On the Law of Importation $(x \wedge y) \longrightarrow z \equiv (x \longrightarrow (y \longrightarrow z))$ in Fuzzy Logic

Balasubramaniam Jayaram, *Member, IEEE*

***Abstract*—The law of importation, given by the equivalence $(x \wedge y) \longrightarrow z \equiv (x \longrightarrow (y \longrightarrow z))$, is a tautology in classical logic. In A-implications defined by Turksen *et al.*, the above equivalence is taken as an axiom. In this paper, we investigate the general form of the law of importation $J(T(x,y),z) \equiv J(x,J(y,z))$, where T is a t-norm and J is a fuzzy implication, for the three main classes of fuzzy implications, i.e., R-, S- and QL-implications and also for the recently proposed Yager's classes of fuzzy implications, i.e., f- and g-implications. We give necessary and sufficient conditions under which the law of importation holds for R-, S-, f- and g-implications. In the case of QL-implications, we investigate some specific families of QL-implications. Also, we investigate the general form of the law of importation in the more general setting of uninorms and t-operators for the above classes of fuzzy implications. Following this, we propose a novel modified scheme of compositional rule of inference (CRI) inferencing called the *hierarchical CRI,* which has some advantages over the classical CRI. Following this, we give some sufficient conditions on the operators employed under which the inference obtained from the classical CRI and the hierarchical CRI become identical, highlighting the significant role played by the law of importation.**

***Index Terms*—Approximate reasoning, compositional rule of inference, fuzzy implications, fuzzy inference, law of importation, rule reduction.**

I. Introduction

THE equation $(x \wedge y) \longrightarrow z \equiv (x \longrightarrow (y \longrightarrow z))$, known as the law of importation, is a tautology in classical logic. The general form of the above equivalence is given by

$$J(T(x,y),z) \equiv J(x,J(y,z)) \tag{LI}$$

where T is a t-norm and J a fuzzy implication.

In the framework of fuzzy logic, the law of importation has not been studied so far in isolation. In A-implications defined by Turksen *et al.* [51], (LI) with T as the product t-norm $T_{\mathbf{P}}(x,y) = x \cdot y$ was taken as one of the axioms. Baczyński [1] has studied the law of importation in conjunction with the general form of the following distributive property of fuzzy implications:

$$[x \longrightarrow (y \wedge z)] \equiv [(x \longrightarrow y) \wedge (x \longrightarrow z)] \tag{1}$$

and has given a characterization. Bouchon-Meunier and Kreinovich [11] have characterized fuzzy implications that have the law of importation (LI) as one of the axioms along with (1). They have considered $T_{\mathbf{M}}(x,y) = \min(x,y)$ for the t-norm T and claim that Mamdani's choice of implication "*min*" is "not so strange after all."

A. Motivation for This Work

Recently, there have been many attempts to examine classical logic tautologies involving fuzzy implication operators, both from a theoretical perspective and from possible applicational value. There were a number of correspondences [10], [13]–[15], [18], [34], [48] on (1), the following equivalence:

$$(x \wedge y) \longrightarrow z \equiv (x \longrightarrow z) \vee (y \longrightarrow z) \tag{2}$$

and related distributive properties in fuzzy logic, in the context of *inference invariant complexity reduction* in fuzzy rule based systems (see [8], [9], [45], and [53]). Daniel-Ruiz and Torrens have investigated the distributivity of fuzzy implications over uninorms (see Section II-D) in [43]. Fodor [20] has investigated the contrapositive symmetry of the three classes of fuzzy implications R-, S- and QL-implications with respect to a strong negation. Igel and Tamme [28] have dealt with chaining of implication operators in a syllogistic fashion. Thus the investigation of these properties of fuzzy implications has interesting applications in *approximate reasoning*.

In this paper, we consider the general form of the law of importation in the setting of fuzzy logic and explore its potential applications in the field of approximate reasoning along the following lines.

The compositional rule of inference (CRI) proposed by Zadeh [57] is one of the earliest and most important inference schemes in approximate reasoning involving fuzzy propositions. Inferencing in CRI involves fuzzy relations that are multidimensional and hence is resource consuming—both memory and time. We propose a novel modified scheme of the classical CRI inferencing called the *hierarchical CRI* that alleviates some of the drawbacks of the classical CRI. Next we investigate when the inference obtained from the classical CRI and the hierarchical CRI become identical. Toward this end, we give some sufficient conditions on the operators employed in the inferencing, conditions that highlight the significant role played by the law of importation.

Manuscript received June 11, 2006; revised October 26, 2006 and January 8, 2007.

The author is with the Department of Mathematics and Computer Sciences, Sri Sathya Sai University, Prasanthi Nilayam, A.P. 515134, India. (e-mail: jbala@ieee.org).

Digital Object Identifier 10.1109/TFUZZ.2007.895969

1063-6706/$25.00 © 2007 IEEE

Authorized licensed use limited to: IEEE Xplore. Downloaded on January 14, 2009 at 23:40 from IEEE Xplore. Restrictions apply.

B. Main Contents of the Paper

The three established and well-studied classes of fuzzy implications in the literature are the R-, S- and QL-implications. Recently, Yager [54] has proposed two new classes of fuzzy implications—f- and g-implications—that can be seen to be obtained from the additive generators of continuous Archimedean t-norms and t-conorms (see Section II-C), respectively. In this paper, we consider the law of importation (LI) for these classes of fuzzy implications: R-, S-, QL-, f-, and g-implications. We give necessary and sufficient conditions under which the above equivalence can be established for R-, S-, f-, and g-implications. In the case of QL-implications, we investigate some specific families of QL-implications. Also we have investigated the general form of law of importation in the more general setting of uninorms and t-operators and show that they get reduced to t-norms.

We propose a novel modified scheme of CRI inferencing called the *hierarchical CRI* that has many advantages over classical CRI. Subsequently, we give sufficient conditions on the operators employed in the hierarchical CRI under which the inference obtained from the classical CRI and the hierarchical CRI become identical. The law of importation plays a significant role in the above equivalence. We illustrate the above concepts through some numerical examples. It is not uncommon to use a t-norm for relating antecedents to consequents in CRI, in which case we have shown that under even less restrictive conditions the equivalence between the inferences in the classical CRI and hierarchical CRI can be obtained.

C. Outline of This Paper

In Section II, we review the basic fuzzy logic operators and their different properties. The main theoretical results of this paper are contained in Section III, wherein we investigate the general form of law of importation for some well-known classes of fuzzy implications, and in Section IV, where we investigate the general form of law of importation in the more general setting of uninorms and t-operators. In Section V, we propose a novel scheme of inferencing called *hierarchical CRI* and highlight its advantages. In Section VI, we illustrate the applicational values of the theoretical results in Sections III and IV by demonstrating the significant role the law of importation plays in the equivalence of the outputs obtained from classical CRI and hierarchical CRI. Section VII gives a few concluding remarks.

II. Preliminaries

To make this paper self-contained, we briefly mention some of the concepts and results employed in the rest of this paper.

Definition 1: Let $\varphi : [0,1] \to [0,1]$ be an increasing bijection.

- If $f\colon [0,1] \to [0,1]$ is any function, then the φ-conjugate of f is given by

$$f_\varphi(x) = \varphi^{-1}(f(\varphi(x))), \qquad x \in [0,1]. \qquad (3)$$

- If $F\colon [0,1] \times [0,1] \to [0,1]$ is any binary function, then the φ-conjugate of F is

$$F_\varphi(x,y) = \varphi^{-1}(F(\varphi(x),\varphi(y))), \qquad x,y \in [0,1]. \qquad (4)$$

A. Negations

Definition 2 ([21, Def. 1.2–1.4]:

i) A function $N\colon [0,1] \to [0,1]$ is called a fuzzy negation if $N(0) = 1$, $N(1) = 0$, and N is nonincreasing.

ii) A fuzzy negation N is called strict if, in addition, N is strictly decreasing and N is continuous.

iii) A fuzzy negation N is called strong if it is an involution, i.e., $N(N(x)) = x$ for all $x \in [0,1]$.

Theorem 1 [47], [21, Theorem 1.1]: A function $N : [0,1] \to [0,1]$ is a strong negation if and only if there exists an increasing bijection $\varphi : [0,1] \to [0,1]$, such that $N(x) = \varphi^{-1}(1 - \varphi(x))$.

