infty$ then $N_{J_f}(x) = J_f(x,0) = f^{-1}(x\cdot f(0))$, $\forall x \in (0,1)$. Since f and, thus, f^{-1} are strictly decreasing continuous functions, we have that N_{J_f} is a strict negation. For N_{J_f} to be strong, we need that $N_{J_f}(N_{J_f}(x)) = x$, $\forall x \in [0,1]$, which is not the case always (see Example 3).

Example 3. Consider the f-generated implication $J_{f_{\mathbf{Y}}}(x,y) = 1 - x^{\frac{1}{\lambda}}(1-y)$ obtained from the Yager's class of f-generators $f(x) = (1-x)^{\lambda}$ with $f(0) = 1 < \infty$ (see Table 2). Now, if $\lambda = 0.5$, i.e. $\frac{1}{\lambda} = 2$, then $N_{J_{f_{\mathbf{Y}}}}(x) = J_{f_{\mathbf{Y}}}(x,0) = 1 - x^2$ is a strict negation. That it is not strong can be seen by letting $x = 0.5$ in which case $N_{J_{f_{\mathbf{Y}}}}(N_{J_{f_{\mathbf{Y}}}}(x)) = 1 - [1-x^2]^2 = 1 - (1-0.25)^2 = 0.4375 \neq 0.5 = x$. On the other hand, if $\lambda = 1$, then $N_{J_{f_{\mathbf{Y}}}}(x) = J_{f_{\mathbf{Y}}}(x,0) = 1 - x$, which is a strong negation.

Case II: $f(0)=\infty$

In the case when $f(0)=\infty$ it is easy to see that N_{J_f} is not even strict, since $\forall x\in(0,1]$, we have $J_f(x,0)=N_{J_f}(x)=f^{-1}(x\cdot f(0))=f^{-1}(x\cdot\infty)=f^{-1}(\infty)=0$. Quite obviously, it is not strong either. In fact, N_{J_f} is a vanishing but a non-filling negation.

Thus, as per Definition 16, J_f does not have contrapositive symmetry with respect to its natural negation. The following result gives a sufficient condition under which this happens.

Theorem 11. *Let the f-generator be such that $f(0)=1$ and $f^{-1}=f$. Then the natural negation of J_f, N_{J_f}, is a strong negation and J_f has CP(N_{J_f}).*

Proof. Let the f-generator be such that $f(0)=1$ and $f^{-1}=f$. Then the pseudo-inverse of f from (10) is given by

$$f^{(-1)}(x)=\begin{cases} f^{-1}(x) & \text{if } x\in[0=f(1),f(0)=1],\\ 0 & \text{if } x\in[1,\infty].\end{cases} \tag{14}$$

Now, the natural negation of J_f is given by $J_f(x,0)=N_{J_f}(x)=f^{(-1)}(x\cdot f(0))=f^{(-1)}(x\cdot 1)=f^{-1}(x)$ for any $x\in[0,1]$. Since f is strictly decreasing so is N_{J_f}. Also, $N_{J_f}(N_{J_f}(x))=f^{-1}\circ f^{-1}(x)=f\circ f^{-1}(x)=x$, since $f^{-1}=f$. Hence N_{J_f}, is a strong negation.

From the following string of equalities we note that J_f has CP(N_{J_f}):

$$\begin{aligned} J_f(N_{J_f}(y),N_{J_f}(x)) &= f^{(-1)}[N_{J_f}(y)\cdot f(N_{J_f}(x))]\\ &= f^{(-1)}[f^{-1}(y)\cdot f\circ f^{-1}(x)]\\ &= f^{(-1)}[f^{-1}(y)\cdot x]\\ &= f^{(-1)}[f(y)\cdot x]=J_f(x,y). \qquad\square \end{aligned}$$

Corollary 19. *If N is any strong negation then J_N has CP(N).*

Any strong N can be thought of as a decreasing bijection ϕ on the unit interval $[0,1]$ with $\phi=\phi^{-1}$ and hence is a multiplicative generator of a strict t-conorm. Also note that for a strong N, J_N has CP(N), (EP) and (NP). Thus by Theorem 10 J_N can be represented as an S-implication. Now, the t-conorm S_J obtained from J_N according to Theorem 10 is $S_J(x,y)=J_N(N(x),y)=N[N(x)\cdot N(y)]=\phi^{-1}[\phi(x)\cdot\phi(y)]$, which by Definition 8 is nothing but the t-conorm obtained using ϕ as the multiplicative generator. Hence, in the case $f(0)=1$ and $f^{-1}=f$ we do not obtain any new fuzzy implications but only S-implications from a strict t-conorm S and the strong negation N_S which is also the multiplicative generator ϕ of S.

