



Rule reduction for efficient inferencing in similarity based reasoning

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Abstract

The two most important models of inferencing in approximate reasoning with fuzzy sets are Zadeh's Compositional Rule of Inference (CRI) and Similarity Based Reasoning (SBR). It is known that inferencing in the above models is resource consuming (both memory and time), since these schemes often consist of discretisation of the input and output spaces followed by computations in each point. Also an increase in the number of rules only exacerbates the problem. As the number of input variables and/or input/output fuzzy sets increases, there is a combinatorial explosion of rules in multiple fuzzy rule based systems. In this paper, given a fuzzy if–then rule base that is used in an SBR inference mechanism, we propose to reduce the number of rules by combining the antecedents of the rules that have the same consequent. We also present some sufficient conditions on the operators employed in SBR inference schemes such that the inferences obtained using the original rule base and the reduced rule base obtained as above are identical. Subsequently, these conditions are investigated and many solutions are presented for some specific SBR inference schemes.

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1. Introduction

1.1. Inference in approximate reasoning

One of the best known application areas of fuzzy logic is approximate reasoning, wherein from imprecise inputs and fuzzy premises or rules we obtain, often, imprecise conclusions. Approximate reasoning with fuzzy sets encompasses a wide variety of inference schemes and have been readily applied in many fields, especially among others, decision making, expert systems and control.

Of all the various approaches taken in such schemes in approximate reasoning, two of them have been prevalent in the literature, viz., reasoning methods based on the

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- (i) Combination-Projection principle, of which Zadeh's Compositional Rule of Inference (CRI) [55] is a good example.
- (ii) Similarity between inputs and antecedents and the subsequent modification of the consequent, usually called Similarity Based Reasoning (SBR) or plausible reasoning [23], of which Compatibility Modification Inference (CMI) [15] and Turksen's Approximate Analogical Reasoning Scheme (AARS) [48] are some representative samples.

Of course, there are many more that do not strictly fall under these two categories, for example, Swapan Raha et al., proposed an inference that is a combination of both the above approaches in [43], Baldwin's Fuzzy Truth Value Modification inference [5], the scheme proposed by Ughetto et al. for implication based rules in [49], etc. Also see earlier papers of Mizumoto [35,36], Roger [44], etc.

An inference scheme proposed under Approximate Reasoning (AR) is validated or assessed mainly based on the reasonableness of inference and the complexity of the algorithm. For example, they are used in fuzzy control primarily to approximate a function, which usually describes the system under consideration. On the other hand, in the areas of decision making and expert systems, AR techniques are employed for their inferential capabilities that conform to the basic rules of Generalised Modus Ponens (GMP) as envisaged in fuzzy logic. Given a fuzzy if-then rule of the type $A \rightarrow B$ and a fuzzy input A' , GMP allows us to infer the output fuzzy set B' even if $A' \not\equiv A$. Hence the different schemes under AR are evaluated based on their approximation abilities in the former, while in the latter they are assessed based on the "goodness" of inference as given by how well they satisfy the "axioms" of GMP as listed in [6,25,30,35], etc.

1.2. Motivation for this work

It is known that (see [20,45,49]) the inferencing schemes in AR are generally resource consuming (both memory and time), since these schemes often consist of discretisation of the input and output spaces followed by computations in each point. Also an increase in the number of rules only exacerbates the problem. As the number of input variables and/or input/output fuzzy sets increases, there is a combinatorial explosion of rules in multiple fuzzy rule based systems.

Many works have appeared towards reducing the complexity of the inference procedure, see, for example [1,42] for an excellent coverage. Of the many ways of reducing complexity, rule reduction methods are prevalent in the literature and have been proposed for fuzzy systems employed in fuzzy control, where the main aim is to approximate the behaviour of a system under consideration, which is a function of its inputs. For a good survey on many of these techniques we refer the readers to [1,7,41,42,53].

Currently, there is an increased awareness that the approximation accuracy achieved should not be sacrificed in the process of complexity reduction. In [8], Baranyi et al. discuss the trade off between approximation accuracy and complexity. See also [32] for a discussion on the trade off between computation time and precision. All these necessitate rule reduction techniques that are lossless with respect to inference, i.e., the inference obtained from the original rule base and that obtained from the reduced rule base should be identical. Some works have appeared along these lines, see, for example [9,12,14,31].

In this work, we consider only inference schemes in AR that can be grouped under Similarity Based Reasoning (SBR). In inferences in SBR, given a fuzzy if-then rule of the type $A \rightarrow B$ and a fuzzy input A' , the input is matched to the antecedent A to obtain a measure of similarity $s = M(A, A')$. The output fuzzy set B' is obtained by modifying the consequent B using this similarity measure s and a modification function J .

In this paper we address the issue of efficient inferencing through rule reduction. The rule reduction technique we propose here is a simple technique of combining the antecedents of rules with same consequents. To this end, we propose some sufficient conditions on the different operators employed in SBR inferencing that ensure that the inference obtained from the original rule base is identical to that obtained from the reduced rule base.

1.3. Outline of the paper

In Section 2, we give some preliminaries on the fuzzy logic operators required for the rest of the paper. This section also includes a brief background on fuzzy if-then rules. Sections 3–5 constitute the main parts of this

work. While Section 3 discusses the structure and inference in SBR, Section 4 proposes some sufficient conditions on the different operators employed in SBR that ensure inference invariant rule reduction of the above mentioned type. Subsequently, in Section 5 we investigate the conditions of the previous section and present as solutions inference operators employed in many SBR inference schemes. In Section 6, we illustrate the rule reduction method with a numerical example. In Section 7, some concluding remarks are given.

2. Preliminaries: basic fuzzy logic connectives

To make this work self-contained, we briefly mention some of the concepts and results employed in the rest of the work.

2.1. Negations, T -norms and T -conorms

Definition 1 [26, Definitions 1.2–1.4]. A function $N: [0, 1] \rightarrow [0, 1]$ is called a fuzzy negation if $N(0) = 1$, $N(1) = 0$ and N is non-increasing. N is called strict if, in addition, N is strictly decreasing and N is continuous. N is called strong if it is an involution, i.e., $N(N(x)) = x$ for all $x \in [0, 1]$.

Definition 2 (cf. [40,29, Definition 1.1])

- (i) An associative, commutative and increasing operation $T: [0, 1]^2 \rightarrow [0, 1]$ is called a t -norm if it has the neutral element 1.
- (ii) An associative, commutative and increasing operation $S: [0, 1]^2 \rightarrow [0, 1]$ is called a t -conorm if it has the neutral element 0.

If F is an associative binary operation on a domain $\mathbb{X}$ then by the notation, $x_F^{(n)}$ we mean $F(x, \underbrace{F(x, \dots, x)}_{n-1 \text{ times}})$ for an $x \in \mathbb{X}$ and $n \geq 2$. Also $x_F^{(1)} = x$.

Definition 3 [29, Definitions 2.9 and 2.13]. A t -norm T (t -conorm S , resp.) is said to be

- continuous if it is continuous in both the arguments;
- Archimedean if T (S , resp.) is such that for every $x, y \in (0, 1]$ ($x, y \in [0, 1)$ resp.) there is an $n \in \mathbb{N}$ with $x_T^{(n)} < y$ ($x_S^{(n)} > y$);
- strict if T (S , resp.) is continuous and strictly monotone, i.e., $T(x, y) < T(x, z)$ ($S(x, y) < S(x, z)$) whenever $x > 0$ ($x < 1$ resp.) and $y < z$.

It is well known that if T and S are continuous Archimedean t -norm and t -conorm, then they have continuous additive generators (see [29, Theorem 5.1 and Corollary 5.5]). Table 1 lists the basic t -norms and t -conorms. The set of all t -norms and t -conorms will be denoted by $\mathcal{T}$, $\mathcal{S}$, respectively.