B. T-Norms and T-Conorms

Definition 3 [44], [29, Definition 1.1]: An associative, commutative, increasing operation $T\colon [0,1]^2 \to [0,1]$ is called a t-norm if it has neutral element equal to one.

Definition 4 [44], [29, Def. 1.13]: An associative, commutative, increasing operation $S\colon [0,1]^2 \to [0,1]$ is called a t-conorm if it has neutral element equal to zero.

If F is an associative binary operation on a domain $\mathbb{X}$, then by the notation $x_F^{(n)}$ we mean $F\left(x, x_F^{(n-1)}\right) = F(x, F(\underbrace{x,\ldots,x}_{n-1\ times}))$ for an $x \in \mathbb{X}$ and $n \geq 2$. Also $x_F^{(1)} = x$.

Definition 5 [29, Def. 2.9 and 2.13]: A t-norm T (t-conorm S, respectively) is said to be:

- continuous if it is continuous in both the arguments;
- Archimedean if T (S, respectively) is such that for every $x,y \in (0,1]$ ($x,y \in [0,1)$, respectively) there is an $n \in \mathbb{N}$ with $x_T^{(n)} < y$ $\left(x_S^{(n)} > y\right)$;
- strict if T (S, respectively) is continuous and strictly monotone, i.e., $T(x,y) < T(x,z)$ $\big(S(x,y) < S(x,z)\big)$ whenever $x > 0$ ($x < 1$, respectively) and $y < z$;
- nilpotent if T (S, respectively) is continuous and if each $x \in (0,1)$ is such that $x_T^{(n)} = 0$ $\left(x_S^{(n)} = 1\right)$ for some $n \in \mathbb{N}$.

Table I lists the basic t-norms and t-conorms along with their properties.

Theorem 2 ([29, Theorem 5.1]: For a function $T : [0,1]^2 \to [0,1]$, the following are equivalent.

i) T is a continuous Archimedean t-norm.

ii) T has a continuous additive generator, i.e., there exists a continuous, strictly decreasing function $f : [0,1] \to [0,\infty]$ with $f(1) = 0$, which is uniquely determined up to a positive multiplicative constant, such that for all $x,y \in [0,1]$

$$T(x,y) = f^{(-1)}(f(x) + f(y)) \qquad (5)$$

Authorized licensed use limited to: IEEE Xplore. Downloaded on January 14, 2009 at 23:40 from IEEE Xplore. Restrictions apply.

TABLE I
BASIC t-NORMS AND t-CONORMS WITH THEIR PROPERTIES

Name	t-norm	t-conorm	Properties
minimum / maximum	$T_{\mathbf{M}}(x,y)=\min(x,y)$	$S_{\mathbf{M}}:\max(x,y)$	idempotent, continuous
algebraic product/ algebraic sum	$T_{\mathbf{P}}: x\cdot y$	$S_{\mathbf{P}}: x+y-x\cdot y$	strict, continuous
Łukasiewicz	$T_{\mathbf{LK}}:\max(x+y-1,0)$	$S_{\mathbf{LK}}:\min(x+y,1)$	nilpotent, continuous
drastic product / drastic sum	$T_{\mathbf{D}}:\begin{cases}x, & \text{if } y=1\\ y, & \text{if } x=1\\ 0, & \text{otherwise}\end{cases}$	$S_{\mathbf{D}}:\begin{cases}x, & \text{if } y=0\\ y, & \text{if } x=0\\ 1, & \text{otherwise}\end{cases}$	Archimedean, non-continuous
nilpotent minimum/ nilpotent maximum	$T_{\mathbf{nM}}:\begin{cases}0, & \text{if } x+y\le 1\\ \min(x,y), & \text{otherwise}\end{cases}$	$S_{\mathbf{nM}}:\begin{cases}1, & \text{if } x+y\ge 1\\ \max(x,y), & \text{otherwise}\end{cases}$	non-Archimedean, left-continuous / right-continuous

where $f^{(-1)}$ is the pseudoinverse of f and is defined as:

$$f^{(-1)}(x)=\begin{cases}f^{-1}(x), & \text{if } x\in[0,f(0)]\\ 0, & \text{if } x\in(f(0),\infty]\end{cases}. \tag{6}$$

If $f(0)=\infty$, then T is strict and if $f(0)<\infty$, then T is nilpotent.

Theorem 3 [29, Corollary 5.5]: For a function $S:[0,1]^2\to[0,1]$, the following are equivalent.

i) S is a continuous Archimedean t-conorm.

ii) S has a continuous additive generator, i.e., there exists a continuous, strictly increasing function $g:[0,1]\to[0,\infty]$ with $g(0)=0$, which is uniquely determined up to a positive multiplicative constant, such that for all $x,y\in[0,1]$ we have

$$S(x,y)=g^{(-1)}(g(x)+g(y))$$

where $g^{(-1)}$ is the pseudoinverse of g and is given by

$$g^{(-1)}(x)=\begin{cases}g^{-1}(x), & \text{if } x\in[0,g(1)]\\ 1, & \text{if } x\in(g(1),\infty]\end{cases}. \tag{7}$$

Theorem 4 [21, Theorem 1.8]: A continuous t-conorm S satisfies $S(N(x),x)=1$, for all $x\in[0,1]$ with a strict negation N, if and only if there exists a strictly increasing bijection φ of the unit interval [0,1] such that S and N have the following representations:

$$S(x,y)=\varphi^{-1}\big(\min(\varphi(x)+\varphi(y),1)\big) \tag{8}$$

$$N(x)\ge\varphi^{-1}(1-\varphi(x)), \qquad x\in[0,1]. \tag{9}$$

C. Uninorms and t-Operators

Definition 6 [22, Definition 1]: A uninorm is a two-place function $U:[0,1]^2\to[0,1]$ that is associative, commutative, nondecreasing in each place, and such that there exists some element $e\in[0,1]$ called the neutral element such that $U(e,x)=x$, for all $x\in[0,1]$.

If $e=0$, then U is a t-conorm; and if $e=1$, then U is a t-norm. For any uninorm U, $U(1,0)\in\{0,1\}$. A uninorm U such that $U(1,0)=0$ is called a *conjunctive* uninorm and if $U(1,0)=1$ it is called a *disjunctive* uninorm. Note that it can be easily shown that a uninorm U that is continuous on the whole of $[0,1]^2$ is either a t-norm or a t-conorm. There are three main classes of uninorms that have been well studied in the literature.

1) Pseudocontinuous uninorms (see [32]), i.e., uninorms U that are continuous on $[0,1]^2$ except on the segments $(0,e),(1,e)$ and $(e,0),(e,1)$. These are precisely the uninorms such that both functions $U(x,1)$ and $U(x,0)$ are continuous except at the point $x=e$. The class of conjunctive uninorms having this property are usually referred to as $\mathcal{U}_{\mathbf{Min}}$ and the disjunctive ones by $\mathcal{U}_{\mathbf{Max}}$ (see [22]).
2) Idempotent uninorms, i.e., uninorms U such that $U(x,x)=x$ for all $x\in[0,1]$ (see [55], [5], [30], and [42]).
3) Representable uninorms that have additive generators and are continuous everywhere on the $[0,1]^2$ except at the points (0, 1) and (1, 0) (see [22]).

Definition 7 [31, Definition 3.1]: A t-operator is a two-place function $F:[0,1]^2\to[0,1]$ which is associative, commutative, nondecreasing in each place, and such that:

- $F(0,0)=0; F(1,1)=1;$
- the sections $F_0,F_1:[0,1]\to[0,1]$ are continuous functions, where $F_0(x)=F(0,x)$; $F_1(x)=F(1,x)$, for all $x\in[0,1]$.

D. Fuzzy Implications

Definition 8 [21, Definition 1.15]: A function $J:[0,1]^2\to[0,1]$ is called a fuzzy implication if, for all $x,y,z\in[0,1]$, it satisfies

$$J(x,z)\ge J(y,z), \qquad \text{if } x\le y \tag{J1}$$

$$J(x,y)\le J(x,z), \qquad \text{if } y\le z \tag{J2}$$

$$J(0,y)=1 \tag{J3}$$

$$J(x,1)=1 \tag{J4}$$

$$J(1,0)=0. \tag{J5}$$

A fuzzy implication J is said to be *neutral* if

$$J(1,y)=y, \qquad y\in[0,1]. \tag{NP}$$

Remark 1: Though the above axioms are the usually required set in literature, they are not mutually exclusive. For example, it can be shown that (J1) implies (J4). Since a fuzzy implication operator coincides with the Boolean implication on {0, 1}, we

Authorized licensed use limited to: IEEE Xplore. Downloaded on January 14, 2009 at 23:40 from IEEE Xplore. Restrictions apply.