5.2. J_f and contrapositivisation

From the discussions in Section 5.1 we observe that the natural negations of J_f, in general, are not strong and thus, as per Definition 16, do not have contrapositive symmetry with respect to their natural negation. In fact, the natural negation N_{J_f} of J_f is, in general, only a strict negation if $f(0)<\infty$, while it is a vanishing and a non-filling negation if $f(0)=\infty$.

Towards imparting contrapositive symmetry to such fuzzy implications J with respect to a strong negation N the following two contrapositivisation techniques – upper and lower contrapositivisation – have been proposed by Bandler and Kohout in [10], whose definitions we give below.

Definition 20. Let J be any fuzzy implication and N a strong negation. The upper and lower contrapositivisations of J with respect to N, denoted herein as $\overset{U:N}{\Longrightarrow}$ and $\overset{L:N}{\Longrightarrow}$, respectively, are defined as follows:

$$x\overset{U:N}{\Longrightarrow}y=\max\{J(x,y),J(N(y),N(x))\}, \tag{15}$$

$$x\overset{L:N}{\Longrightarrow}y=\min\{J(x,y),J(N(y),N(x))\} \tag{16}$$

for any $x,y\in[0,1]$.

As can be seen, $\overset{U:N}{\Longrightarrow}$ and $\overset{L:N}{\Longrightarrow}$ are both fuzzy implications, as per Definition 9, and always have the contrapositive symmetry with respect to the strong negation N employed in their definitions.

Definition 21. Let J be a fuzzy implication and N a strong negation. A contrapositivisation technique $\overset{*:N}{\Longrightarrow}$ is said to be *N-compatible* if the contrapositivisation of J with respect to N, denoted as $J^*(x,y) = x \overset{*:N}{\Longrightarrow} y$ for all $x, y \in [0,1]$, is such that the natural negation of J^*, given by $N_{J^*}(\cdot) = J^*(\cdot, 0)$, is equal to the strong negation N employed.

Definition 21, in essence, is asking for J^* to have CP(N_{J^*}).

The following result has been proven in [9]:

Proposition 22. *Let J be a neutral fuzzy implication with natural negation $J(x,0) = N_J(x)$ and N a strong negation.*

(i) *The upper contrapositivisation of J with respect to N is N-compatible if and only if $N(x) \geqslant N_J(x)$, for all $x \in [0,1]$.*
(ii) *The lower contrapositivisation of J with respect to N is N-compatible if and only if $N(x) \leqslant N_J(x)$, for all $x \in [0,1]$.*

Proof. We give the proof of part (i) as that of part (ii) is similar.

(i) Let $x \in [0,1]$. By definition of $\overset{U:N}{\Longrightarrow}$ we have

$$\begin{aligned} \overset{U:N}{\Longrightarrow} \text{ is } N\text{-Compatible} \quad & \text{iff } N(x) = x \overset{U:N}{\Longrightarrow} 0 \\ & \text{iff } N(x) = \max\{J(x,0), J(1,N(x))\} \\ & \text{iff } N(x) = \max(N_J(x), N(x)) \\ & \text{iff } N(x) \geqslant N_J(x), \quad \text{for all } x \in [0,1]. \qquad \square \end{aligned}$$

If the upper contrapositivisation of J with respect to a strong N is N-compatible, then from Proposition 22 we know $N \geqslant N_J$. Since N is strong $N(x) = 1$ if and only if $x = 0$ and we have that for all $x \in (0,1]$, $1 > N(x) \geqslant N_J(x)$ and N_J is a non-filling negation. In other words, if the natural negation of the fuzzy implication J is a *filling* negation we cannot find any strong N with which the upper contrapositivisation of J becomes N-compatible. Similarly, if the natural negation of the fuzzy implication J is a *vanishing* negation we cannot find any strong N with which the lower contrapositivisation of J becomes N-compatible.

