On the Distributivity of Implication Operators Over T and S Norms

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***Abstract*—In this paper, we explore the distributivity of implication operators [especially Residuated (R)- and Strong (S)-implications] over Takagi (T)- and Sugeno (S)-norms. The motivation behind this work is the on going discussion on the law $[(p \wedge q) \rightarrow r] \equiv [(p \rightarrow r) \vee (q \rightarrow r)]$ in fuzzy logic as given in the title of the paper by Trillas and Alsina. The above law is only one of the four basic distributive laws. The general form of the previous distributive law is $J(T(p,q),r) \equiv S(J(p,r),J(q,r))$. Similarly, the other three basic distributive laws can be generalized to give equations concerning distribution of fuzzy implications J on T- and S-norms. In this paper, we study the validity of these equations under various conditions on the implication operator J. We also propose some sufficiency conditions on a binary operator under which the general distributive equations are reduced to the basic distributive equations and are satisfied. Also in this work, we have solved one of the open problems posed by M. Baczynski (2002).**

***Index Terms*—Distributivity, fuzzy implication, fuzzy logic, S-norms, T-norms.**

I. Introduction

RECENTLY, there has been a lot of discussion [1]–[6] centred around a paper by Combs and Andrews [1] where they attempt to exploit the equivalence

$$(p \wedge q) \rightarrow r \equiv (p \rightarrow r) \vee (q \rightarrow r) \tag{1}$$

toward eliminating combinatorial rule explosion in fuzzy systems. (1) is only one of four such equations as listed in [7], which deals with the distributivity of Implication operators with respect to T- and S-norms, the rest of them being

$$(p \vee q) \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r) \tag{2}$$

$$r \rightarrow (t \wedge s) \equiv (r \rightarrow t) \wedge (r \rightarrow s) \tag{3}$$

$$r \rightarrow (t \vee s) \equiv (r \rightarrow t) \vee (r \rightarrow s). \tag{4}$$

In [6], Trillas and Alsina have investigated the conditions under which the following general form of (1):

$$J(T(p,q),r) \equiv S(J(p,r),J(q,r)) \tag{5}$$

holds for the main four different types of implication operators, viz., R-, S-, QL-, and ML-implications, where T and S denote any T- and S-norms, respectively. The generalizations of (2)–(4) are as follows:

$$J(S(p,q),r) \equiv T(J(p,r),J(q,r)) \tag{6}$$

$$J(r,T_1(s,t)) \equiv T_2(J(r,s),J(r,t)) \tag{7}$$

$$J(r,S_1(s,t)) \equiv S_2(J(r,s),J(r,t)). \tag{8}$$

The general forms of these equations have a role in lossless rule reduction for Control/Expert systems in Fuzzy Logic. Work on these lines has appeared in [8]. Equation (7) has been discussed in [9] under the assumption that $T = T_1 = T_2$ is a strict t-norm and the implication J is continuous except at (0, 0).

In this paper, we study the validity of (2)–(4) under various conditions. We propose some sufficiency conditions on a binary operator under which the general distributive equations (5)–(8) are reduced to the basic distributive equations (1)–(4)and are satisfied. Also in this paper, we have solved one of the open problems posed by Baczynski [10].

A. Few Definitions

To make this paper self-contained, we briefly mention some of the concepts and results employed in the rest of the paper.

Definition 1: A t-norm T is a binary operation from $I \times I \rightarrow I$ such that $\forall a,b,c \in I = [0,1]$,

- $T(a,1) = a$;
- $T(a,0) = 0$;
- $T(a,b) = T(b,a)$;
- $T(a,T(b,c)) = T(T(a,b),c)$;
- $T(a,b)$ is monotonic in both the variable;
- T is continuous.

Definition 2: An S-norm S is a binary operation from $I \times I \rightarrow I$ such that $\forall a,b,c \in I = [0,1]$,

- $S(a,1) = 1$;
- $S(a,0) = a$;
- $S(a,b) = S(b,a)$;
- $S(a,S(b,c)) = S(S(a,b),c)$;
- $S(a,b)$ is monotonic in both the variable;
- S is continuous.

Definition 3: An implication J is a binary operation from $I \times I \rightarrow I$ such that the following properties hold:

- $J(p,r) \geq J(q,r)$, if $q \geq p$;
- $J(p,r) \leq J(p,s)$, if $r \leq s$;
- $J(1,t) = t, \forall t \in I$-Neutrality principle;
- $J(0,t) = 1, \forall t \in I$-Falsity principle;
- $J(p,J(q,r)) = J(q,J(p,r))$-Exchange principle.