TABLE II
SOME S-IMPLICATIONS WITH THEIR CORRESPONDING t-CONORMS AND THEIR N-DUAL t-NORMS, WHERE N IS THE STANDARD NEGATION $1-x$

Name	S	$J_{S,N}(x,y)$	T : N-dual of S
Kleene-Dienes	$S_{\mathbf{M}}$	$J_{\mathbf{KD}}(x,y) = \max(1-x, y)$	$T_{\mathbf{M}}$
Reichenbach	$S_{\mathbf{P}}$	$J_{\mathbf{RC}}(x,y) = 1 - x + x \cdot y$	$T_{\mathbf{P}}$
Łukasiewicz	$S_{\mathbf{LK}}$	$J_{\mathbf{LK}}(x,y) = \min(1, 1-x+y)$	$T_{\mathbf{LK}}$

know that $J(1,1) = 1$. Now by (J1) we have $x \leq 1$ implies $J(x,1) \geq J(1,1) = 1$, i.e., (J1) implies (J4). Similarly, from $J(0,0) = 1$ and (J2), we obtain (J3).

The following are the two important classes of fuzzy implications well established in the literature:

Definition 9 [21, Definition 1.16]: An S-implication $J_{S,N}$ is obtained from a t-conorm S and a strong negation N as follows:

$$J_{S,N}(x,y) = S(N(x), y), \qquad x, y \in [0,1]. \tag{10}$$

Definition 10 [21, Definition 1.16]: An R-implication J_T is obtained from a t-norm T as follows:

$$J_T(x,y) = Sup\left\{t \in [0,1] : T(x,t) \leq y\right\}, \qquad x, y \in [0,1]. \tag{11}$$

A t-norm T and R-implication J_T obtained from T are said to have the residuation principle if they satisfy the following:

$$T(x,y) \leq z \text{ iff } J_T(x,z) \geq y, \qquad x, y \in [0,1]. \tag{RP}$$

Remark 2: It is important to note that (RP) is a characterizing condition for a left-continuous t-norm T (see [21, p. 25] and [27, Proposition 5.4.2]. In this paper, we only consider R-implications J_T obtained from t-norms T such that the pair (J_T, T) satisfy the residuation principle (RP) or, equivalently, T is a left-continuous t-norm. Also if T is a left-continuous t-norm, then $\sup$ in (11) reduces to $\max$.

Any R-implication J_T obtained from a left-continuous t-norm T satisfies (NP) and has the ordering property (OP) (see, for example, [50]) for all $x, y \in [0,1]$

$$x \leq y \Longleftrightarrow J_T(x,y) = 1, \qquad \text{Ordering Property.} \tag{OP}$$

Since $T(x,y) \leq T(x,y)$ for any $x, y \in [0,1]$, we have that (see also [39] and [40])

$$J_T(x, T(x,y)) \geq y, \qquad x, y \in [0,1]. \tag{12}$$

Proposition 1 [2, Propositions 12 and 21]: Let $\varphi : [0,1] \rightarrow [0,1]$ be an increasing bijection.

i) If $J_{S,N}$ is an S-implication obtained from a t-conorm S and a strong negation N, then its φ-conjugate $(J_{S,N})_\varphi$ is also an S-implication obtained from the t-conorm S_φ and the strong negation N_φ.
ii) If J_T is an R-implication obtained from a left continuous t-norm T, then its φ-conjugate $(J_T)_\varphi$ is also an R-implication obtained from the left continuous t-norm T_φ.

A third important class of fuzzy implications studied in the literature is the following.

TABLE III
SOME R-IMPLICATIONS AND THEIR CORRESPONDING t-NORMS

Name	T^*	$J_{T^*}(x,y)$
Godel	$T_{\mathbf{M}}$	$J_{\mathbf{GD}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{otherwise} \end{cases}$
Łukasiewicz	$T_{\mathbf{LK}}$	$J_{\mathbf{LK}}(x,y) = \min(1, 1-x+y)$
Goguen	$T_{\mathbf{P}}$	$J_{\mathbf{GG}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ \frac{y}{x}, & \text{otherwise} \end{cases}$

Definition 11 [21, p. 24]: A QL-implication is obtained from a t-conorm S, t-norm T, and strong negation N as follows:

$$J_{QL}(x,y) = S(N(x), T(x,y)), \qquad x, y \in [0,1]. \tag{13}$$

Remark 3: It should be emphasized that not all QL-implications satisfy the condition (J1) of Definition 8. For example, consider the function $J_{\mathbf{Z}}(x,y) = \max(1-x, \min(x,y))$, called the Zadeh implication in the literature. Let $x_1 = 0.7 < 0.8 = x_2$ and $y = 0.9$. Then $J_{\mathbf{Z}}(x_1, y) = 0.7 < 0.8 = J_{\mathbf{Z}}(x_2, y)$ and hence does not satisfy (J1), but it is a QL-implication obtained from the triple $(T_{\mathbf{M}}, S_{\mathbf{M}}, 1 - .)$.

Further, a QL-implication satisfies (J4), and hence (J1), only if the strong negation N and the t-conorm S in Definition 11 are such that $S(x, N(x)) = 1$ for all $x \in [0,1]$.

We say that a QL-implication J_{QL} obtained from the triple (T, S, N) is a fuzzy implication only if J_{QL} satisfies (J1). For more recent works on QL-implications, we refer the readers to [33] and [49].

Remark 4: An S-implication $J_{S,N}$ (Definition 9) or a QL-implication J_{QL} can be considered for noninvolutive negations also (see [4]), but in this paper we only consider the stronger definition as given in [21].

All the above three classes of R-, S-, and QL-implications are neutral. Tables II–IV list a few of the well-known S-, R-, and QL-implications, respectively.

E. Yager's Classes of Fuzzy Implications

Recently, Yager [54] has proposed two new classes of fuzzy implication operators—f- and g-generated implications—that cannot be strictly categorized to fall in the above classes. Reference [3] discusses the intersection of the above classes of fuzzy implications with the classes of R- and (S,N)-implications. They can be seen to be obtained from additive generators of t-norms and t-conorms, whose definitions we give below along with a few of their properties.

Definition 12 [54, p. 196]): An f-generator is a function $f : [0,1] \rightarrow [0,\infty]$ that is a strictly decreasing and continuous function with $f(1) = 0$. Also we denote its pseudoinverse by $f^{(-1)}$ given by (6).

Authorized licensed use limited to: IEEE Xplore. Downloaded on January 14, 2009 at 23:40 from IEEE Xplore. Restrictions apply.

TABLE IV
SOME QL-IMPLICATIONS WITH THEIR CORRESPONDING t-CONORMS AND t-NORMS, WHERE N IS THE STANDARD NEGATION $1-x$

Name	S	T	$J_{QL}(x,y)$
Łukasiewicz	$S_{\mathbf{LK}}$	$T_{\mathbf{M}}$	$J_{\mathbf{LK}}(x,y)=\min(1,1-x+y)$
Kleene-Dienes	$S_{\mathbf{LK}}$	$T_{\mathbf{LK}}$	$J_{\mathbf{KD}}(x,y)=\max(1-x,y)$
Fodor	$S_{\mathbf{nM}}$	$T_{\mathbf{M}}$	$J_{\mathbf{FD}}(x,y)=\begin{cases}1, & \text{if } x\le y\\ \max(1-x,y), & \text{if } x>y\end{cases}$

TABLE V
SOME J_f IMPLICATIONS WITH THEIR f-GENERATORS

Name	$f(x)$	$f(0)$	$J_f(x,y)$
Yager	$-\ log\ x$	∞	y^x
Frank	$-ln\left\{\frac{s^x-1}{s-1}\right\};\ s>0, s\neq 1$	∞	$log_s\ \left\{1+(s-1)^{1-x}\cdot(s^y-1)^x\right\}$
Trigonometric	$cos\left(\frac{\pi}{2}\cdot x\right)$	1	$cos^{-1}\left[x\cdot cos\left(\frac{\pi}{2}\cdot y\right)\right]$
-	$(1-x)^\lambda;\ \lambda>0$	1	$1-x^{\frac{1}{\lambda}}\cdot(1-y)$