Now, in the case $f(0) < \infty$, we have that the natural negation of J_f is at least strict and so both upper and lower contrapositivisation techniques are N-compatible, with respect to strong negations N, depending on whether $N \geqslant N_{J_f}$ or $N \leqslant N_{J_f}$, respectively. On the other hand, when $f(0) = \infty$, N_{J_f} is a non-filling but a *vanishing* negation and thus we cannot have any strong negation $N \leqslant N_{J_f}$. Therefore, only the upper contrapositivisation technique is N-compatible with respect to strong negations $N \geqslant N_{J_f}$.

A summary of results in this section is given in Table 6 in Section 7.

In the following section, taking cue from the Yager's f-generated implications, we propose a new class of h-generated implications, denoted J_h, where h is defined on $[0,1]$ to $[0,1]$, unlike f which is from $[0,1]$ to $[0,\infty]$ and study its properties. We also show that one can obtain natural negations N_{J_h} of h-generated implications that are non-vanishing and hence the lower contrapositivisation technique is N-compatible with respect to strong negations $N \leqslant N_{J_h}$.

6. A new class of implications: *h*-generated implications – J_h

Definition 23. An h-generator is a function $h : [0,1] \to [0,1]$, that is strictly decreasing and continuous such that $h(0) = 1$. Let $h^{(-1)}$ be its pseudo-inverse given by

$$h^{(-1)}(x) = \begin{cases} h^{-1}(x) & \text{if } x \in [h(1), 1] \\ 1 & \text{if } x \in [0, h(1)] \end{cases} \tag{17}$$

Lemma 3. *Let the function J_h from $[0,1]\times[0,1]$ to $[0,1]$ be defined as*

$$J_h(x,y) =_{def} h^{(-1)}(x\cdot h(y)),\quad x,y\in[0,1]. \tag{18}$$

J_h is a fuzzy implication and called the h-generated implication.

Proof. That J_h is a fuzzy implication can be seen from the following:

- $J_h(1,0) = h^{(-1)}(1\cdot h(0)) = h^{(-1)}(1\cdot 1) = 0.$
- $J_h(0,1) = h^{(-1)}(0\cdot h(1)) = h^{(-1)}(0) = 1 = J_h(0,0).$
- $J_h(1,1) = h^{(-1)}(1\cdot h(1)) = h^{(-1)}(h(1)) = 1$, since $h^{(-1)}\circ h = id$.
- For any $x,x',y\in[0,1]$ we have $x\leqslant x' \Rightarrow x\cdot h(y)\leqslant x'\cdot h(y) \Rightarrow h^{(-1)}(x\cdot h(y)) \geqslant h^{(-1)}(x'\cdot h(y)) \Rightarrow J_h(x,y)\geqslant J_h(x',y)$. Thus J_h is non-increasing in the first variable.
- For any $x,y,y'\in[0,1]$ we have $y\leqslant y' \Rightarrow x\cdot h(y)\geqslant x\cdot h(y') \Rightarrow h^{(-1)}(x\cdot h(y)) \leqslant h^{(-1)}(x\cdot h(y')) \Rightarrow J_h(x,y)\leqslant J_h(x,y')$. Thus J_h is non-decreasing in the second variable.
- Since $0\leqslant x\cdot h(1)\leqslant h(1)$, $\forall x\in[0,1]$, we have $J_h(x,1) = h^{(-1)}(x\cdot h(1)) = 1$, by definition of $h^{(-1)}$.
- $J_h(0,y) = h^{(-1)}(0\cdot h(y)) = h^{(-1)}(0) = 1$, for all $y\in[0,1]$. □

Without explicitly using the pseudo-inverse (18) can be written in the following form:

$$J_h(x,y) = h^{-1}(\max(x\cdot h(y), h(1))),\quad x,y\in[0,1]. \tag{19}$$

Also, J_h has the following desirable properties:

- *Neutrality (NP):* $J_h(1,x) = h^{(-1)}(1\cdot h(x)) = x$, since $h^{(-1)}\circ h = id$.
- *Exchange principle (EP):* For every h-generator h and $x,y,z\in[0,1]$ we get

$$\begin{aligned} J_h(x,J_h(y,z)) &= h^{(-1)}(x\cdot h(J_h(y,z))) = h^{-1}(\max(x\cdot h(h^{-1}(\max(y\cdot h(z),h(1)))),h(1)))\\ &= h^{-1}(\max(x\cdot\max(y\cdot h(z),h(1)),h(1))) = h^{-1}(\max(x\cdot y\cdot h(z),x\cdot h(1),h(1)))\\ &= h^{-1}(\max(x\cdot y\cdot h(z),h(1))),\end{aligned}$$

  since $x\cdot h(1)\leqslant h(1)$. Similarly we get that

$$J_h(y,J_h(x,z)) = h^{-1}(\max(y\cdot x\cdot h(z),h(1))).$$

  Thus J_h satisfies the exchange principle.

Table 4 gives a few examples from the above class J_h.

6.1. J_h and its natural negation N_{J_h}

The natural negation of J_h, $N_{J_h}(x) = J_h(x,0) = h^{(-1)}(x\cdot h(0)) = h^{(-1)}(x)$, for all $x\in[0,1]$ is, in general, only a negation. But,

- N_{J_h} is a strict negation if $h(1)=0$;
- N_{J_h} is a strong negation iff $h=h^{-1}$, in which case $N_{J_h} = h^{(-1)} = h$.

When $h=h^{-1}$ from Corollary 19 we see that J_h has CPS(N_{J_h}). Let $h\neq h^{-1}$. Then, if $h(1)=0$ we have that the natural negation N_{J_h} is strict and hence is both a non-vanishing and non-filling negation. When $h(1)>0$ then N_{J_h} is a non-vanishing but a filling negation.

Table 4
Examples of some J_h implications with their h-generators

Name	$h(x)$	$h(1)$	$J_h(x,y)$
Schweizer–Sklar	$1-x^p;\ p\neq 0$	0	$[1-x+x\cdot y^p]^{\frac{1}{p}}$
Yager's	$(1-x)^\lambda;\ \lambda>0$	0	$1-x^{\frac{1}{\lambda}}(1-y)$
–	$1-\frac{x^n}{n};\ n\geqslant 1$	$1-\frac{1}{n}$	$\min\left\{[n-nx+x\cdot y^n]^{\frac{1}{n}},1\right\}$

6.2. J_h and contrapositivisation

Let $h \neq h^{-1}$. Then, if $h(1) = 0$ we have that the natural negation N_{J_h} is strict and hence there exist strong negations N such that both the upper and lower contrapositivisation of J_h are N-compatible, depending on whether $N \geqslant N_{J_h}$ or $N \leqslant N_{J_h}$, respectively. On the other hand, if $h(1) > 0$ then N_{J_h} is a non-vanishing but a *filling* negation and only the lower contrapositivisation technique is N-compatible with respect to strong negations $N \leqslant N_{J_h}$.

Fig. 1 shows plots of the fuzzy implication $J_{\mathbf{Y}_{\frac{1}{\lambda}}}(x,y) = 1 - x^{\frac{1}{\lambda}} + x^{\frac{1}{\lambda}} \cdot y$ obtained from the Yager's class of h-generators for $\lambda = 0.5$ or $\frac{1}{\lambda} = 2$ (see Table 4) along with its natural negation $N_{J_{\mathbf{Y}_2}}(x) = 1 - x^2$, the lower and upper contrapositivised implications with respect to negations $N_1(x) = (1 - \sqrt{x})^2$ and $N_2(x) = \sqrt{1 - x^2}$, respectively.

For more details on contrapositivisation and significance of N-compatibility, see [9,10,20,21].