2.2. Uninorms

Definition 4 [27, Definition 1]. A uninorm is a two-place function $U: [0, 1]^2 \rightarrow [0, 1]$ which is associative, commutative, non-decreasing in each place and such that there exists some element $e \in [0, 1]$ called the neutral element such that $U(e, x) = x$, for all $x \in [0, 1]$.

Table 1
Examples of t -norms and t -conorms

t -norm T	Formula	t -conorm S	Formula f
T_M : minimum	$\min(x, y)$	S_M : maximum	$\max(x, y)$
T_P : product	$x \cdot y$	S_P : probabilistic sum	$x + y - x \cdot y$
T_{LK} : Łukasiewicz	$\max(x + y - 1, 0)$	S_{LK} : Łukasiewicz	$\min(x + y, 1)$

Remark 5

- (i) If $e = 0$ then U is a t -conorm and if $e = 1$ then U is a t -norm. For any uninorm U , $U(1, 0) \in \{0, 1\}$ (see [27] Corollary 1).
- (ii) A uninorm U such that $U(1, 0) = 0$ is called a *conjunctive* uninorm and if $U(1, 0) = 1$ it is called a *disjunctive* uninorm.
- (iii) It is known that a uninorm U behaves as a t -norm on the square $[0, e] \times [0, e]$ and as a t -conorm on the square $[e, 1] \times [e, 1]$. Hence a uninorm U with the neutral element $e \in (0, 1)$ is typically denoted as $U = (T, S, e)$, where T and S are the underlying t -norm and t -conorm.

There are three main classes of uninorms in the literature, viz.,

- (i) Pseudo-continuous uninorms (see [34]), i.e., uninorms U that are continuous on $[0, 1]^2$ except on the segments $(0, e)$, $(1, e)$ and $(e, 0)$, $(e, 1)$. These are precisely the uninorms for which both the functions $U(x, 1)$ and $U(x, 0)$ are continuous except at the point $x = e$.
- (ii) Idempotent uninorms, i.e., uninorms U such that $U(x, x) = x$ for all $x \in [0, 1]$ (see [21, 33, 51]).
- (iii) Representable (also called almost continuous) uninorms that have additive generators and are continuous everywhere on the $[0, 1]^2$ except at the points $(0, 1)$, $(1, 0)$ (see [27]).

Analogous to the representation theorems for continuous Archimedean t -norms and t -conorms, Fodor et al. [27] have proven the following:

Proposition 6 (cf. [27]). *A uninorm U is an almost continuous uninorm with neutral element $e \in (0, 1)$ if and only if there exists a strictly increasing continuous function $r : [0, 1] \rightarrow [0, +\infty]$ with $r(0) = 0$, $r(e) = 1$ and $r(1) = +\infty$ such that U is given by*

$$U(x, y) = r^{-1}(r(x) \cdot r(y)), \quad (x, y) \in [0, 1]^2 \setminus \{(0, 1), (1, 0)\} \quad (1)$$

and $U(0, 1) = U(1, 0) = 0$ or $U(0, 1) = U(1, 0) = 1$. Such a function r is called a *multiplicative generator* of U .

The set of all uninorms will be denoted by $\mathcal{U}$.

2.3. Fuzzy implication operators

Definition 7 [26, Definition 1.15]. A function $I : [0, 1]^2 \rightarrow [0, 1]$ is called a fuzzy implication if for all $x, y, z \in [0, 1]$, it satisfies

$$I(x, z) \geq I(y, z), \quad \text{if } x \leq y, \quad (11)$$

$$I(x, y) \leq I(x, z), \quad \text{if } y \leq z, \quad (12)$$

$$I(0, y) = 1, \quad (13)$$

$$I(x, 1) = 1, \quad (14)$$

$$I(1, 0) = 0. \quad (15)$$

The following are the two important classes of fuzzy implications well established in the literature:

Definition 8 [26, Definition 1.16]. An S -implication I_S is obtained from a t -conorm S and a strong negation N as follows:

$$I_S(x, y) = S(N(x), y), \quad x, y \in [0, 1]. \quad (2)$$

Definition 9 [26, Definition 1.16]. An R -implication I_T is obtained from a t -norm T as its residuation as follows:

$$I_T(x, y) = \sup\{t \in [0, 1] : T(x, t) \leq y\}, \quad x, y \in [0, 1]. \quad (3)$$

Along the lines of Definition 8 we can obtain fuzzy implications from a uninorm U .

Definition 10. A U -implication I_U is obtained from a uninorm U , with neutral element $e \in (0, 1)$, and a strong negation N as follows:

$$I_U(x, y) = U(N(x), y), \quad x, y \in [0, 1]. \quad (4)$$

Definition 11 (cf. [26,28]). A fuzzy implication I is said to have

(i) the left neutrality property or is said to be left neutral, if

$$I(1, y) = y, \quad y \in [0, 1]; \quad (\text{NP})$$

(ii) the ordering property, if

$$x \leq y \iff I(x, y) = 1, \quad x \in [0, 1]. \quad (\text{OP})$$

Remark 12

- (i) While all S - and R -implications satisfy (NP), U -implications do not.
- (ii) An R -implication I_T obtained from a left-continuous t -norm has (OP).
- (iii) The function I_U as defined in (4) is a fuzzy implication if and only if U is a disjunctive uninorm.

For some well known S -, R -, and U -implications we refer the readers, for example, to [26,18]. We need the following result in the sequel:

Proposition 13 [11, Proposition 1]. Let $I : [0, 1]^2 \rightarrow [0, 1]$. Then the following are equivalent:

- (i) I satisfies (I2);
- (ii) $I(x, \min(y, z)) = \min(I(x, y), I(x, z))$ for all $x, y, z \in [0, 1]$;
- (iii) $I(x, \max(y, z)) = \max(I(x, y), I(x, z))$ for all $x, y, z \in [0, 1]$.

2.4. Fuzzy if-then rules

A linguistic statement “ $\tilde{x}$ is A ” is interpreted as the linguistic variable $\tilde{x}$ taking the linguistic value A . For example, if $\tilde{x}$ denotes “Temperature” (on a suitable domain X), then it can assume the following linguistic values A , viz., high, very high, medium, cool, hot, etc. Each of the linguistic values (say cool) is represented by a fuzzy set on the domain X of the linguistic variable $\tilde{x}$.

A Single Input Single Output (SISO) fuzzy if-then rule is of the form,

If $\tilde{x}$ is A Then $\tilde{y}$ is B

where $\tilde{x}, \tilde{y}$ are linguistic variables and A, B are linguistic expressions/values assumed by the linguistic variables. For example,

If $\tilde{x}$ (temperature) is A (High) Then $\tilde{y}$ (Pressure) is B (Low).

A Mutli Input Single Output (MISO) fuzzy if-then rule is of the form,

If $\tilde{x}_1$ is A and $\tilde{x}_2$ is B Then $\tilde{y}$ is C ,

where $\tilde{x}_1, \tilde{x}_2, \tilde{y}$ are linguistic variables and A, B, C are linguistic expressions.

The following definition will be helpful in the sequel:

Definition 14. Let $X = \{x_1, x_2, \dots, x_n\}$ be a finite set. Let $A, B : X \rightarrow [0, 1]$, and F be any binary operation on $[0, 1]$, i.e., $F : [0, 1] \times [0, 1] \rightarrow [0, 1]$.

- (i) $F(A, B)$ is a fuzzy set on X , i.e., $F(A, B) : X \rightarrow [0, 1]$, defined as $F(A, B)(x) = F(A(x), B(x)) \quad \forall x \in X$.
- (ii) If $\alpha \in [0, 1]$ then $F(\alpha, B)$ is a fuzzy set on X , i.e., $F(\alpha, B) : X \rightarrow [0, 1]$, defined as $F(\alpha, B)(x) = F(\alpha, B(x)), \quad \forall x \in X$.