Definition 4: A strong Negation N is a unary operator from $I \rightarrow I$ such that

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- $N(a)$ is a decreasing function;
- $N(N(a)) = a, \forall a \in I$;
- $N(0) = 1; N(1) = 0$.

Definition 5: An S-implication is obtained from an S-norm S and a strong negation N as follows: $a \to b = S(N(a), b), \forall a, b \in I$.

Definition 6: An R-implication is obtained from a T-norm T as its residuation as follows: $a \to b = \bigvee\{x \in I : T(a,x) \leq b\}, \forall a, b \in I$.

II. On the General Forms of (1)–(4)

In this section, we consider each of the general forms (5)–(8) of the basic distributive equations (1)–(4).

A. On the Equation $J(T(p,q),r) \equiv S(J(p,r), J(q,r))$ (5)

In [6], it is shown that (5) reduces to (9) under any of the implication operators R-,S-, QL-, and ML-implications

$$J(T(p,q),r) = \max(J(p,r), J(q,r)) \tag{9}$$

i.e., $S = \max$, in (5). Interpreting J as S-, R-, or a QL- implication reduces (9) to (10)

$$J(\min(p,q),r) = \max(J(p,r), J(q,r)), \tag{10}$$

i.e., $T = \min$. Also it is shown that all R- and S- implications satisfy (10) which is (5) with $T = \min$ and $S = \max$, but only a certain restricted class of QL-implications satisfy (10). Also, ML-implications do not satisfy (10), but if $T = S = \min$ in (5) then J interpreted as an ML-implication satisfies the following law:

$$J(\min(p,q),r) \equiv \min(J(p,r), J(q,r)). \tag{11}$$

Due to these results, in the remaining part of this work we will be concerned with only R- and S-implications, the implication operators for which (5) always holds.

B. On the Equation $J(S(p,q),r) \equiv T(J(p,r), J(q,r))$ (6)

Letting $p = q = 1$, we have

$$J(1,r) = T(J(1,r), J(1,r)). \tag{12}$$

We know that $J(1,r) = r, \forall r \in$ I, for S- and R- implications. Thus, we have from (12), $r = T(r,r), \forall r \in$ I. This implies $T = \min$, the only idempotent t-norm. Thus, (6) reduces to (13) in the case of R- and S-implications

$$J(S(p,q),r) = \min(J(p,r), J(q,r)). \tag{13}$$

Also, (6) can be shown to reduce to (13) for QL- and ML-implications. Let us investigate (13) separately for S- implications and R-implications.

1) Case I: S-Implications: Let J_S^* be an S-implication obtained from an S-norm S^* and a strong negation N^* as follows:

$$J_S^*(p,q) = S^*(N^*(p), q)$$

We know $J_S^*(p,0) = N^*(p)$. Taking $r = 0$ in (13), we have

$$\begin{aligned} N^*(S(p,q)) &= \min(N^*(p), N^*(q)) \\ &\Rightarrow S(p,q) = \max(p,q) \\ &\Rightarrow S = \max \end{aligned}$$

the dual of min. Also

$$\begin{aligned} \text{LHS of (13)} &= S^*(N^*(\max(p,q)), r) \\ &= S^*(\min(N^*(p), N^*(q)), r) \\ &= \min(S^*(N^*(p),r), S^*(N^*(q),r)) \\ &\quad [\text{by distributivity of } S^* \text{ over } \min] \\ &= \min(J_S^*(p,r), J_S^*(q,r)) \\ &= \text{RHS of (13)}, \end{aligned}$$

$$\text{i.e., } J_S^*(\max(p,q),r) = \min(J_S^*(p,r), J_S^*(q,r) \tag{14}$$

where J_S^* is an S-implication. Thus, (14) holds for all S-implications.