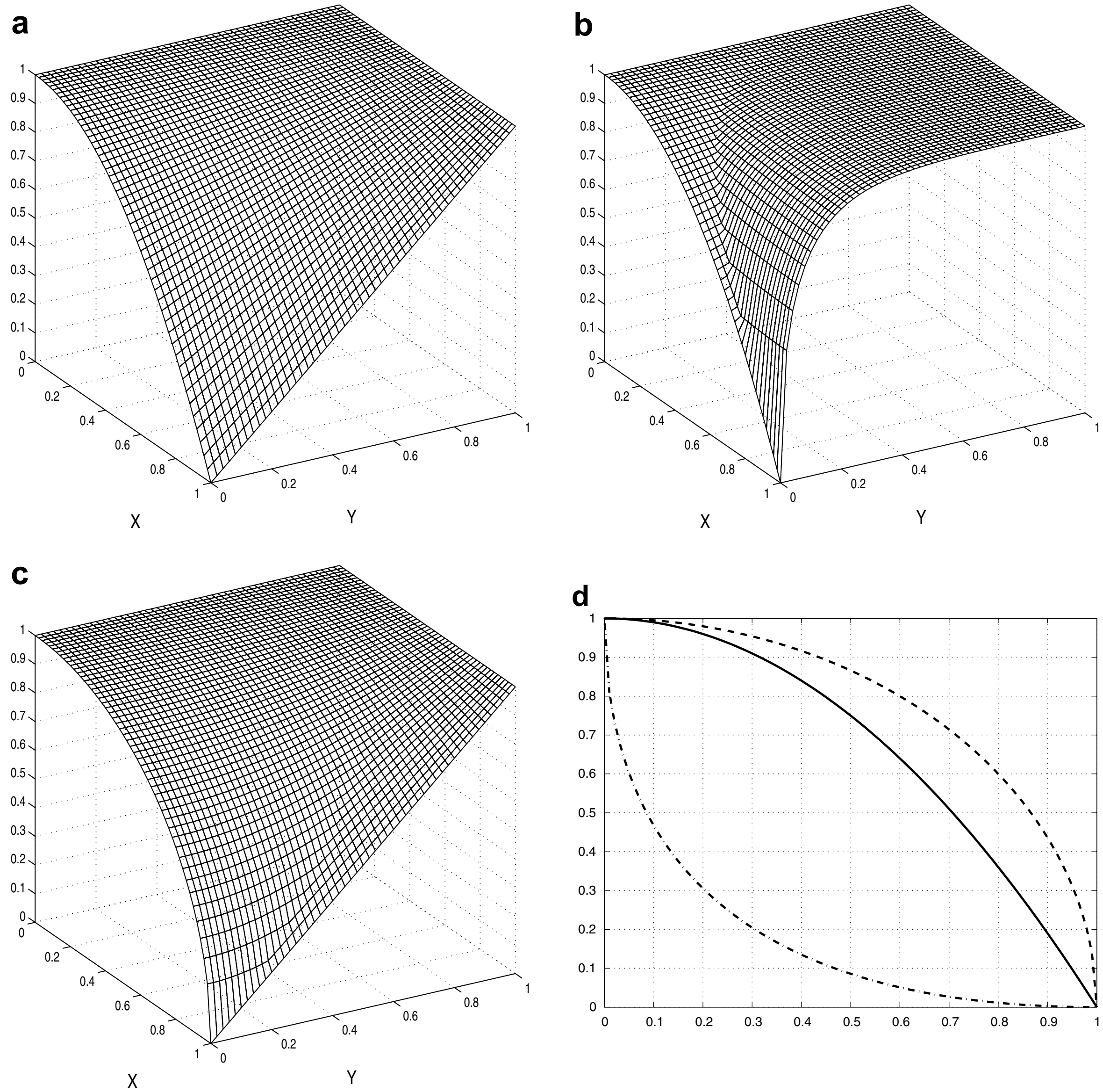

Fig. 1. Fuzzy implication $J_{\mathbf{Y}_2}(x,y) = 1 - x^2 + x^2 \cdot y$ with $\frac{1}{\lambda} = 2$ whose natural negation is the strict negation $N_{J_{\mathbf{Y}_2}}(x) = 1 - x^2$ with lower and upper contrapositivisations. (a) Yager's implication $J_{\mathbf{Y}_2}(x,y) = 1 - x^2 + x^2 \cdot y$ with $\frac{1}{\lambda} = 2$, (b) lower contrapositivisation of $J_{\mathbf{Y}_2}$ with $N_1(x) = (1 - \sqrt{x})^2$, (c) upper Contrapositivisation of $J_{\mathbf{Y}_2}$ with $N_2(x) = \sqrt{1 - x^2}$ and (d) plots of negations N_1 (-·-), N_2 (- - -), $N_{J_{\mathbf{Y}_2}}$ (—).