3. Structure and inference in similarity based reasoning

Let *If $\tilde{x}$ is A Then $\tilde{y}$ is B* be a given fuzzy if–then rule and the given input be $\tilde{x}$ is A' . Inference in Similarity Based Reasoning (SBR) schemes in AR is based on the calculation of a measure of compatibility or similarity $M(A, A')$ of the input A' to the antecedent A of the rule, and the use of a modification function J to modify the consequent B , according to the value of $M(A, A')$.

Some of the well-known examples of SBR are Compatibility Modification Inference (CMI) [15], “Approximate Analogical Reasoning Scheme” (AARS) in [48] and “Consequent Dilation Rule” (CDR) in [37], Smets and Magrez [46], Chen [13], etc. For a comparative study of many SBR inference schemes see [54]. In this section, we detail the typical inferencing mechanism in SBR, both in the case of SISO and MISO fuzzy rule bases.

3.1. Matching function M

Given two fuzzy sets, say A, A' , on the same domain, a matching function M compares them to get a degree of similarity, which is expressed as a real in the $[0, 1]$ interval. We refer to M as the Matching Function in the sequel. Formally, it can be defined as follows:

Definition 15. A matching function $M : \mathcal{F}(X) \times \mathcal{F}(X) \rightarrow [0, 1]$, where $\mathcal{F}(X)$ is the fuzzy power set of a non-empty set X , i.e., $\mathcal{F}(X) = \{A \mid A : X \rightarrow [0, 1]\}$.

Example 1. Let X be a non-empty set and $A, A' \in \mathcal{F}(X)$. Below we list a few of the matching functions employed in the literature.

- Zadeh’s max–min: $M_Z(A, A') = \max_x \min(A(x), A'(x))$.
- Magrez – Smets’ Measure [46]: $M_M(A, A') = \max_x \min(\overline{A(x)}, A'(x))$, where $\overline{A(x)}$ is the negation of $A(x)$.
- Measure of Subsethood [37]: $M_S(A, A') = \min_x I(A'(x), A(x))$, where I is a fuzzy implication.
- Scalar Product [13]: $M_C(A, A') = \frac{A \cdot A'}{\max(A \cdot A, A' \cdot A')}$, where the domain X is discretized into n points, i.e., $X = \{x_1, x_2, \dots, x_n\}$ and hence $A, A' \in [0, 1]^n$ with ‘ $\cdot$ ’ is the scalar product of the ‘vectors’ A, A' .
- Disconsistency Measure [48]: $M_{Tk}(A, A') = \left[\frac{\sum_{i=1}^n (A(x_i) - A'(x_i))^2}{n} \right]^{1/2}$, once again the domain X is discretized into n points.

Remark 16

- (i) Zwick et al. [56] have compared 19 such similarity measures based on a few parameters. Also see [10, 38, 39, 50, 13] for more such measures.
- (ii) Note that a matching function M is not required to be symmetric. For example, since a fuzzy implication I is not commutative, the subsethood measure M_S of A' in A is different from than that of A in A' .

3.2. SBR inference for SISO fuzzy rule base

3.2.1. Modification function J

Let us again consider *If $\tilde{x}$ is A Then $\tilde{y}$ is B* to be the given SISO fuzzy if–then rule and $\tilde{x}$ is A' the observed fuzzy input. Let $s = M(A, A') \in [0, 1]$ be a measure of the compatibility of A' to A .

Let Y be a non-empty set and $B \in \mathcal{F}(Y)$. The modification function J is again a function from $[0, 1]^2$ to $[0, 1]$ and produces a modification $B' \in \mathcal{F}(Y)$ based on s and B , i.e., the consequence in SBR, using the modification function J , is given by

$$B'(y) = J(s, B(y)) = J(M(A, A'), B(y)), \quad y \in Y. \quad (5)$$

In AARS [48] the following modification operators have been proposed, for any $y \in Y$:

- (i) *More or less*: $J_{ML}(s, B) = B'(y) = \min\{1, B(y)/s\}$;
- (ii) *Membership value reduction*: $J_{MVR}(s, B) = B'(y) = B(y) \cdot s$.

Table 2

Some SBR inference schemes along with their inference operators, where T is any t -norm, I is any fuzzy implication and Avg. is the averaging operator

SBR scheme	G	J	K	M
CMI [15]	T	I	T	M_Z
AARS [48]	S_M	J_{MVR}, J_{ML}	Avg.	M_{TK}, M_Z
CDR [37]	T_M	I	–	M_S

Once again in J_{MVR} ‘.’ can be generalised to any t -norm T . In CMI [15] and CDR [37] J is taken to be a fuzzy implication operator. Note that J need not be either commutative or associative.

3.2.2. Aggregation function G

In the case of multiple rules

$$R_i : \text{If } \tilde{x} \text{ is } A_i \text{ Then } \tilde{y} \text{ is } B_i, \quad i = 1, 2, \dots, m,$$

we infer the final output by aggregating over the rules, using an associative aggregation operator $G : [0, 1]^2 \rightarrow [0, 1]$:

$$B'(y) = G_{i=1}^m (J(M(A_i, A'_i), B_i(y))), \quad y \in Y. \quad (6)$$

Usually, G is either a t -norm, t -conorm or a uninorm, i.e., $G \in \mathcal{T} \cup \mathcal{S} \cup \mathcal{U}$.

3.3. SBR Inference for MISO fuzzy rule base

3.3.1. Combiner function K

On the other hand, if we consider a Mutli Input Single Output (MISO) fuzzy if–then rule of the form,

$$\text{If } \tilde{x}_1 \text{ is } A_1 \text{ and } \dots \text{ and } \tilde{x}_n \text{ is } A_n \text{ Then } \tilde{y} \text{ is } C,$$

then given the input that $(\tilde{x}_1 \text{ is } A'_1; \dots; \tilde{x}_n \text{ is } A'_n)$, the consequence in SBR is given by

$$C'(y) = J(K(M(A_1, A'_1), \dots, M(A_n, A'_n)), C(y)), = J(K_{i=1}^n (M(A_i, A'_i)), C(y)), \quad y \in Y, \quad (7)$$

where $K : [0, 1]^2 \rightarrow [0, 1]$, referred to as ‘Combiner’ in the sequel, is an associative and commutative function that combines the matching degrees of A_i to A'_i , for all $i = 1, 2, \dots, n$. Once again, typically, $K \in \mathcal{T} \cup \mathcal{S} \cup \mathcal{U}$.

In the case of MISO multiple rules

$$R_j : \text{If } \tilde{x}_1 \text{ is } A_{1j} \text{ and } \dots \text{ and } \tilde{x}_n \text{ is } A_{nj} \text{ Then } \tilde{y} \text{ is } C_j, \quad j = 1, 2, \dots, m,$$

given the input that $(\tilde{x}_1 \text{ is } A'_1; \dots; \tilde{x}_n \text{ is } A'_n)$, we infer the final output by aggregating over the rules,

$$C'(y) = G_{j=1}^m (J(K_{i=1}^n (M(A_{ij}, A'_i)), C_j(y))), \quad y \in Y. \quad (8)$$

Table 2 lists some SBR inference schemes along with their inference operators.

4. Rule reduction in SBR

In this section, we propose a simple rule reduction technique of combining the antecedents of rules with identical consequents. To this end, we propose some sufficient conditions on the different operators employed in SBR that ensure that the inferences obtained from the original rule base and the reduced rule base are identical.