TABLE VI
SOME J_g IMPLICATIONS WITH THEIR g-GENERATORS

$g(x)$	$g^{(-1)}(z)$	$g(1)$	$J_g(x,y)$
$-\ log(1-x)$	$1-e^{-z}$	∞	$1-(1-y)^{\frac{1}{x}}$
$\tan\left(\frac{\pi}{2}x\right)$	$\frac{2}{\pi}\tan^{-1}(z)$	∞	$\frac{\pi}{2}\tan^{-1}\left[\frac{1}{x}\cdot\tan\left(\frac{\pi}{2}\cdot y\right)\right]$
x	$\min(1,z)$	1	$\min\left(1,\frac{y}{x}\right)$
x^λ	$\min\left(1,z^{\frac{1}{\lambda}}\right);\lambda\in(0,\infty)$	1	$\min\left(1,\frac{y}{x^{\frac{1}{\lambda}}}\right)$

Definition 13 [54, p. 197]): A function from $[0,1]^2$ to [0, 1] defined by an f-generator as

$$J_f(x,y)=f^{(-1)}(x\cdot f(y)) \tag{14}$$

with the understanding that $0\times\infty=0$ is called an f-generated implication.

It can easily be shown that J_f is a fuzzy implication (see [54, p. 197]).

Definition 14 [54, p. 201]: A g-generator is a function $g:[0,1]\to[0,\infty]$ that is a strictly increasing and continuous function with $g(0)=0$. Also we denote its pseudoinverse by $g^{(-1)}$ given by (7).

Definition 15 (Yager [54, p. 201]): A function from $[0,1]^2$ to [0, 1] defined by a g-generator as

$$J_g(x,y)=g^{(-1)}\left(\frac{1}{x}\cdot g(y)\right) \tag{15}$$

with the understanding that $0\times\infty=\infty$ is a fuzzy implication and is called a g-generated implication.

Note that for any $x,y\in[0,1]$, since $x\le 1, x\cdot f(y)\le f(y)\le f(0)$ and hence $f^{(-1)}=f^{-1}$ in (14). As can be seen from Theorems 2 and 3, the f- and g-generators can be used as additive generators for generating t-norms and t-conorms, respectively. We will use the terms f-generated (g-generated, respectively) implication and f-implication (g-implication, respectively) interchangeably.

It is easy to see that J_f, J_g are neutral fuzzy implications, and a few examples from these two classes are given in Tables V and VI (see [54, pp. 199–203]).

Lemma 1: Let J be:

i) an f-implication J_f obtained from an f-generator; or

ii) a g-implication J_g obtained from a g-generator such that $g(1)=\infty$.

Then $J(x,y)=1$ iff either $x=0$ or $y=1$.

Proof:

i) Let J be an f-implication J_f. Then the reverse implication is obvious. On the other hand

$$\begin{aligned}J_f(x,y)=1 &\Longrightarrow f^{(-1)}(x\cdot f(y))=1\\ &\Longrightarrow x\cdot f(y)=f(1)=0\end{aligned}$$

which implies either $x=0$ or $f(y)=0$, which by the strictness of f means $y=1$.

ii) Let J be a g-implication J_g obtained from a g-generator such that $g(1)=\infty$. Again, the reverse implication is obvious. On the other hand, since $g(1)=\infty$, we have $g^{(-1)}=g^{-1}$ and

$$\begin{aligned}J_g(x,y)=1 &\Longrightarrow g^{-1}\left(\frac{1}{x}\cdot g(y)\right)=1\\ &\Longrightarrow \frac{1}{x}\cdot g(y)=g(1)=\infty\end{aligned}$$

which implies either $x=0$, since if $x\in(0,1]$ then $1/x$ cannot be ∞, or $g(y)=\infty$, i.e., $y=1$.

■

Lemma 2: Let J be:

i) an f-implication J_f obtained from an f-generator with $f(0)=\infty$; or

ii) a g-implication J_g.

Authorized licensed use limited to: IEEE Xplore. Downloaded on January 14, 2009 at 23:40 from IEEE Xplore. Restrictions apply.

Then

$$J(x,0) = \begin{cases} 0, & \text{if } x \in (0,1] \\ 1, & \text{if } x = 0 \end{cases}.$$

Proof: If $x=0$, then $J_f(x,0) = J_g(x,0) = 1$ since J_f, J_g are fuzzy implications. If $x \neq 0$, then:

i) $J_f(x,0) = f^{-1}(x \cdot f(0)) = f^{-1}(x \cdot \infty) = f^{-1}(\infty) = 0$, by definition of f.

ii) $1/x \neq \infty$, and since g is strictly increasing and $g(0)=0$, we have $J_g(x,0) = g^{(-1)}(1/x \cdot g(0)) = g^{(-1)}(0) = 0$. ■

III. The Law of Importation

In the following sections we discuss the validity of (LI) when J is an R-, S-, QL-, f-, and g-implication.

A. R- and S-Implications and the Law of Importation

Theorem 5: An S-implication $J_{S,N}$, obtained from a t-conorm S and a strong negation N satisfies (LI) with a t-norm T iff T is the N-dual of S.

Proof: Let $J_{S,N}$ be an S-implication obtained from a t-conorm S and a strong negation N.

($\Longrightarrow$:) Let $J_{S,N}$ satisfy (LI) with a t-norm T. If S^* is the N-dual t-conorm of T, then

$$\begin{aligned} \text{LHS (LI)} &= J_{S,N}(T(x,y),z) \\ &= S[N(T(x,y)),z] \\ &= S[S^*(N(x),N(y)),z] \qquad (16) \\ \text{RHS (LI)} &= J_{S,N}(x, J_{S,N}(y,z)) \\ &= S[N(x), J_{S,N}(y,z)] \\ &= S[N(x), S(N(y),z)]. \qquad (17) \end{aligned}$$

Since (16) = (17), taking $z=0$, we have $S^*(N(x),N(y) = S(N(x),N(y)$ for all $x,y \in [0,1]$. Since N is a strong negation and hence both one-to-one and onto on [0, 1], for every $x,y \in [0,1]$, there exists $x',y' \in [0,1]$ such that $x = N(x')$; $y = N(y')$. Now it follows that $S^*(x,y) = S(x,y)$ for all $x,y \in [0,1]$, and that S is the N-dual t-conorm of T.

($\Longleftarrow$:) If T is the N-dual of S, then $S = S^*$ and (16) = (17) by the associativity of S. Thus an S-implication $J_{S,N}$ and a t-norm T satisfy (LI) iff T is the N-dual of S. ■

Theorem 6 [38], [21, Theorem 1.14]: An R-implication J_{T^*} obtained from a left-continuous t-norm T^* satisfies (LI) with a t-norm T iff $T = T^*$.

Proof: Let J_{T^*} be the R-implication obtained as the residuation of the left-continuous t-norm T^*. Then $J_{T^*}(x,y) = \max\{t \in [0,1] : T^*(x,t) \leq y\}$, for all $x,y \in [0,1]$. Since (J_{T^*}, T^*) satisfy the residuation principle (RP), we have $T^*(x,y) \leq z \iff x \leq J_{T^*}(y,z)$. Also by (OP), we have $J_{T^*}(x,y) = 1$ iff $x \leq y$.