6.3. Relation between f- and h-generators

Let f be an f-generator. Then let us define an $\hat{h} : [0,1] \to [0,1]$ as follows

$$\hat{h}(x) =_{def} \exp\{-f(1-x)\}. \tag{20}$$

Then $\hat{h}$ is a strictly decreasing function on the unit interval $[0,1]$, such that $\hat{h}(0) = \exp\{-f(1-0)\} = \exp\{-f(1)\} = 1$ since $f(1) = 0$. Now, if $f(0) = \infty$ then $\hat{h}(1) = 0$ while if $f(0) < \infty$ then $\hat{h}(1) > 0$. In either case, the $\hat{h}$ obtained as in (20) can act as an h-generator.

Similarly, from an h-generator one can obtain an $\hat{f}$-generator as follows:

$$\hat{f}(x) =_{def} -\ln h(1-x). \tag{21}$$

While an f-generator can be seen as the additive generator of some continuous Archimedean t-norm T, an h-generator can be seen as the multiplicative generator of some continuous Archimedean t-conorm S. Thus (20) and (21) are how one obtains the multiplicative generator of the N – dual t-conorm S from the additive generator of the t-norm T and viceversa (see [18], pp. 80–81), where N is the classical negation $N(x) = 1 - x$.

Also note that if the range of the f-generator is $[0,1]$, i.e., $f(0) = 1$, then f itself can act as the h-generator and $J_h = J_f$ and $h(1) = 0$. This equivalence can be readily seen in the case of Yager's class of generators from both Tables 2 and 4. On the other hand, we can still obtain the h-generator from f as in (20) (see Example 4).

Example 4. Consider the f-generator given by $f(x) = 1 - x$. Then $f(1) = 0$ and also $f(0) = 1$ and thus letting $h = f$ we get that $J_f = J_h(x,y) = 1 - x + x \cdot y$. On the other hand, by employing (20) we obtain the following:

$$\hat{h}(x) = \exp\{-f(1-x)\} = \exp\{-x\},$$

$$\hat{h}^{(-1)}(x) = \begin{cases} -\ln x & \text{if } x \in [\hat{h}(1) = \frac{1}{e}, \hat{h}(0) = 1], \\ 1 & \text{if } x \in [0, \hat{h}(1)], \end{cases}$$

$$J_{\hat{h}}(x,y) = \hat{h}^{(-1)}(x \cdot \exp\{-y\}) = \min\{-\ln(x \cdot \exp\{-y\}), 1\} = \min\{y - \ln x, 1\},$$

$$N_{J_{\hat{h}}}(x) = \min\{-\ln x, 1\},$$

whereas $J_f(x,y) = 1 - x + x \cdot y$ and $N_{J_f}(x) = 1 - x$.

When $h(1) \neq 0$ or $f(0) \neq 1$, one cannot take $f = h$ and by applying (21) and (20) one gets different $\hat{f}$ and $\hat{h}$, respectively (see Example 5).

Example 5. Consider the h-generator $h(x) = 1 - \frac{x^2}{2}$. Then clearly $h(1) = 0.5 \neq 0$ and thus is not suitable to be employed as an f-generator directly. Also $J_h(x,y) = \min\left\{1, \sqrt{2 - 2 \cdot x + x \cdot y^2}\right\}$ as can be seen from Table 4.