2) Case II: R-Implications: Let J_R be an R-implication given by a t-norm T^* as follows:

$$J_R(p,q) = J^{T^*}(p,q) = \bigvee\{x \in I : T^*(p,x) \leq q\}. \tag{15}$$

Now, letting $p = q$ in (13) we have

$$\begin{aligned} J_R(S(p,p),r) &= \min(J_R(p,r), J_R(p,r)) \\ &= J_R(p,r). \end{aligned} \tag{16}$$

Since S is an s-norm, $S(p,p) \geq p$. If $S(p_0,p_0) > p_0$ for some $p_0 \in I$ then for this $p_0 \,\exists\, t_0 \in I \ni S(p_0,p_0) = s_0 > t_0 > p_0$. Then, from (16) we have

$$\begin{aligned} J_R(S(p_0,p_0),t_0) &= J_R(p_0,t_0), \\ \text{i.e., } J_R(s_0,t_0) &= J_R(p_0,t_0). \end{aligned} \tag{17}$$

However, J_R being an R-implication has the identity principle, i.e., $J_R(a,b) = 1 \Leftrightarrow a \leq b, \forall a, b \in I$, and thus $J_R(p_0,t_0) = 1$ and therefore $J_R(s_0,t_0) = 1$, which is not true since this would imply $s_0 \leq t_0$. Therefore, $s_0 = p_0$, i.e., $S(p,p) = p, \forall p \in I$. This implies $S = \max$, the only idempotent s-norm. Thus, (13) reduces to (18) when J is an R-implication

$$J(\max(p,q),r) = \min(J(p,r), J(q,r)). \tag{18}$$

We know in a Residuated Lattice L, for the adjoin couple $(\star, \to)$ we have

$$a \star b \leq c \Leftrightarrow a \leq b \to c \qquad \forall a,b,c \in L \tag{19}$$

where $\to$ is an R-implication obtained from the t-norm $\star$. The left-hand side of (18), using (19), becomes

$$\begin{aligned} &t \leq J_R(\max(p,q),r) \\ &\Leftrightarrow t \star \max(p,q) \leq r \\ &\Leftrightarrow \max(t \star p, t \star q) \leq r \\ &\Leftrightarrow t \star p \leq r \text{ and } t \star q \leq r \\ &\Leftrightarrow t \leq J_R(p,r) \text{ and } t \leq J_R(q,r) \\ &\Leftrightarrow t \leq \min(J_R(p,r), J_R(q,r)). \end{aligned}$$

Since $t \in I$ is arbitrary, we have (18). Thus, (18) holds for all R-implications.

In the rest of this paper, so as to use an unobtrusive notation and since the context itself will make it clear, we will denote by J both an S- and an R-implication.

C. On the Equation $J(r, T_1(s,t)) \equiv T_2(J(r,s), J(r,t))$ (7)

Let us consider S- and R-implications for the (7). We know for any S- and R-implication, $J(1,t) = t, \forall t \in I$. Putting $r = 1$ in

(7), we get $J(1, T_1(t,s)) = T_2(J(1,t), J(1,s))$ i.e., $T_1(t,s) = T_2(t,s), \forall t, s \in I$. Hence $T_1 \equiv T_2 = T$ on I^2.

Thus, (7) becomes

$$J(r, T(s,t)) = T(J(r,s), J(r,t)). \tag{20}$$

1) Case I: S-Implication: Taking $t = s = 0$ in (20) we obtain

$$J(r,0) = T(J(r,0), J(r,0)). \tag{21}$$

We know that for S-implications $J(r,0) = N(r), \forall r \in I$, thus (21) becomes $N(r) = T(N(r), N(r)), \forall r \in I$. Since N is a strong negation, we have that $t = T(t,t), \forall t \in I$. Hence, $T = \min$. (20) now becomes

$$J(r, \min(s,t)) = \min(J(r,s), J(r,t)). \tag{22}$$

Also

$$\begin{aligned} \text{LHS of (22)} &= S(N(r), \min(t,s)) \\ &= \min(S(N(r),t), S(N(r),s) \\ &\quad \text{[by distributivity of } S \text{ over } \min] \\ &= \min(J(r,t), J(r,s)) \\ &= \text{RHS of (22)} \end{aligned}$$

Thus, (22) holds for all S-implications.