($\Longleftarrow$:) Let $T = T^*$. Now, since T is left-continuous we have by the associativity of T and (RP)

$$\begin{aligned} \text{LHS (LI)} &= J_{T^*}(T(x,y),z) = J_T(T(x,y),z) \\ &= \max\{t \in [0,1] : T(T(x,y),t) \leq z\} \\ &= \max\{t \in [0,1] : T(T(t,x),y) \leq z\} \\ &= \max\{t \in [0,1] : T(t,x) \leq J_T(y,z)\} \\ &= \max\{t \in [0,1] : t \leq J_T(x, J_T(y,z))\} \\ &= J_T(x, J_T(y,z)) \\ &= \text{RHS (LI)}. \end{aligned}$$

($\Longrightarrow$:) Let J_{T^*} satisfy (LI) for some t-norm T. Let $x,y \in (0,1)$ be arbitrary but fixed and $z = T^*(x,y)$. Now, by employing the following properties—commutativity of T^*, (12), and (OP)—we have

$$\begin{aligned} J_{T^*}(T(x,y), T^*(x,y)) &= J_{T^*}(x, J_{T^*}(y, T^*(x,y))) \\ &= J_{T^*}(x, J_{T^*}(y, T^*(y,x))) \\ &\geq J_{T^*}(x,x) = 1 \\ &\Longrightarrow T(x,y) \leq T^*(x,y). \qquad (18) \end{aligned}$$

On the other hand, for the above fixed x,y, let $z = T(x,y)$. Then we have, from (OP) and (RP)

$$\begin{aligned} J_{T^*}(T(x,y),T(x,y)) &= 1 \\ &\Longrightarrow J_{T^*}(x, J_{T^*}(y, T(x,y))) = 1 \\ &\Longrightarrow x \leq J_{T^*}(y, T(y,x))) \\ &\Longrightarrow T^*(x,y) \leq T(x,y). \qquad (19) \end{aligned}$$

From (18) and (19) and since x,y are arbitrary, we see that $T = T^*$. ■

In Table II, the t-norms under column T and in Table III the t-norms under column T^* are the t-norms corresponding to the S-implications and R-implications (under the columns $J_{S,N}$ and J_{T^*}), respectively, that satisfy (LI).

From Theorems 5 and 6 and Proposition 1, we have the following corollary.

Corollary 1: Let $\varphi : [0,1] \to [0,1]$ be an increasing bijection.

i) The φ-conjugate of an S-implication $J_{S,N}$ satisfies (LI) iff T is the N_φ-dual of S_φ.

ii) The φ-conjugate of an R-implication J_{T^*} obtained from a left-continuous t-norm T^* satisfies (LI) iff $T = T^*_\varphi$, where T^*_φ is the φ-conjugate of the t-norm T^*.

B. QL-Implications and the Law of Importation

Let the QL-implication J_{QL} obtained from a t-conorm S^*, t-norm T^*, and strong negation N^* be a fuzzy implication; then $S^*(N^*(x),x) = 1$, for all $x \in [0,1]$ (see Remark 3). In the rest of this section, we consider only continuous t-conorms S^* and hence by Theorem 4 we have that S^* and N^* have the representation as given in (8) and (9), respectively, i.e.,

$$\begin{aligned} S^*(x,y) &= S_{\mathbf{LK}_\varphi}(x,y) = \varphi^{-1}(\min(\varphi(x)+\varphi(y),1)) \\ N^*(x) &\geq \varphi^{-1}(1-\varphi(x)) \end{aligned}$$

for all $x,y \in [0,1]$.

Let us consider the extreme case when $N^*(x) = \varphi^{-1}(1 - \varphi(x))$ (from the same bijection φ), in which case from Theorem 1 we have that N^* is a strong negation.

Now, if we consider the J_{QL} obtained from the triple $(T, S_{\mathbf{LK}_\varphi}, N_\varphi)$ where T is any t-norm, then since $T(x,y) \leq x$

Authorized licensed use limited to: IEEE Xplore. Downloaded on January 14, 2009 at 23:40 from IEEE Xplore. Restrictions apply.

for any t-norm T and $x \in [0,1]$, we have, for all $x, y \in [0,1]$, the following (see also [33] and [49]):

$$\begin{aligned} J_{QL}(x,y) &= S_{\mathbf{LK}_\varphi}(N_\varphi(x), T(x,y)) \\ &= \varphi^{-1}(\min\{\varphi(N(x)) + \varphi(T(x,y)), 1\}) \\ &= \varphi^{-1}(1 - \varphi(x) + \varphi(T(x,y))). \end{aligned} \tag{20}$$

If we consider T^* to be either a φ-conjugate of a continuous Archimedean t-norm or $\min$, it can be easily verified that J_{QL} obtained from the triple $(T^*, S_{\mathbf{LK}_\varphi}, N_\varphi)$ all satisfy (J1) and hence are fuzzy implications. In fact, Fodor [20, Corollary 4] has shown that an "if and only if" relation exists between the different t-norms T employed below and the resulting QL-implications.

i) If the t-norm T in (20) is conjugate with the Łukasiewicz t-norm $T_{\mathbf{LK}}$, then J_{QL} is the (S,N)-implication obtained from $S = S_{\mathbf{M}}$ and $N = N_\varphi$, i.e.,

$$J_{QL}(x,y) = \max(N_\varphi(x), y). \tag{21}$$

ii) If the t-norm T in (20) is conjugate with the product t-norm $T_{\mathbf{P}}$, then J_{QL} is conjugate with the Reichenbach implication $J_{\mathbf{RC}}$, i.e., $J_{QL} = J_{\mathbf{RC}_\varphi}$ and is given by

$$J_{QL}(x,y) = \varphi^{-1}(1 - \varphi(x) + \varphi(x)\varphi(y)). \tag{22}$$

iii) If the t-norm T in (20) is the minimum t-norm $T_{\mathbf{M}}$, then J_{QL} is conjugate with the Łukasiewicz implication $J_{\mathbf{LK}}$, i.e., $J_{QL} = J_{\mathbf{LK}_\varphi}$ and is given by

$$J_{QL}(x,y) = \min\{1, \varphi^{-1}(1 - \varphi(x) + \varphi(y))\}. \tag{23}$$

Theorem 7: Let S^* be a continuous t-conorm with representation (8) for an increasing bijection $\varphi : [0,1] \to [0,1]$ and N^* be the associated strong negation, i.e., $N^*(x) = \varphi^{-1}(1-\varphi(x))$. Let J_{QL} be a QL-implication obtained from S^*, T^*, N^*.

i) If the t-norm T^* is the φ-conjugate of the Łukasiewicz t-norm $T_{\mathbf{LK}}$, then J_{QL} satisfies (LI) with a t-norm T iff $T = T_{\mathbf{M}}$, the minimum t-norm.

ii) If the t-norm T^* is the φ-conjugate of the product t-norm $T_{\mathbf{P}}$, then J_{QL} satisfies (LI) with a t-norm T iff $T = T^*$.

iii) If the t-norm $T^* = T_{\mathbf{M}}$, the minimum t-norm, then J_{QL} satisfies (LI) with a t-norm T iff T is the φ-conjugate of the Łukasiewicz t-norm $T_{\mathbf{LK}}$.

Proof:

i) We know from (21) that J_{QL} is the φ-conjugate of the Łukasiewicz implication $J_{\mathbf{LK}}(x,y) = \min(1, 1-x+y)$. Now by Corollary 1–ii, we have that J_{QL} satisfies (LI) if and only if T is the φ-conjugate of the Łukasiewicz t-norm $T_{\mathbf{LK}}$, i.e., $T(x,y) = \varphi^{-1}(\max\{\varphi(x) + \varphi(y) - 1, 0\})$.

ii) From (22), we see that J_{QL} is an S-implication obtained from the algebraic sum t-conorm $S_{\mathbf{P}}(x,y) = x + y - x \cdot y$ and the strong negation $N(x) = 1 - x$. Now by Corollary 1–i, we have that J_{QL} satisfies (LI) if and only if T is the N_φ-dual of the φ-conjugate of the algebraic sum t-conorm $S_{\mathbf{P}_\varphi}(x,y) = \varphi^{-1}(\varphi(x) + \varphi(y) - \varphi(x) \cdot \varphi(y))$, where $N_\varphi(x) = \varphi^{-1}(1 - \varphi(x)) = N^*(x)$. Now by an easy verification, we can see that

$$\begin{aligned} T(x,y) &= N_\varphi[S_\varphi(N_\varphi(x), N_\varphi(y))] = \varphi^{-1}(\varphi(x) \cdot \varphi(y)) \\ &= T^*(x,y). \end{aligned}$$

iii) From (23), we see that J_{QL} is an S-implication with $S = \max$. Hence by Theorem 5, we have that J_{QL} satisfies (LI) iff $T = \min$, which is the N-dual of $\max$ for any strong N. ■

C. Yager's f- and g-Implications and the Law of Importation

Theorem 8: Let J_f be an f-generated implication. J_f satisfies the law of importation (LI) with a t-norm T if and only if $T = T_{\mathbf{P}}$, the product t-norm.