Such a procedure of combining antecedents in fuzzy rules with identical consequents was considered by Dubois and Prade [24]. The focus of their study was the conditions on the underlying possibility distributions that enabled meaningful combination, whereas our agenda here is to study the conditions on the operators used in the SBR inference mechanisms that allows combining antecedents without losing the obtained inference.

In SBR the steps involved are the following:

- (i) Selection of a matching function M to match the antecedent A of the rule to the current input/observation A' .
- (ii) Selection of the modification function J to modify the consequent B according to the degree of compatibility between A and A' to obtain B' .
- (iii) In the case of MISO fuzzy rule bases, an additional step employing a commutative and associative operator K is required for combining the matching degrees of the antecedents A_i to the given inputs A'_i .
- (iv) When there are more than one rule, an associative aggregation operator G is employed over the rules and the inference is obtained by (6) or (8), using J , M and K .

We denote the SBR inference scheme employed in the case of SISO fuzzy rule base by the quadruple $(\mathcal{R}, G, J, M)$, where $\mathcal{R}$ denotes the SISO fuzzy rule base given and the inference is given by (6). Similarly, we denote the SBR inference scheme employed in the case of MISO fuzzy rule base by the quintuple $(\mathcal{R}, G, J, K, M)$, where $\mathcal{R}$ denotes the MISO fuzzy rule base given and the inference is given by (8).

Theorem 1. *Let a MISO fuzzy rule base $\mathcal{R}$ be given with the non-empty input universes of discourses X_i , for $i = 1, 2, \dots, n$ and an output universe of discourse Y . Let the inference be drawn using the SBR inference scheme $(\mathcal{R}, G, J, K, M)$, viz., (8). If the operators K, J, G, M are such that the following distributive equations hold:*

$$G(J(x, z), J(y, z)) = J(K(x, y), z), \quad (\text{C1})$$

$$M(K(A_1, A_2), A') = K(M(A_1, A'), M(A_2, A')), \quad (\text{C2})$$

where $A_1, A_2, A \in \mathcal{F}(X)$ and $x, y, z \in [0, 1]$, then inference invariant rule reduction is possible by combining antecedents of those rules in $\mathcal{R}$ whose consequents are identical.

Proof. For the sake of clarity we consider a 2-input–1-output MISO rule base with just three rules as given in $(\mathcal{R}_O)$, where $\tilde{x}_1, \tilde{x}_2, \tilde{y}$ are linguistic variables assuming the linguistic values $A_1, A_2, A_3 \in \mathcal{F}(X_1)$, $B_1, B_2, B_3 \in \mathcal{F}(X_2)$ and $C, D \in \mathcal{F}(Y)$, respectively, and X_1, X_2 are the non-empty input domains while Y is the non-empty output domain.

If $\tilde{x}_1$ is A_1 and $\tilde{x}_2$ is B_1 Then $\tilde{y}$ is C ,

If $\tilde{x}_1$ is A_2 and $\tilde{x}_2$ is B_2 Then $\tilde{y}$ is C , ($\mathcal{R}_O$)

If $\tilde{x}_1$ is A_3 and $\tilde{x}_2$ is B_3 Then $\tilde{y}$ is D .

In the presence of an input ($\tilde{x}_1$ is A' ; $\tilde{x}_2$ is B'), where $A' \in \mathcal{F}(X_1)$ and $B' \in \mathcal{F}(X_2)$ the inference is given by (8), for every $y \in Y$, as follows:

$$\begin{aligned} C'(y) &= G_{i=1}^3 (J(K[M(A_i, A'), M(B_i, B')], C_i(y))) \\ &= G(J(K(M(A_1, A'), M(B_1, B')), C(y)), J(K(M(A_2, A'), M(B_2, B')), C(y)), J(K(M(A_3, A'), M(B_3, B')), D(y))). \end{aligned} \quad (9)$$

Let the operators K, J, G, M be such that (C1) and (C2) hold. We claim that the above rule base $(\mathcal{R}_O)$ can be reduced to the following rule base $(\mathcal{R}_R)$ with two rules:

If $\tilde{x}_1$ is $K(A_1, A_2)$ and $\tilde{x}_2$ is $K(B_1, B_2)$ Then $\tilde{y}$ is C ,

If $\tilde{x}_1$ is A_3 and $\tilde{x}_2$ is B_3 Then $\tilde{y}$ is D , ($\mathcal{R}_R$)

such that the inference obtained from the reduced rule base $(\mathcal{R}_R)$ for the identical input ($\tilde{x}_1$ is A' ; $\tilde{x}_2$ is B') is equivalent to (9). Indeed, the inference obtained in this case as given by (8) is, for every $y \in Y$,

$$\begin{aligned}
C'(y) &= G(J(K(M(K(A_1, A_2), A'), M(K(B_1, B_2), B')), C(y)), J(K(M(A_3, A'), M(B_3, B')), D(y))) \\
&= G(J(K(K(M(A_1, A'), M(A_2, A')), K(M(B_1, B'), M(B_2, B'))), C(y)), \\
&\quad J(K(M(A_3, A'), M(B_3, B')), D(y))) \quad [\text{By (C2)}] \\
&= G(J(K(M(A_1, A'), M(A_2, A'), M(B_1, B'), M(B_2, B')), C(y)), \\
&\quad J(K(M(A_3, A'), M(B_3, B')), D(y))) \quad [\text{By associativity of } K] \\
&= G(J(K(K(M(A_1, A'), M(B_1, B')), K(M(A_2, A'), M(B_2, B'))), C(y)), \\
&\quad J(K(M(A_3, A'), M(B_3, B')), D(y))) \\
&= G(J(K(M(A_1, A'), M(B_1, B')), C(y)), J(K(M(A_2, A'), M(B_2, B')), C(y)), \\
&\quad J(K(M(A_3, A'), M(B_3, B')), D(y))) \quad [\text{By (C1)}] \\
&= ((9)).
\end{aligned}$$

Thus when K, J, G, M are such that (C1) and (C2) hold, inference invariant rule reduction as proposed above is possible in the SBR inference scheme $(\mathcal{R}, G, J, K, M)$. $\square$

Notice that in the case of SISO rules, the combiner operator K , though does not play a role in inferencing, does play a role in rule reduction, as can be seen from the following result, which follows immediately from Theorem 1 above.

Theorem 2. *Let a SISO fuzzy rule base $\mathcal{R}$ be given with the input and output universes of discourses being non-empty sets X, Y , respectively. Let the inference be drawn using the SBR inference scheme $(\mathcal{R}, G, J, M)$, viz., (6). If there exists an associative and commutative operator $K : [0, 1]^2 \rightarrow [0, 1]$ such that (C1) and (C2) hold, then inference invariant rule reduction is possible by combining antecedents of those rules in $\mathcal{R}$ whose consequents are identical.*

5. Some solutions of conditions (C1) and (C2)

In this section we investigate the sufficient conditions (C1) and (C2) obtained in the previous section and present some solutions.

5.1. Some solutions of equivalence (C1)

In this section we investigate the equivalence (C1)

$$G(J(x, z), J(y, z)) = J(K(x, y), z), \quad (\text{C1})$$

for some modification functions J and associative and commutative operators G, K .

In the case when $J = J_{\mathbf{MVR}}$ as in AARS [48] then it can be easily verified that J satisfies (C1) with $G = K = S_{\mathbf{M}}$ – the *max* t -conorm.

The case when J is a fuzzy implication as in CMI or CDR is more interesting, since it presents more solutions as we show from our investigations in the following sub-sections, where $J = I$, a fuzzy implication, and G, K are one of t -norms, t -conorms or uninorms in the equivalence (C1). Also, since the main focus of this section is to show that the above equivalence has many solutions, whence there are many operators that enable rule reduction in SBR, we limit our study to the three families of fuzzy implications introduced in Section 2.3, viz., S -, R - and U -implications.