2) Case II: R-Implication: Let $T_1 = T_2 = T \neq \min$. Then there exist s and t such that $T(s,t) < s < t$. Let r be so chosen that $T(s,t) < r < s < t$. Now, LHS of (20) $= J(r, T(s,t))$. RHS of (20) $= T(J(r,s), J(r,t)) = T(1,1) = 1$, by the identity principle of R-implications, i.e., $J_R(a,b) = 1 \Leftrightarrow a \leq b, \forall a, b \in I$. Thus, LHS of (20) $= 1$, which again by the identity principle implies $r \leq T(s,t)$, a contradiction to our assumption. Therefore, we see that when J is an R-implication $T_1 \equiv T_2 \equiv T \equiv \min$ and (20) reduces to

$$J(r, \min(s,t)) = \min(J(r,s), J(r,t)). \tag{23}$$

We have to show that when J is an R-implication (23) holds. Again, by the definition of a residuated lattice, we have

$$\begin{aligned} & u \leq J(r, \min(t,s)) \\ & \Leftrightarrow u \star r \leq \min(t,s) \\ & \Leftrightarrow u \star r \leq t \text{ and } u \star r \leq s \\ & \Leftrightarrow u \leq J(r,t) \text{ and } u \leq J(r,s) \\ & \Leftrightarrow u \leq \min(J(r,t), J(r,s). \end{aligned}$$

Since $u \in I$ is arbitrary, we have (23). Thus, (23) holds for all R-implications.

In [10], the following was posed as an open problem.

Open Problem 2 [10]: What is the solution of the functional equation

$$J(r, T(s,t)) = T(J(r,s), J(r,t)) \tag{24}$$

when J is a given fuzzy implication, for example, R-implication or S-implication?

As can be seen, (24) is only a special case of (7), when $T_1 \equiv T_2 \equiv T$.

D. *On the Equation $J(r, S_1(s,t)) \equiv S_2(J(r,s), J(r,t))$ (8)*

We know for any S- and R-implication, $J(1,t) = t, \forall t \in I$. Putting $r = 1$ in (8), we get $J(1, S_1(t,s)) = S_2(J(1,t), J(1,s))$, i.e., $S_1(t,s) = S_2(t,s), \forall t, s \in I$. Hence, $S_1 \equiv S_2 = S$ on I^2.

Thus, (8) becomes

$$J(r, S(s,t)) = S(J(r,s), J(r,t)). \tag{25}$$

1) Case I: S-Implications: Taking $t = s = 0$ in (25) we obtain, $J(r,0) = S(J(r,0), J(r,0)), \forall r \in I$. We know that for S-implications $J(r,0) = N(r), \forall r \in I$, thus we have from above $N(r) = S(N(r), N(r)), \forall r \in I$. Again, since N is a strong negation, we have that $t = S(t,t), \forall t \in I$. Hence, $S = \max$. (25) now becomes

$$J(r, \max(s,t)) = \max(J(r,s), J(r,t)). \tag{26}$$

Further, it can be shown (as has been shown for (22) in Section II.C.1) that if J is an S-implication then it satisfies (26).

2) Case II: R-Implications: We know that if J is an R-implication obtained from a nilpotent t-norm then J is continuous in both the variables [11]. We also have that $J(1,0) = 0$ and $J(0,0) = 1$ and since J is continuous J takes all values in $I = [0,1]$ when the second coordinate of J is fixed, i.e., $\forall r \in I \, \exists \, s \in I \ni J(s,0) = r$. Then, (25) becomes

$$\begin{aligned} J(s, S(0,0)) &= S(J(s,0), J(s,0)) \\ r &= S(r,r) \qquad \forall \, r \in I \\ &\Rightarrow S \equiv \max \end{aligned}$$

the only idempotent s-norm.

From this discussion, the following results arise.

Theorem 1: An S-implication or an R-implication satisfies (5) if and only if $S = \max$ and $T = \min$.

Theorem 2: An S-implication or an R-implication satisfies (6) if and only if $S = \max$ and $T = \min$.

Theorem 3: An S-implication or an R-implication satisfies (7) if and only if $T_1 = T_2 = \min$.

Theorem 4: An S-implication or an R-implication obtained from a nilpotent t-norm satisfies (8) if and only if $S_1 = S_2 = \max$.

Even though the proof of reduction of (8) to (4) has been given for the case where the R-implication was obtained from a nilpotent t-norm, the authors have a strong feeling that it holds for the case when the R-implication is obtained from a strict t-norm, and that Theorem 4 will hold for any R-implication.

III. Some Sufficiency Conditions on the General Forms of the Distributive Equations

In this section, we consider the general equations (5)–(8) and propose sufficiency conditions on a binary operator $J : I \times I \rightarrow I$ that reduces them to (1)–(4), the basic distributive equations.

A. *On the Equation $J(T(r,s), t) \equiv S(J(r,t), J(s,t))$ (5)*

Let J be a binary operator from $I \times I$ to I. Then, we have the following propositions.

Proposition 1: If J satisfies neutrality, i.e., $J(1,t) = t, \forall t \in I$, then $S = \max$.