Proof: ($\Longleftarrow$:) Let T be the product t-norm. Then for any $x, y, z \in [0,1]$

$$\begin{aligned} \text{RHS (LI)} &= J_f(x, J_f(y,z)) \\ &= f^{-1}[x \cdot f(J_f(y,z))] \\ &= f^{-1}[x \cdot f \circ f^{-1}(y \cdot f(z))] \\ &= f^{-1}[x \cdot (y \cdot f(z))] \\ &= J_f(x \cdot y, z) = \text{LHS (LI)}. \end{aligned}$$

($\Longrightarrow$:) Let J_f obey the law of importation (LI) with a t-norm T. Then for any $x, y, z \in [0,1]$, we have

$$\begin{aligned} \text{LHS (LI)} &= \text{RHS (LI)} \\ &\Longrightarrow J_f(T(x,y), z)) = J_f(x, J_f(y,z)) \\ &\Longrightarrow f^{-1}[T(x,y) \cdot f(z)] \\ &\quad = f^{-1}[x \cdot f \circ f^{-1}(y \cdot f(z))] \\ &\Longrightarrow [T(x,y) \cdot f(z)] = x \cdot y \cdot f(z). \end{aligned} \tag{24}$$

Now, if $z \in (0,1)$, then, since f is strictly decreasing, $0 < f(z) < \infty$ and hence from (24) we have $T(x,y) = x \cdot y$ for all $x, y \in [0,1]$. ■

Lemma 3: If a g-implication J_g satisfies the law of importation (LI) with a t-norm T, then $T(x,y) \neq 0$ for any $x, y \in (0,1)$.

Proof: On the contrary, let $T(x_0, y_0) = 0$ for some $x_0, y_0 \in (0,1)$. Let $z \in [0,1)$. Then $J_g(T(x_0,y_0), z) = J_g(0,z) = 1$ by (J3). Since J_g satisfies the law of importation (LI) with respect to T, we have $1 = J_g(x_0, J_g(y_0, z))$. In particular, if $z = 0$, then $J_g(y_0, z) = J_g(y_0, 0) = 0$ and $J_g(x_0, J_g(y_0,z)) = J_g(x_0, 0) = 0$ from Lemma 2–ii, from which we have $1 = 0$, a contradiction. Hence $T(x,y) \neq 0$ for any $x, y \in (0,1)$. ■

Theorem 9: Let J_g be a g-generated implication with $g : [0,1] \to [0,\infty]$ such that $g(1) = \infty$. J_g satisfies the law of importation (LI) with a t-norm T if and only if $T = T_{\mathbf{P}}$, the product t-norm.

Proof: Let J_g be a g-generated implication obtained from a g-generator such that $g(1) = \infty$. Then we have $g^{(-1)} = g^{-1}$. First, since J_g is a neutral fuzzy implication, if $x \in \{0,1\}$ or $y \in \{0,1\}$ or $z = 1$, then (LI) holds for any t-norm T. Again from Lemma 2–ii, $J_g(x,0) = 0$, for all $x \in (0,1]$, and hence (LI) holds for J_g and any t-norm T such that $T(x,y) \neq 0$ for all

Authorized licensed use limited to: IEEE Xplore. Downloaded on January 14, 2009 at 23:40 from IEEE Xplore. Restrictions apply.

$x, y \in (0,1)$, when $z = 0$. Hence in the following, we consider (LI) only for $x, y, z \in (0,1)$, which also implies $g(z) \in (0, \infty)$.

($\Longleftarrow$:) Let T be the product t-norm. Then for all $x, y, z \in (0,1)$

$$\begin{aligned}\text{RHS (LI)} &= J_g(x, J_g(y,z))\\ &= g^{-1}\left(\frac{1}{x} \cdot g\left[J_g(y,z)\right]\right)\\ &= g^{-1}\left(\frac{1}{x} \cdot g \circ g^{-1}\left[\frac{1}{y} \cdot g(z)\right]\right)\\ &= g^{-1}\left(\frac{1}{x \cdot y} \cdot g(z)\right) = g^{-1}\left[\frac{1}{T(x,y)} \cdot g(z)\right]\\ &= J_g(T(x,y), z)) = \text{LHS (LI)}.\end{aligned}$$

($\Longrightarrow$:) Let J_g obey the law of importation (LI) with a t-norm T. Then for all $x, y, z \in (0,1)$, we note that from Lemma 3 $T(x,y) \neq 0$, and also

$$\begin{aligned}J_g(T(x,y), z) &= J_g(x, J_g(y,z))\\ \Longrightarrow g^{-1}\left[\frac{1}{T(x,y)} \cdot g(z)\right] &= g^{-1}\left(\frac{1}{x} \cdot g\left[J_g(y,z)\right]\right)\\ \text{i.e.,} \frac{1}{T(x,y)} \cdot g(z) &= \frac{1}{x \cdot y} \cdot g(z)\end{aligned}$$

from which we obtain $T(x,y) = x \cdot y$, for all $x, y \in [0,1]$, since $g(z) \in (0, \infty)$. ■

Theorem 10: Let g be a g-generator such that $g(1) < \infty$. Then J_g, the g-generated implication, satisfies the law of importation (LI) if and only if $T = T_{\mathbf{P}}$, the product t-norm.

Proof: Let g be a g-generator such that $g(1) < \infty$.

($\Longleftarrow$:) Clearly if $T = T_{\mathbf{P}}$, then it can be easily verified that J_g satisfies (LI) with $T_{\mathbf{P}}$.

($\Longrightarrow$:) To prove the converse, let T be a t-norm for which J_g satisfies the law of importation and note that from Lemma 3, $T(x,y) \neq 0$ for any $x, y \in (0,1)$.

Suppose first that for some $x, y \in (0,1)$, we have $T(x,y) > x \cdot y$, and let $z = g^{-1}(x \cdot y \cdot g(1))$. Then we have

$$\frac{g(z)}{y} \leq \frac{g(z)}{T(x,y)} < \frac{g(z)}{x \cdot y} = g(1)$$

and so

$$\begin{aligned}J_g(x, J_g(y,z)) &= g^{(-1)}\left(\frac{1}{x} \cdot g \circ g^{(-1)}\left(\frac{g(z)}{y}\right)\right)\\ &= g^{(-1)}\left(\frac{1}{x} \cdot g \circ g^{-1}\left(\frac{g(z)}{y}\right)\right)\\ &= g^{(-1)}\left(\frac{g(z)}{x \cdot y}\right) = 1\end{aligned}$$

whereas

$$J_g(T(x,y), z) = g^{(-1)}\left(\frac{g(z)}{T(x,y)}\right) < 1$$

a contradiction.

On the other hand, if for some $x, y \in (0,1)$ we have $T(x,y) < x \cdot y < y$ and let $z = g^{-1}(T(x,y) \cdot g(1))$, since $T(x,y) \neq 0$, we have

$$\frac{g(z)}{y} < \frac{g(z)}{x \cdot y} < \frac{g(z)}{T(x,y)} = g(1)$$

and so

$$J_g(x, J_g(y,z)) = g^{(-1)}\left(\frac{g(z)}{x \cdot y}\right) < 1$$

whereas

$$J_g(T(x,y), z) = g^{(-1)}\left(\frac{g(z)}{T(x,y)}\right) = 1$$

a contradiction. Hence necessarily $T(x,y) = x \cdot y$ for all $x, y \in (0,1)$, i.e., $T = T_{\mathbf{P}}$ in (LI). ■

IV. The Law of Importation in the Setting of Uninorms and t-Operators

In this section, we investigate the equivalence (LI) when T is either a uninorm or a t-operator.

A. *Law of Importation With a Uninorm U Instead of a t-Norm T*

In this section, we consider the following equality, where U is a uninorm:

$$J(U(x,y), z) = J(x, J(y,z)). \tag{LI.U}$$

Lemma 4: If a neutral fuzzy implication J satisfies (LI.U), then $U(0,1) \neq 1$, i.e., U is conjunctive.

Proof: Let J be a neutral fuzzy implication and $U(0,1) = 1$. Letting $x = 0$, $y = 1$, $z \neq 1$, we have

$$\begin{aligned}\text{LHS (LI.U)} &= J(U(0,1), z) = J(1,z) = z\\ \text{RHS (LI.U)} &= J(0, J(1,z)) = J(0,z) = 1.\end{aligned}$$

LHS (LI.U) = RHS (LI.U) $\Longrightarrow z = 1$, a contradiction. ■

Since all of the fuzzy implications considered in this work are neutral, in this section, we only consider conjunctive uninorms U for (LI.U).