5.1.1. In the setting of t -norms and t -conorms, i.e., $G, H \in \mathcal{T} \cup \mathcal{S}$

Proposition 17. *Let I have (NP). If K is a t -norm in (C1) then G is a t -conorm and vice versa.*

Proof. Let K be a t -norm. Then taking $x = 1, y = 0$ and for any $z \in [0, 1]$ we have $I(T(1, 0), z) = I(0, z) = 1$, while $G(I(1, z), I(0, z)) = G(z, 1)$, since I is neutral. Hence (C1) is satisfied only if $G(z, 1) = 1$ for all $z \in [0, 1]$, i.e., G is a t -conorm. $\square$

Since all S - and R -implications satisfy (NP), in the setting of t -norms and t -conorms (C1) reduces to the following two equations.

$$I(S(x, y), z) = T(I(x, z), I(y, z)) \quad (10)$$

$$I(T(x, y), z) = S(I(x, z), I(y, z)). \quad (11)$$

In [2,47] the following has been proven:

Theorem 3. An S -implication I_S or an R -implication I_{T^*} obtained from a left-continuous t -norm T^* satisfies (10) or (11) if and only if $S = S_M$ and $T = T_M$.

Theorem 4 [18, Theorems 3 and 8]. An U -implication I_U , T a t -norm, and S a continuous t -conorm satisfy

- (i) (10) if and only if T and S are N -dual and we have one of the following two cases:
 - (a) $S = S_M$, $T = T_M$, or
 - (b) T is strict and U is representable and such that, if t is the additive generator of T with $t(e) = 1$, then $\frac{1}{t}$ is also a multiplicative generator of U .
- (ii) (11) if and only if T and S are N -dual and we have one of the following two cases:
 - (a) $S = S_M$, $T = T_M$, or
 - (b) S is strict and U is representable and such that, if s is the additive generator of S with $s(e) = 1$, then s is also a multiplicative generator of U .

Example 2. Let T be the product t -norm T_P which is strict with additive generator $t(x) = -\ln x$ for $x \in [0, 1]$. Now,

$$r(x) = \frac{1}{t(x)} = -\frac{1}{\ln x}; \quad r^{-1}(z) = \exp\left(-\frac{1}{z}\right).$$

Using r as the multiplicative generator (see Proposition 6) we obtain the disjunctive uninorm

$$U_t(x, y) = r^{-1}(r(x) \cdot r(y)) = r^{-1}\left(-\frac{1}{\ln x \cdot \ln y}\right) = \exp(-\ln x \cdot \ln y), \quad x, y \in [0, 1].$$

The neutral element of U_t is $e = \exp(-1)$ and $t(e) = 1$. Consider the U -implication obtained from U_t and the strong negation $N(x) = 1 - x$ given by

$$I_{U_t}(x, y) = U_t(1 - x, y) = \exp(-\ln(1 - x) \cdot \ln y), \quad x, y \in [0, 1].$$

Then it can be easily verified that I_{U_t} satisfies (10) for the N -dual t -conorm of the product t -norm T_P , viz., probabilistic sum t -conorm S_P .

Example 3 (cf. [17, Example 1]). Let S be the probabilistic sum t -conorm S_P which is strict with additive generator $s(x) = -\log(1 - x)$. Using s as the multiplicative generator (see Proposition 6) we obtain the disjunctive uninorm

$$U_s(x, y) = 1 - \exp(-\log(1 - x) \cdot \log(1 - y)), \quad x, y \in [0, 1].$$

The neutral element of U_s is $1 - e^{-1}$ and $s(e) = 1$. Consider the U -implication obtained from U_s and the strong negation $N(x) = 1 - x$ given by

$$I_{U_s}(x, y) = U_s(1 - x, y) = 1 - e^{(\log x \cdot \log(1 - y))}, \quad x, y \in [0, 1].$$

Then it can be easily verified that I_{U_s} satisfies (11) for the N -dual t -norm of the probabilistic sum t -conorm S_P , viz., the product t -norm T_P .

5.1.2. In the setting of Uninorms, i.e., $G, K \in \mathcal{U}$

Exactly along the same lines as in Proposition 17 the following can be proven:

Proposition 18. Let I have (NP). If K is a conjunctive uninorm U_c in (C1) then G is a disjunctive uninorm U_d and vice versa.

Once again since all S - and R -implications satisfy (NP), in the setting of uninorms (C1) reduces to the following two equations.

$$I(U_d(x, y), z) = U_c(I(x, z), I(y, z)) \quad (12)$$

$$I(U_c(x, y), z) = U_d(I(x, z), I(y, z)). \quad (13)$$

Proposition 19. If I has (NP) then U_c in (12) and U_d in (13) are idempotent uninorms.

Proof. Let I have (NP). Then taking $x = 1 = y$ and for any $z \in [0, 1]$ we have $I(U_d(1, 1), z) = I(1, z) = z$, since I is neutral, while $U_c(I(1, z), I(1, z)) = U_c(z, z)$. Now, the equivalence (12) holds if and only if $U_c(z, z) = z$ for all $z \in [0, 1]$, i.e., U_c is an idempotent conjunctive uninorm.

That U_d in (13) is an idempotent disjunctive uninorm can be shown along similar lines. $\square$

Theorem 5. An S -implication I_S , a disjunctive uninorm $U_d = (e_d, T_d, S_d)$ and a conjunctive uninorm $U_c = (e_c, T_c, S_c)$ satisfy the equivalence

- (i) (12) if and only if U_c and U_d are idempotent, N -dual of each other and S is distributive over U_c .
- (ii) (13) if and only if U_c and U_d are idempotent, N -dual of each other and S is distributive over U_d .

Proof. Let I_S be an S -implication, U_d a disjunctive uninorm and U_c a conjunctive uninorm.

- (i) ($\Rightarrow$): Let I_S, U_d, U_c satisfy (12). Since I_S has (NP) by Proposition 19 we have that U_c is idempotent. We know that $I_S(x, 0) = N(x)$ for any $x \in [0, 1]$. Now, taking $z = 0$ in (12), for any $x, y \in [0, 1]$ we have that

$$\text{LHS (12)} = I_S(U_d(x, y), 0) = N(U_d(x, y)),$$

$$\text{RHS (12)} = U_c(I_S(x, z), I_S(y, z)) = U_c(N(x), N(y)),$$

from whence we surmise that U_d is the N -dual of U_c and hence is also idempotent. Since N is strong and hence a bijection on $[0, 1]$ we have from the equivalence (12), for any $x, y \in [0, 1]$

$$\text{LHS (12)} = S(N(U_d(x, y)), z) = S(U_c(N(x), N(y)), z),$$

$$\text{RHS (12)} = U_c(S(N(x), z), S(N(y), z)),$$

and hence S is distributive over U_c .

($\Leftarrow$): The sufficiency can be obtained by retracing the above arguments.

- (ii) Can be shown as above. $\square$

Theorem 6. An R -implication I_{T^*} obtained from a left-continuous t -norm T^* , a disjunctive uninorm $U_d = (e_d, T_d, S_d)$ and a conjunctive uninorm $U_c = (e_c, T_c, S_c)$ satisfy the equivalence

- (i) (12) if and only if $U_c = T_{\mathbf{M}}$ and $U_d = S_{\mathbf{M}}$ are idempotent.
- (i) (13) if and only if U_d is idempotent and $U_c = T_{\mathbf{M}}$.

Proof. Let I_{T^*} be an R -implication obtained from a left-continuous t -norm T^* , U_d a disjunctive uninorm and U_c a conjunctive uninorm.

- (i) ($\Rightarrow$): Let I_{T^*}, U_d, U_c satisfy (12).