Proof: $r = s = 1$ in (5) reduces it to $J(1,t) = S(J(1,t), J(1,t)), \forall t \in I$, which implies $t = S(t,t), \forall t \in I$. Hence, $S = \max$, the only idempotent s-norm. ■

Proposition 2: If J satisfies the neutrality principle and is one-to-one in the first variable, then $T = \min$ in (5).

Proof: As shown by neutrality of J, (5) becomes $J(T(r,s),t) = \max(J(r,t), J(s,t))$. Letting $r = s$ in the above equation with a fixed t, we have, $J(T(r,r),t) = \max(J(r,t), J(r,t)$, i.e., $J(T(r,r),t = J(r,t)$, which implies $T(r,r) = r, \forall r \in I$, since J is one-to-one in the first variable. Hence, $T = \min$, the only idempotent t-norm. ■

Proposition 3: A binary operator J is nonincreasing in the first variable if and only if J satisfies

$$J(\min(r,s),t) = \max(J(r,t), J(s,t)) \tag{27}$$

which is nothing but (1).

Proof: Let J be nonincreasing in the first variable, i.e., $r < s \Rightarrow J(r,t) \geq J(s,t), \forall r,s \in I$. Then the two sides of (27) are equal. On the other hand, let J satisfy (27) and let $r < s$ for any $t \in I$. Then from (27) we have, $J(r,t) = \max(J(r,t), J(s,t))$, which implies $J(r,t) \geq J(s,t)$, i.e., J is nonincreasing in the first variable. ■

From the definition of an implication operator and the aforementioned propositions, we have the following.

Theorem 5: Any implication operator J that is one-to-one in the first variable reduces (5) to (1) and satisfies (1).

B. On the Equation $J(S(r,s),t) \equiv T(J(r,t), J(s,t))$ (6)

Let J be a binary operator from $I \times I$ to I. Then, we have the following propositions, the proofs of which can be obtained as in Section III-A.

Proposition 4: If J satisfies neutrality, i.e., $J(1,t) = t, \forall t \in I$ then $T = \min$.

Proposition 5: If J satisfies the neutrality principle and is one-to-one in the first variable, then $S = \max$ in (6).

Proposition 6: A binary operator J is nonincreasing in the first variable if and only if J satisfies

$$J(\max(r,s),t) = \min(J(r,t), J(s,t)) \tag{28}$$

which is nothing but (2).

From the definition of an implication operator and the aforementioned propositions, we have the following.

Theorem 6: Any implication operator J that is one-to-one in the first variable reduces (6) to (2) and satisfies (2).

C. On the Equation $J(r, T_1(s,t)) \equiv T_2(J(r,s), J(r,t))$ (7)

Let J be a binary operator from $I \times I$ *to* I. Then, we have the following propositions.

Proposition 7: If J satisfies the neutrality principle, i.e., $J(1,t) = t, \forall t \in I$ then $T_1 \equiv T_2$ on I^2.

Proof: Taking $r = 1$ in (7) we have, $J(1, T_1(s,t)) = T_2(J(1,s), J(1,t))$, which implies $T_1(s,t) = T_2(s,t), \forall s,t \in I$. Hence, $T_1 \equiv T_2$ on I^2. ■

Proposition 8: If J is neutral and onto in the first variable, then $T_1 = T_2 = \min$ in (7).

Proof: As shown before, by the neutrality of J we have $T_1 \equiv T_2$ on I^2 in (7), i.e.,

$$J(r, T(s,t)) = T(J(r,s), J(r,t)) \tag{29}$$

Let $s = t = 0$, then from (29) we have, $J(r,0) = T(J(r,0), J(r,0))$. Since J is onto in the first variable, $\forall k \in I \; \exists r \in I \ni J(r,0) = k$. Hence, from above, $k = T(k,k), \forall\, k \in I$, which implies $T = \min$. ■

Proposition 9: A binary operator J is nondecreasing in the second variable if and only if J satisfies (30)

$$J(r, \min(s,t)) = \min(J(r,s), J(r,t)) \tag{30}$$

which is nothing but (3).

Proof: Proof is similar to that of Proposition 3. ■

From the definition of an Implication operator and the above propositions, we have the following:

Theorem 7: Any implication operator J that is onto in the first variable reduces (7) to (3) and satisfies (3).