Theorem 11: An S-implication $J_{S,N}$, obtained from a t-conorm S and a strong negation N, satisfies (LI.U) with a conjunctive uninorm U only if the identity e of U is one, or equivalently $U = T$, the t-norm that is N-dual of S.

Proof: Let U be a conjunctive uninorm with the identity element $e \in (0,1]$ and $J_{S,N}$ be the S-implication obtained from a t-conorm S and a strong negation N. Letting $x = e$, $y = 1$, $z = 0$, we have

$$\begin{aligned}\text{LHS (LI.U)} &= J_{S,N}[U(e,1), 0] = J_{S,N}(1,0) = 0.\\ \text{RHS (LI.U)} &= J_{S,N}[e, J_{S,N}(1,0)] = J_{S,N}(e,0) = N(e).\end{aligned}$$

LHS (LI.U) = RHS (LI.U) $\Longrightarrow N(e) = 0 \Longrightarrow e = 1 \Longrightarrow U = T$, a t-norm. Now from Theorem 5, we have that U is the t-norm that is N-dual of S. ■

Authorized licensed use limited to: IEEE Xplore. Downloaded on January 14, 2009 at 23:40 from IEEE Xplore. Restrictions apply.

From the proof of the above theorem and noting that $J_{QL}(x,0) = S(N(x),T(x,0)) = N(x)$ for all $x \in [0,1]$, we have that a QL-implication J_{QL} satisfies (LI.U) with a conjunctive uninorm U only if the identity e of U is one, i.e., U is a t-norm.

Theorem 12: An R-implication J_{T^*} obtained from a left-continuous t-norm T^* satisfies (LI.U) with a conjunctive uninorm U only if the identity e of U is one, or equivalently $U = T^*$, the t-norm from which J_{T^*} was obtained.

Proof: Let U be a conjunctive uninorm with the identity element $e \in (0,1]$. Let J_{T^*} be an R-implication obtained from a left-continuous t-norm T^*. We know from (OP) that $J_{T^*}(x,x) = 1$, for all $x \in [0,1]$. Letting $x = 1$, $y = z = e$ in (LI.U)

$$\text{LHS (LI.U)} = J_{T^*}[U(1,e),e] = J_{T^*}(1,e) = e.$$
$$\text{RHS (LI.U)} = J_{T^*}[1, J_{T^*}(e,e)] = J_{T^*}(1,1) = 1.$$

Now, LHS (LI.U) = RHS (LI.U) $\Longrightarrow e = 1$, i.e., U is a t-norm. Now from Theorem 6, we have that $U = T^*$ is the t-norm from which J_{T^*} was obtained. ■

Theorem 13: An f-implication J_f satisfies (LI.U) with a conjunctive uninorm U only if the identity element e of U is one, or equivalently, U is the product t-norm $T_{\mathbf{P}}$.

Proof: Let J_f be an f-implication obtained from an f-generator. Let $x = z = e$, $y = 1$. Then

$$\text{LHS (LI.U)} = J_f(U(e,1),e) = J_f(1,e) = e$$
$$\text{RHS (LI.U)} = J_f(e, J_f(1,e)) = f^{-1}(e \cdot f(e)).$$

J_f satisfies (LI.U) only if $f^{-1}(e \cdot f(e)) = e \Longrightarrow e \cdot f(e) = f(e)$, which implies $e = 1$ or $f(e) = 0$ or $f(e) = \infty$. Now, $f(e) = \infty \Longrightarrow e = 0$, a contradiction to the fact that U is a conjunctive uninorm and $e \in (0,1]$. Hence we have $e = 1$. Now, from Theorem 8, we have that U is the product t-norm $T_{\mathbf{P}}$. ■

Theorem 14: A g-implication J_g satisfies (LI.U) with a conjunctive uninorm U only if the identity element e of U is one, or equivalently U is the product t-norm $T_{\mathbf{P}}$.

Proof: Let J_g be a g-implication obtained from a g-generator. Let $x = z = e$, $y = 1$. Then LHS (LI.U) $= J_g(U(e,1),e) = J_g(1,e) = e$ while RHS (LI.U) $= J_g(e, J_g(1,e)) = J_g(e,e) = g^{(-1)}(1/e \cdot g(e))$. J_g satisfies (LI.U) only if $g^{(-1)}(1/e \cdot g(e)) = e$. Here again we consider the following two cases.

Case 1: Let $1/e \cdot g(e) < g(1)$. Then $g^{(-1)}(1/e \cdot g(e)) = g^{-1}(1/e \cdot g(e))$. Hence

$$g^{-1}\left(\frac{1}{e} \cdot g(e)\right) = e \Longrightarrow \frac{1}{e} \cdot g(e) = g(e)$$
$$\Longrightarrow \frac{1}{e} = 1 \text{ or } g(e) = 0 \text{ or } g(e) = \infty$$
$$\Longrightarrow e = 1 \text{ or } e = 0.$$

Now, $e = 0$ implies U is a t-conorm, contradicting the fact that U is a conjunctive uninorm, and so we have $e = 1$ and $U = T$, a t-norm.

Case 2: Let $1/e \cdot g(e) \geq g(1)$. Then $g^{(-1)}(1/e \cdot g(e)) = 1 = e$ and $U = T$, a t-norm.

Now, from Theorems 9 and 10, we have that U is the product t-norm $T_{\mathbf{P}}$. ■

B. Law of Importation With a t-Operator F Instead of a t-Norm T

In this section, we consider the following equality, where F is a t-operator:

$$J(F(x,y),z) = J(x, J(y,z)). \tag{LI.F}$$

Lemma 5: Let J be a fuzzy implication operator such that $J(x,y) = 1$ iff either $x = 0$ or $y = 1$. Then J satisfies (LI.F) only if $k = 0$ or, equivalently, $F = T$, a t-norm.

Proof: Let $x = 0$, $y = 1$, $z \in [0,1)$. Then LHS (LI.F) $= J(F(0,1),z) = J(k,z)$ and RHS (LI.F) $= J(0, J(1,z)) = 1$. Hence for J to satisfy (LI.F), we need $J(k,z) = 1$, which since $z \neq 1$ by the hypothesis we have $k = 0$ or, equivalently, $F = T$, a t-norm. ■

Lemma 6: Let J be a fuzzy implication operator such that $J(x,0) = 1$ if and only if $x = 0$. Then J satisfies (LI.F) only if $k = 0$ or, equivalently, $F = T$, a t-norm.

Proof: Let $x = 1$, $y = z = 0$. Then LHS (LI.F) $= J(F(1,0),0) = J(k,0)$ and RHS (LI.F) $= J(1, J(0,0)) = 1$. Hence for J to satisfy (LI.F), we need $J(k,0) = 1$, and from the given condition on J, we have $k = 0$ or, equivalently, $F = T$, a t-norm. ■

Theorem 15: Let J be:

i) an S-implication $J_{S,N}$;
ii) an R-implication J_{T^*} obtained from a left-continuous t-norm T^*;
iii) an f-implication J_f;
iv) a g-implication J_g.

Then J satisfies (LI.F) only if the absorption element k of F is zero, i.e., $F = T$, a t-norm.

Proof: Let F be a t-operator with the absorption element $F(1,0) = k$.

i) Let $J_{S,N}$ be an S-implication obtained from a t-conorm S and strong negation N. Then $J_{S,N}(x,0) = N(x)$, the strong negation, and hence $J_{S,N}(x,0) = 1 \Longleftrightarrow x = 0$. Now, by Lemma 6, we have $k = 0$, i.e., F is a t-norm.
ii) Let J_{T^*} be an R-implication obtained from a left-continuous t-norm T^*. We know, from the ordering property (OP), that $J_{T^*}(x,x) = 1$, for any $x \in [0,1]$. Let $0 \leq x < k$. Then by definition $F(1,x) = k$. Now

$$\text{LHS (LI.F)} = J_{T^*}[F(1,x),x] = J_{T^*}(k,x).$$
$$\text{RHS (LI.F)} = J_{T^*}[1, J_{T^*}(x,x)] = J_{T^*}(1,1) = 1.$$

Hence LHS (LI.F) = RHS (LI.F)$\Longrightarrow J_{T^*}(k,x) = 1 \Longrightarrow k \leq x$, a contradiction to the fact that $x < k$. Thus there does not exist any x such that $0 \leq x < k \Longrightarrow k = 0$, i.e., F is a t-norm.
iii) Let J_f be an f-implication. Then the result is obvious from Lemmas 5 and 1.
iv) Let J_g be a g-implication. Then the result is obvious from Lemmas 6 and 2.