$U_c = T$, a t -norm: Let $x = 1, y = 0, z = e_c$. Then since U_d is disjunctive and I_{T^*} has (NP), we have

$$\text{LHS (12)} = I_{T^*}(U_d(1, 0), e_c) = I_{T^*}(1, e_c) = e_c,$$

$$\text{RHS (12)} = U_c(I_{T^*}(1, e_c), I_{T^*}(0, e_c)) = U_c(e_c, 1) = 1.$$

From the equivalence (12) we obtain $e_c = 1$, i.e., $U_c = T$, a t -norm.

$U_d = S$, a **t-conorm**: We show that $e_d = 0$. If not, then there exists an $x, z \in (0, 1)$ such that $x < z < e_d$ and by the ordering property (OP) we have

$$\text{LHS (12)} = I_{T^*}(U_d(x, e_d), z) = I_{T^*}(x, z) = 1,$$

$$\text{RHS (12)} = U_c(I_{T^*}(x, z), I_{T^*}(e_d, z)) = T(1, I_{T^*}(e_d, z)) = I_{T^*}(e_d, z).$$

Once again, from the equivalence (12) we obtain $I_{T^*}(e_d, z) = 1$ or that $e_d \leq z$, a contradiction. Hence $e_d = 0$ and $U_d = S$, a t -conorm. Now (12) reduces to (10) which, from Theorem 3 we know is satisfied for an R -implication I_{T^*} if and only if $T = T_M$ and $S = S_M$

($\Leftarrow$): The sufficiency can be obtained by retracing the above arguments.

(ii) ($\Rightarrow$): Let I_{T^*}, U_d, U_c satisfy (13).

U_d is **idempotent**: Since I_{T^*} has (NP) by Proposition 19 we have that U_d is idempotent.

$U_c = T_M$: To see this, firstly, we show that the neutral element of U_c , $e_c = 1$. If not, let $e_c \in (0, 1)$. Then there exist $z, x \in (0, 1)$ such that $y = e_c < z < x$. Then by the ordering property (OP) we have $I_{T^*}(e_c, z) = 1$ but $I_{T^*}(x, z) \neq 1$. Now,

$$\text{LHS (13)} = I_{T^*}(U_c(x, e_c), z) = I_{T^*}(x, z),$$

$$\text{RHS (13)} = U_d(I_{T^*}(x, z), I_{T^*}(e_c, z)) = U_d(I_{T^*}(x, z), 1) = 1,$$

since U_d is disjunctive. From the equivalence (13) we obtain $I_{T^*}(x, z) = 1$, a contradiction. Hence $U_c = T$, a t -norm. We claim that $U_c = T$ is idempotent. If not, then there exists an $x_0, z \in (0, 1)$ such that $T(x_0, x_0) < z < x_0$ and by the ordering property (OP) we have

$$\text{LHS (13)} = I_{T^*}(T(x_0, x_0), z) = 1,$$

$$\text{RHS (13)} = U_c(I_{T^*}(x_0, z), I_{T^*}(x_0, z)) = I_{T^*}(x_0, z).$$

Once again, from the equivalence (13) we obtain $I_{T^*}(x_0, z) = 1$, a contradiction.

($\Leftarrow$): The sufficiency can be obtained by retracing the above arguments. $\square$

Ruiz and Torrens [16] have proven the following:

Theorem 7 [16, Theorems 1 and 7]. *An U -implication I_U , U_d a disjunctive uninorm and U_c a conjunctive uninorm satisfy*

- (i) (12) if and only if U_c and U_d are N -dual of each other and U is distributive over U_c .
- (ii) (13) if and only if U_c and U_d are N -dual of each other and U is distributive over U_d .

5.2. Some solutions of equivalence (C2)

In this section, we investigate some solutions of the equivalence (C2)

$$M(K(A_1, A_2), A') = K(M(A_1, A'), M(A_2, A')), \quad (\text{C2})$$

where M is a matching function and K is any associative and commutative operator on $[0, 1]$.

5.2.1. The measure of subethood matching function of CDR

In the case of CDR [37] the matching function is the subethood measure given by

$$M_S(A, A') = \min_x I(A'(x), A(x)),$$

where I is a fuzzy implication.

Theorem 8. *Let X be a finite set and $A_1, A_2, A' \in \mathcal{F}(X)$. M_S distributes over $K = \min$, i.e., M_S satisfies (C2) with $K = \min$.*

Proof. Let $K = \min$, then

$$\begin{aligned}
 \text{LHS (C2)} &= M_S(K(A_1, A_2), A') \\
 &= \min_x I(A'(x), \min(A_1(x), A_2(x))) \\
 &= \min_x (\min(I(A'(x), A_1(x)), I(A'(x), A_2(x)))) \quad [\text{By Proposition 13}] \\
 &= \min(\min_x I(A'(x), A_1(x)), \min_x I(A'(x), A_2(x))) \\
 &= K(M_S(A_1, A'), M_S(A_2, A')) \\
 &= \text{RHS (C2)}. \quad \square
 \end{aligned}$$

5.2.2. The class $\mathcal{M}_{\mathbf{v}, \mathbf{w}}$ of matching functions and their distributivity

Let us consider the following class of matching functions which can be seen as generalisations of many of the matching functions in [Example 1](#).

Definition 20. Let V, W be any two commutative and associative functions from $[0, 1]^2$ to $[0, 1]$. Then by $\mathcal{M}_{\mathbf{v}, \mathbf{w}}$ we denote the class of matching functions given by

$$M_{\mathbf{v}, \mathbf{w}}(A, A') = V_{x \in X} W(A(x), A'(x)),$$

where $A, A' \in \mathcal{F}(X)$ of a non-empty set X .

Now the condition (C2) reduces to the following distributive equation,

$$V^*(M_{\mathbf{v}, \mathbf{w}}(A_1, A'), M_{\mathbf{v}, \mathbf{w}}(A_2, A')) = M_{\mathbf{v}, \mathbf{w}}(V^*(A_1, A_2), A') \quad (14)$$

where $V^* : [0, 1]^2 \rightarrow [0, 1]$, $V^*(A_1, A_2) \in \mathcal{F}(X)$ and is as given in [Definition 14](#).

Theorem 9. Let X be a finite set and $A_1, A_2, A' \in \mathcal{F}(X)$. If the associative operators V^*, V, W are such that $V^* = V$ and W distributes over V , then $M_{\mathbf{v}, \mathbf{w}}$ distributes over V^* , i.e., $M_{\mathbf{v}, \mathbf{w}}$ satisfies (14).

Proof. Let $V^* = V$ and W distribute over V , i.e.,

$$W(V(x, y), z) = V(W(x, z), W(y, z)).$$

For ease of understanding and notation, let $X = \{x_1, x_2\}$ and let $a_{ij} = A_i(x_j)$, $a'_i = A'(x_i)$ for $i, j \in \{1, 2\}$. By the associativity of V^*, V, W the proof can be extended to arbitrary arguments.

$$\begin{aligned}
 \text{LHS (14)} &= V(V_{x \in X} W(A_1(x), A'(x)), V_{x \in X} W(A_2(x), A'(x))) \\
 &= V(V(W(A_1(x_1), A'(x_1)), W(A_1(x_2), A'(x_2))), V(W(A_2(x_1), A'(x_1)), W(A_2(x_2), A'(x_2)))) \\
 &= V(V(W(a_{11}, a'_1), W(a_{12}, a'_2)), V(W(a_{21}, a'_1), W(a_{22}, a'_2)))) \\
 &= V(W(a_{11}, a'_1), W(a_{12}, a'_2), W(a_{21}, a'_1), W(a_{22}, a'_2))) \\
 &= V(V(W(a_{11}, a'_1), W(a_{21}, a'_1)), V(W(a_{12}, a'_2), W(a_{22}, a'_2)))) \\
 &= V(W(V(a_{11}, a_{21}), a'_1), W(V(a_{12}, a_{22}), a'_2)) \\
 &= V(W(V(A_1(x_1), A_2(x_1)), A'(x_1)), W(V(A_1(x_2), A_2(x_2)), A'(x_2))) \\
 &= V_{x \in X} (W(V(A_1(x), A_2(x)), A'(x))) = M_{\mathbf{v}, \mathbf{w}}(V^*(A_1, A_2), A') = \text{RHS (14)}. \quad \square
 \end{aligned}$$