D. On the Equation $J(r, S_1(s,t)) \equiv S_2(J(r,s), J(r,t))$ (8)

Let J be a binary operator from $I \times I$ to I. Then, we have the following propositions,the proofs of which can be obtained as in Section III-C.

Proposition 10: If J satisfies the neutrality principle, i.e., $J(1,t) = t, \forall t \in I$ then $S_1 \equiv S_2$ on I^2.

Proposition 11: If J is neutral and onto in the first variable then $S_1 = S_2 = \max$ in (8).

Proposition 12: A binary operator J is nondecreasing in the second variable if and only if J satisfies (31)

$$J(r, \max(s,t)) = \max(J(r,s), J(r,t)) \tag{31}$$

which is nothing but (4).

From the definition of an implication operator and the above propositions, we have the following:

Theorem 8: Any implication operator J that is onto in the first variable reduces (8) to (4) and satisfies (4).

IV. Conclusion

It may be noted that a fuzzy implication will be of the form "If X is A, then Y is B," where X and Y are the variables and A and B are fuzzy sets on the domain of X and Y, respectively. The implication is represented as $A(x) \rightarrow B(y)$ and often as $p \rightarrow r$. It is common in fuzzy control to have two different antecedents (observations) leading to the same consequent (action). The two rules may be joined by "else" or "and." These lead to the RHS of (1) and (2), respectively. The left-hand side of these equations reduces these to a single rule. Similarly, in the case of fuzzy expert systems, it is possible that one antecedent (symptom) may lead to different consequents (diseases). These lead to the RHS of (3) and (4). The left-hand side of these equations once again enable reduction in number of rules. The advantage of this rule reduction is lossless inferencing, i.e., the inferences drawn from the original system and the reduced system are the same. It is quite satisfying to note that all S-implications and R-implications have this property. The requirement on any binary operator to satisfy these equations is also not too stringent. The discussion in this work is on the framework of single-input–single-output fuzzy systems, but can be extended in an obvious way to multiple-input–single-output systems.

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REFERENCES

[1] W. E. Combs and J. E. Andrews, "Combinatorial rule explosion eliminated by a fuzzy rule configuration," *IEEE Trans. Fuzzy Syst.*, vol. 6, pp. 1–11, Feb. 1998.
[2] J. M. Mendel and Q. Liang, "Comments on "combinatorial rule explosion eliminated by a fuzzy rule configuration"," *IEEE Trans. Fuzzy Syst.*, vol. 7, pp. 369–371, June 1999.
[3] W. E. Combs, "Author's reply," *IEEE Trans. Fuzzy Syst.*, vol. 7, p. 371, June 1999.
[4] S. Dick and A. Kandel, "Comments on "Combinatorial rule explosion eliminated by a fuzzy rule configuration"," *IEEE Trans. Fuzzy Syst.*, vol. 7, pp. 475–477, June 1999.
[5] W. E. Combs, "Author's reply," *IEEE Trans. Fuzzy Syst.*, vol. 7, pp. 478–479, June 1999.
[6] E. Trillas and C. Alsina, "On the law $[(p \wedge q) \rightarrow r] \equiv [(p \rightarrow r) \vee (q \rightarrow r)]$ in fuzzy logic," *IEEE Trans. Fuzzy Syst.*, vol. 10, pp. 84–88, Feb. 2002.
[7] E. Cox, *The Fuzzy Systems Handbook, A Practitioner's Guide to Building, Using and Maintaining Fuzzy Systems*, 2nd ed. Norwell, MA: Academic, 1998.
[8] J. Balasubramaniam and C. J. M. Rao, "R-implication operators and rule reduction in Mamdani-type fuzzy systems," in *6th Joint Conf. Information Sciences, Fuzzy Theory, Technology*, Durham, NC, Mar. 8–12, 2002.
[9] M. Baczynski, "On a class of distributive fuzzy implications," *Int. J. Uncertainty, Fuzziness, Knowledge-Based Systems*, vol. 9, no. 2, pp. 229–238, 2001.
[10] ——, "Contrapositive symmetry of distributive fuzzy implications," *Int. J. Uncertainty, Fuzziness, Knowledge-Based Systems*, vol. 10 (Suppl), pp. 135–147, 2002.
[11] E. Trillas and L. Valverde, "On implication and indistinguishability in the setting of fuzzy logic," *Manage. Decision Support Syst. Using Fuzzy Sets Possibility Theory, Verlag TUV—Rhineland*, pp. 198–212, 1985.

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