On the other hand, using the transformation (21) we have the following:

$$\hat{f}(x) = -\ln h(1-x) = -\ln\left[\frac{1 - x^2 + 2x}{2}\right] = -\ln\frac{1}{2}[1 - x^2 + 2x],$$

$$\hat{f}(1) = 0, \qquad \hat{f}(0) = \ln 2,$$

$$\hat{f}^{(-1)}(x) = \left\{\begin{array}{ll} 1 - [2(1 - e^{-x})]^{\frac{1}{2}}, & x \in [0, \ln 2] \\ 0, & x \in [\ln 2, \infty] \end{array}\right\} = \max\left\{0, 1 - [2(1 - e^{-x})]^{\frac{1}{2}}\right\},$$

$$J_{\hat{f}}(x,y) = \max\left\{0, 1 - \left[2 - 2\left[\frac{1}{2}(1 - y^2 + 2y)\right]^x\right]^{\frac{1}{2}}\right\},$$

$$N_{J_{\hat{f}}}(x) = \max\left\{0, 1 - \left[2 - 2\left[\frac{1}{2}\right]^x\right]^{\frac{1}{2}}\right\}.$$

Similarly, the f-generator $f(x) = -\ln x$ is such that $f(1) = 0$ and $f(0) \neq 1$ and thus $h \neq f$ directly. But by using the transformation (20) we have $\hat{h}(x) = 1 - x$ and $J_{\hat{h}}(x,y) = 1 - x + x \cdot y$ while $J_f(x,y) = J_{\mathbf{Y}}(x,y) = y^x$.

Table 5
Summary of results in Section 3

J_f	Satisfies (5)	Satisfies (6)
$f(0)=\infty$	$\iff S=S_{\mathbf{M}},\ T=T_{\mathbf{M}}$	$\iff S=S_{\mathbf{M}},\ T=T_{\mathbf{M}}$
$f(0)<\infty$	$\iff S=S_{\mathbf{M}},\ T=T_{\mathbf{M}}$	$\iff S=S_{\mathbf{M}},\ T=T_{\mathbf{M}}$
	Satisfies (7)	Satisfies (8)
$f(0)=\infty$	if $T=T_{\mathbf{M}}$/obtained from f	If $S=S_{\mathbf{M}}$
$f(0)<\infty$	$\iff T=T_{\mathbf{M}}$	$\iff S=S_{\mathbf{M}}$

Table 6
Summary of results in Sections 4 and 5

J_f	Satisfies (LI)	$N_{J_f}(x)$	(CP)(N_{J_f})
$f(0)=\infty$	$\iff T=T_{\mathbf{P}}$	NOT strict	Never
$f(0)<\infty$	$\iff T=T_{\mathbf{P}}$	Strict	If $f(0)=1$ and $f^{-1}=f$

A more detailed study of f- and h-generated implications has been carried out in [3,4].

7. Concluding remarks

In this work we have studied the newly proposed Yager's class of f-generated fuzzy implications with respect to three classical tautologies, viz., distributivity over t-norms and t-conorms, law of importation and contrapositive symmetry. The results of the above investigation are given in Tables 5 and 6 for ready reference.

We have also suggested some sufficient conditions under which J_f implications possess contrapositive symmetry with respect to their natural negation. Since the natural negations of J_f, in general, are not strong we resorted to the well-established contrapositivisation techniques, viz., upper and lower contrapositivisation [10]. We have shown that, in general, only the upper contrapositivisation is N-compatible with J_f and hence we have proposed a new class of fuzzy implications called h-generated implications, denoted J_h, along the lines of J_f, for which class the lower contrapositivisation is N-compatible.

In this work both necessary and sufficient conditions have been proposed for J_f to satisfy the considered tautologies (except in the case of (7), (8) and CP(N) when $f(0)=\infty$, where it is only a sufficient condition). Thus determining the necessary conditions so that J_f satisfies these tautologies when $f(0)=\infty$ is likely to be both interesting and important.

Yager in [31] has done an extensive analysis of the impact of this new class of implications in Approximate Reasoning by introducing concepts like strictness of implications and sharpness of inference, among others. For more recent works on the role of fuzzy logic operators in computing with words see [27,32]. This work can be seen as a continuation of the above study on the classical tautologies satisfied by Yager's f-generated implications that have an influence in Approximate Reasoning.

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