5.2.3. Eq. (14) in the setting of t -norm, t -conorm or uninorms

Since the associative and commutative operator K features in both the equivalences (C1) and (C2) and from the previous section we know that usually $K \in \mathcal{T} \cup \mathcal{S} \cup \mathcal{U}$ when $J = I$ – a fuzzy implication – in this section we investigate the cases where $V, W \in \mathcal{T} \cup \mathcal{S} \cup \mathcal{U}$. Then the following Corollary is immediate from [Theorem 9](#):

Corollary 21. Let X be a finite set and $A_1, A_2, A' \in \mathcal{F}(X)$.

- (i) If the t -conorm S distributes over a t -norm T , then $M_{T,S}$ satisfies (14) with $V^* = T$.
- (ii) If the t -norm T distributes over a t -conorm S , then $M_{S,T}$ satisfies (14) with $V^* = S$.
- (iii) If the uninorm U_* distributes over another uninorm U^* , then M_{U^*,U_*} satisfies (14) with $V^* = U^*$.

Theorem 10. Let X be a finite set, $A_1, A_2, A' \in \mathcal{F}(X)$, $U_c = (T_c, S_c, e_c)$, $U_d = (T_d, S_d, e_d)$ be conjunctive and disjunctive uninorms, respectively. There exists no disjunctive uninorm U^* that satisfies (14) with M_{U_c, U_d} .

Proof. Let a disjunctive uninorm U^* satisfy (14) with M_{U_c, U_d} . Once again, we let $X = \{x_1, x_2\}$ and let $a_{ij} = A_i(x_j)$, $a'_i = A'(x_i)$ for $i, j \in \{1, 2\}$.

$$\begin{aligned}
 \text{LHS (14)} &= U^*(U_{c_{x \in X}} U_d(A_1(x), A'(x)), U_{c_{x \in X}} U_d(A_2(x), A'(x))) \\
 &= U^*(U_c(U_d(A_1(x_1), A'(x_1)), U_d(A_1(x_2), A'(x_2))), \\
 &\quad U_c(U_d(A_2(x_1), A'(x_1)), U_d(A_2(x_2), A'(x_2)))) \\
 &= U^*(U_c(U_d(a_{11}, a'_1), U_d(a_{12}, a'_2)), U_c(U_d(a_{21}, a'_1), U_d(a_{22}, a'_2))) \\
 \text{RHS (14)} &= M_{U_c, U_d}(U^*(A_1, A_2), A') \\
 &= U_{c_{x \in X}}(U_d(U^*(A_1(x), A_2(x)), A'(x))) \\
 &= U_c(U_d(U^*(A_1(x_1), A_2(x_1)), A'(x_1)), \\
 &\quad U_d(U^*(A_1(x_2), A_2(x_2)), A'(x_2))) \\
 &= U_c(U_d(U^*(a_{11}, a_{21}), a'_1), U_d(U^*(a_{12}, a_{22}), a'_2))
 \end{aligned}$$

Now, LHS of (14) should be equal to RHS of (14) for the disjunctive uninorm U^* to satisfy (14) with M_{U_c, U_d} . In the case when $a'_1 = a'_2 = e_d$, the above equivalence reduces to

$$U^*(U_c(a_{11}, a_{12}), U_c(a_{21}, a_{22})) = U_c(U^*(a_{11}, a_{21}), U^*(a_{12}, a_{22})) \quad (15)$$

But (15) does not hold for any pair of disjunctive and conjunctive uninorm. To see this, if possible, let for some disjunctive uninorm U^* (i.e., $U^*(1, 0) = 1$), conjunctive uninorm U_* (i.e., $U_*(1, 0) = 0$) and $x, x', y, y' \in [0, 1]$

$$U^*(U_*(x, y), U_*(x', y')) = U_*(U^*(x, x'), U^*(y, y')).$$

Then letting $x = y' = 0$ and $x' = y = 1$ we have

$$U^*(U_*(0, 1), U_*(1, 0)) = U^*(0, 0) = 0 \neq 1 = U_*(U^*(0, 1), U^*(1, 0)) = U_*(1, 1).$$

Hence there exists no disjunctive uninorm U^* that satisfies (14) with M_{U_c, U_d} . $\square$

Theorem 11. Let X be a finite set, $A_1, A_2, A' \in \mathcal{F}(X)$, $U_c = (T_c, S_c, e_c)$, $U_d = (T_d, S_d, e_d)$ be conjunctive and disjunctive uninorms, respectively. There exists no conjunctive uninorm U^* that satisfies (14) with M_{U_d, U_c} .

Corollary 22. Let X be a finite set, $A_1, A_2, A' \in \mathcal{F}(X)$, S a t -conorm and T a t -norm. There exists no

- (i) t -conorm S^* that satisfies (14) with $M_{T,S}$;
- (ii) t -norm T^* that satisfies (14) with $M_{S,T}$.

Summarising the above results, Table 3 gives examples of sets of operators G, J, K, M that satisfy the conditions (C1) and (C2) of Theorem 1.

6. A numerical example

In this section, we present a numerical example to show the efficiency and invariance in the inference obtained when the above rule reduction procedure is employed. Consider a rule base consisting of the following three rules:

- If $\tilde{x}_1$ is A_1 and $\tilde{x}_2$ is B_1 Then $\tilde{y}$ is C ,
 - If $\tilde{x}_1$ is A_2 and $\tilde{x}_2$ is B_2 Then $\tilde{y}$ is C ,
 - If $\tilde{x}_1$ is A_3 and $\tilde{x}_2$ is B_3 Then $\tilde{y}$ is D ,
- ($\mathcal{R}_O$)

Table 3
Some classes of operators obeying Eqs. (C1), (C2)

G	J	K	M	Comments
S_M	J_{MVR}	S_M	M_Z	AARS [48]
S_M	I_{T^*}	T_M	M_S	CDR [37]
T_M	I_S/I_{T^*}	S_M	$M_{S_M,T}$	S_M distributes over any T
S_M	I_S/I_{T^*}	T_M	$M_{T_M,S}$	T_M distributes over any S
U_c	I_S	U_d	$M_{U_d,U}$	S, U_c, U_d satisfy Theorem 5 (i) U_d distributes over U
U_d	I_S	U_c	$M_{U_c,U}$	S, U_c, U_d satisfy Theorem 5 (ii) U_c distributes over U
U_c	I_{T^*}	U_d	$M_{U_d,U}$	S, U_c, U_d satisfy Theorem 6 (i) U_d distributes over U
U_d	I_{T^*}	U_c	$M_{U_c,U}$	S, U_c, U_d satisfy Theorem 6 (ii) U_c distributes over U
T	I_U	S	$M_{S,U}$	T, U, S satisfy Theorem 4 (i) S distributes over U
S	I_U	T	$M_{T,U}$	T, U, S satisfy Theorem 4 (ii) T distributes over U
U_c	I_U	U_d	$M_{U_d,U}$	U, U_c, U_d satisfy Theorem 7 (i) U distributes over U_c
U_d	I_U	U_c	$M_{U_c,U}$	U, U_c, U_d satisfy Theorem 7 (ii) U distributes over U_d

where A_i, B_i for $i=1,2,3$ and C, D are fuzzy sets defined on $X = \{x_1, x_2, x_3, x_4\}$, $Y = \{y_1, y_2, y_3\}$ and $Z = \{z_1, z_2, z_3, z_4\}$, respectively, and are given as follows:

$$\begin{aligned}
 A_1 &= [.3 \ .5 \ 0 \ 1]; \quad B_1 = [.3 \ .4 \ .9]; \quad C = [1 \ .8 \ .4 \ .7] \\
 A_2 &= [.36 \ .25 \ .3 \ .8]; \quad B_2 = [.12 \ .67 \ .99]; \quad D = [.8 \ .7 \ 0 \ 1] \\
 A_3 &= [.9 \ 0 \ .8 \ .5]; \quad B_3 = [.2 \ .7 \ .6].
 \end{aligned}$$

We employ the AARS inference scheme of Turksen et al., [48] with $G = S_M$; $J = J_{MVR}$; $K = S_M$; $M = M_Z$ (see Table 3). Let the given input be ($\tilde{x}_1$ is A' ; $\tilde{x}_2$ is B') where

$$A' = [.4 \ .7 \ .8 \ 0] \quad \& \quad B' = [.2 \ 0 \ 1].$$

In the following we infer both with the original rule base ($\mathcal{R}_O$) and the reduced rule base ($\mathcal{R}_R$) and show that the inferred output is identical in both the cases.

6.1. Inference with the original rule base ($\mathcal{R}_O$)

6.1.1. Calculating the matching degrees

$$\begin{aligned}
 M_Z(A_1, A') &= \max(.3, .5, 0, 0) = 0.5; \quad M_Z(B_1, B') = \max(.2, 0, .9) = 0.9 \\
 M_Z(A_2, A') &= \max(.36, .25, .3, 0) = 0.36; \quad M_Z(B_2, B') = \max(.12, 0, .99) = 0.99 \\
 M_Z(A_3, A') &= \max(.4, 0, .8, 0) = 0.8; \quad M_Z(B_3, B') = \max(.2, 0, .6) = 0.6.
 \end{aligned}$$

6.1.2. Combining the matching degrees to obtain similarity values s_i

Following this we calculate the similarity values using the operator $K = S_M$, as follows:

$$\begin{aligned}
 s_1 &= K(M_Z(A_1, A'), M_Z(B_1, B')) = \max(0.5, 0.9) = 0.9 \\
 s_2 &= \max(0.36, 0.99) = 0.99 \\
 s_3 &= \max(0.8, 0.6) = 0.8.
 \end{aligned}$$

6.1.3. Modifying the consequents based on the similarity values s_i

$$\begin{aligned} J(s_1, C) &= J_{\text{MVR}}(s_1, C) = (0.9) \cdot [1 \quad .8 \quad .4 \quad .7] = [.9 \quad .72 \quad .36 \quad .63] = C'_1 \\ J(s_2, C) &= J_{\text{MVR}}(s_2, C) = (0.99) \cdot [1 \quad .8 \quad .4 \quad .7] = [.99 \quad .79 \quad .396 \quad .69] = C'_2 \\ J(s_3, D) &= J_{\text{MVR}}(s_3, D) = (0.8) \cdot [.8 \quad .7 \quad 0 \quad 1] = [.64 \quad .56 \quad 0 \quad .8] = C'_3. \end{aligned}$$

6.1.4. Combining the obtained consequents for a conclusion

$$C' = G(C'_1, C'_2, C'_3) = S_{\mathbf{M}}(C'_1, C'_2, C'_3) = [.99 \quad .79 \quad .396 \quad .8]. \quad (16)$$

6.2. Inference with the reduced rule base ($\mathcal{R}_{\mathbf{R}}$)

As can be seen, the first two rules in the original rule base ($\mathcal{R}_{\mathbf{O}}$) have the same consequent fuzzy set C and hence can be reduced to the rule base consisting of the following two rules:

$$\begin{aligned} \text{If } \tilde{x}_1 \text{ is } A^* \text{ and } \tilde{x}_2 \text{ is } B^* \text{ Then } \tilde{y} \text{ is } C, \\ \text{If } \tilde{x}_1 \text{ is } A_3 \text{ and } \tilde{x}_2 \text{ is } B_3 \text{ Then } \tilde{y} \text{ is } D, \end{aligned} \quad (\mathcal{R}_{\mathbf{R}})$$

where

$$\begin{aligned} A^* &= K(A_1, A_2) = S_{\mathbf{M}}(A_1, A_2) = [.36 \quad .5 \quad .3 \quad 1], \\ B^* &= K(B_1, B_2) = S_{\mathbf{M}}(B_1, B_2) = [.3 \quad .67 \quad .99]. \end{aligned}$$

Once again, calculating the matching degrees with respect to the same input pair (A' ; B') we obtain

$$\begin{aligned} M_{\mathbf{Z}}(A^*, A') &= M_{\mathbf{Z}}(S_{\mathbf{M}}(A_1, A_2), A') = \max(.36, .5, .3, 0) = 0.5, \\ M_{\mathbf{Z}}(B^*, B') &= M_{\mathbf{Z}}(S_{\mathbf{M}}(B_1, B_2), B') = \max(.2, 0, .99) = 0.99. \end{aligned}$$

Hence the similarity value of the input to the new rule is $s_1^* = \max(0.5, 0.99) = 0.99$, which modifies the consequent C as

$$C_1^* = J(s_1^*, C) = J_{\text{MVR}}(s_1^*, C) = (0.99) \cdot [1 \quad .8 \quad .4 \quad .7] = [.99 \quad .79 \quad .396 \quad .69].$$

Combining the obtained consequents for a conclusion we obtain

$$C'' = G(C_1^*, C'_3) = S_{\mathbf{M}}(C_1^*, C'_3) = [.99 \quad .79 \quad .396 \quad .8],$$

i.e., $C'' = C'$ in (16). Equivalently, we have shown that the inference obtained for the same inputs from the original and reduced rule bases are identical.

Remark 23. In the example above, the operators G , K turned out to be the same because of the choice of the operator J . It should be noted that if $J = I$, a fuzzy implication, then G , K are usually different as can be seen from the results in Section 5.1 and Table 3.

7. Concluding remarks

In this work we have proposed a simple rule reduction technique that of combining the antecedent(s) of rules that have the same consequent. We have shown that this type of rule reduction can be done in Similarity Based Reasoning inference schemes that employ a fuzzy if-then rule base, in such a way that the inferences obtained from the original and the reduced rule bases are identical. Towards this end, some sufficient conditions involving the inference operators employed in these SBR inference schemes were proposed. Subsequently, these conditions were investigated and many solutions were presented for some specific SBR inference schemes.

In fact, it can be shown that the existence of an associative and commutative operator $H: [0, 1]^2 \rightarrow [0, 1]$ satisfying the following condition (C3),

$$H(K(M(A_1, A), M(B_1, B)), K(M(A_2, A), M(B_2, B))) = K(M(H(A_1, A_2), A), M(H(B_1, B_2), B)), \quad (C3)$$

along with (C1), is sufficient to enable inference invariant rule reduction along the proposed approach. Investigations of equivalence (C3) will be taken up in future works.

In this work, though we have only considered three families or classes of fuzzy implications, there are a few more families that have been proposed, viz., the residual implications of uninorms I_{U*} in [22] and the recently proposed families of f -generated implications I_f and g -generated implications I_g by Yager in [52] and h -generated implications I_h in [3]. The distributivity of I_{U*} over uninorms is studied in [19] while that of I_f over t -norms and t -conorms is done in [4]. Hence these families of fuzzy implications also become potential solutions for equivalence (C1).

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