■

Corollary 2:

i) An S-implication $J_{S,N}$ satisfies (LI.F) iff F is the N-dual t-norm of S.
ii) An R-implication J_{T^*} obtained from a left-continuous t-norm T^* satisfies (LI.F) iff $F = T^*$.

Authorized licensed use limited to: IEEE Xplore. Downloaded on January 14, 2009 at 23:40 from IEEE Xplore. Restrictions apply.

iii) An f-implication J_f satisfies (LI.F) iff F is the product t-norm.

iv) A g-implication J_g satisfies (LI.F) iff F is the product t-norm.

Once again, from part i) of the proof of Theorem 15 and noting that $J_{QL}(.,0) = N$, a strong negation, we have that a QL-implication J_{QL} satisfies (LI.F) only if the absorption element k of F is zero or, equivalently, $F = T$, a t-norm.

V. Hierarchical CRI

In this section, we discuss the structure and inference in one of the most established methods of inferencing in *approximate reasoning,* the classical compositional rule of inference (CRI) [57]. After detailing some of the drawbacks of CRI, we propose a novel inferencing scheme called hierarchical CRI, which is a modification of and has many advantages over CRI, most notably computational efficiency. We illustrate these concepts with some numerical examples.

Definition 16: A fuzzy set on a nonempty set X, $A : X \rightarrow [0,1]$, is said to be a "fuzzy singleton" if there exists an $x_0 \in X$ such that A has the following representation:

$$A(x) = \begin{cases} 1, & \text{if } x = x_0 \\ 0, & \text{if } x \neq x_0 \end{cases}. \tag{25}$$

We say A attains normality at $x_0 \in X$.

A. Structure and Inference in Classical CRI

The CRI proposed by Zadeh [57] is one of the earliest implementations of generalized modus ponens. Here, a fuzzy if-then rule of the form

$$\text{If } \widetilde{x} \text{ is } A \text{ Then } \widetilde{y} \text{ is } B \tag{26}$$

is represented as a fuzzy relation $R(x,y) : X \times Y \rightarrow [0,1]$ as follows:

$$R(x,y) = [A(x) \longrightarrow B(y)] = J(A(x), B(y)) \tag{27}$$

where $\longrightarrow$ is any fuzzy implication J and A, B are fuzzy sets on their respective domains X, Y. In this section, unless otherwise explicitly stated, $x \in X$, $y \in Y$, and $z \in Z$. Then given a fact $\widetilde{x}$ *is* A', the inferred output B' is obtained as composition of $A'(x)$ and $R(x,y)$, i.e.,

$$B'(y) = A'(x) \circ R(x,y) \tag{28}$$

where $\circ$ is a Sup–T^* composition given by

$$B'(y) = \text{Sup}_{x \in X}\{T^*[A'(x), R(x,y)]\} \tag{29}$$

where T^* can be any t-norm (see [52]).

In the case when the input A' is a "singleton" (25) attaining normality at an $x_0 \in X$, then

$$\begin{aligned} B'(y) &= A'(x) \circ R(x,y) \\ &= \text{Sup}_{x \in X}\{T^*[A'(x), R(x,y)]\} \\ &= T^*[A'(x_0), R(x_0,y)] = R(x_0,y). \end{aligned} \tag{30}$$

In the case of a multiple-input single-output (MISO) fuzzy rule given by

$$R : \text{If } \widetilde{x} \text{ is } A \text{ and } \widetilde{y} \text{ is } B \text{ Then } \widetilde{z} \text{ is } C \tag{31}$$

the relation R is given by

$$\begin{aligned} R(x,y;z) &= J[ce(A(x,y)) \odot ce(B(x,y)), C(z)] \\ &= J[A(x) \odot B(y), C(z)] \end{aligned} \tag{32}$$

where $ce(A(x,y))$ stands for the "cylindrical extension" of the fuzzy set A with respect to the universe of B, and vice versa, and $\odot$ is the antecedent combiner, which is usually a t-norm.

Consider a single MISO rule of the type (31), denoted $(A,B) \longrightarrow C$, for simplicity. Then, given a multiple-input (A', B'), the infered output C', taking the Sup–T^* composition, is given by

$$C' = (A', B') \circ \big[(A,B) \longrightarrow C\big] \tag{33}$$

$$\begin{aligned} C'(z) = \text{Sup}_{x,y}\, T^*\big\{&\big[A'(x) \odot B'(y)\big], \\ &\big[A(x) \odot B(y)\big] \longrightarrow C(z)\big\}. \end{aligned} \tag{34}$$

As in the single-input single output case, when the inputs are "singleton" fuzzy sets A' and B' attaining normality at the points $x_0 \in X$, $y_0 \in Y$, respectively, we have from (34)

$$C'(z) = [A(x_0) \odot B(y_0)] \longrightarrow C(z) = R(x_0, y_0; z). \tag{35}$$

Once again, $\odot$ is the antecedent combiner which is usually any of the t-norms.

1) Example 1: Let $A = [.9\ .8\ .7\ .7]$, $B = [1\ .6\ .8]$, and $C = [.1\ .1\ .2]$, where A, B, and C are defined on $X = \{x_1, x_2, x_3, x_4\}$, $Y = \{y_1, y_2, y_3\}$, and $Z = \{z_1, z_2, z_3\}$, respectively. Let J be the Reichenbach implication $J_{\mathbf{RC}}(x,y) = 1 - x + x \cdot y$ and the antecedent combiner $\odot$ be the product t-norm $T_{\mathbf{P}}(x,y) = x \cdot y$. Then, taking the cylindrical extensions of A and B with respect to $T_{\mathbf{P}}$, we have that

$$A \odot B = T_{\mathbf{P}}(A,B) = \begin{pmatrix} .9 & .54 & .72 \\ .8 & .48 & .64 \\ .7 & .42 & .56 \\ .7 & .42 & .56 \end{pmatrix}.$$

Now, with $x \longrightarrow y = 1 - x + x \cdot y$, $T_{\mathbf{P}}(A,B) \longrightarrow C$ from (32) will be given by $R(A, B : C) = [C(z_1)\ C(z_2)\ C(z_3)]$, where

$$\begin{aligned} T_{\mathbf{P}}(A,B) \longrightarrow C &= T_{\mathbf{P}}(A,B) \longrightarrow [z_1\ z_2\ z_3] \\ &= T_{\mathbf{P}}(A,B) \longrightarrow [.1\ .1\ .2] \end{aligned}$$

and $C(z_i) = T_{\mathbf{P}}(A,B) \longrightarrow z_i$ are given as follows:

$$C(z_1) = C(z_2) = \begin{pmatrix} .19 & .514 & .352 \\ .28 & .568 & .424 \\ .37 & .622 & .496 \\ .37 & .622 & .496 \end{pmatrix}$$

$$C(z_3) = \begin{pmatrix} .28 & .568 & .424 \\ .36 & .616 & .488 \\ .44 & .664 & .552 \\ .44 & .664 & .552 \end{pmatrix}.$$

Authorized licensed use limited to: IEEE Xplore. Downloaded on January 14, 2009 at 23:40 from IEEE Xplore. Restrictions apply.

Let $A' = [0\ 0\ 1\ 0]$, $B' = [0\ 1\ 0]$ be the given fuzzy singleton inputs. Then

$$A' \odot B' = T_{\mathbf{P}}(A', B') = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{36}$$

Taking the Sup–$T_{\mathbf{M}}$ composition, we have

$$\begin{aligned} C' &= T_{\mathbf{P}}(A', B') \circ [T_{\mathbf{P}}(A, B) \longrightarrow C] \\ &= [.622\ .622\ .664]. \end{aligned} \tag{37}$$

B. Drawbacks of CRI

Despite the prevalent use of CRI, the following are usually cited as its drawbacks (see, for example, [16] and [17]).

- *Computational Complexity:* The calculation of the supremum in (34) is a time-consuming process. When $|X| = m$ and $|Y| = n$, the complexity of a single inference amounts to $\mathcal{O}(m \cdot n)$.
- *Explosion in Dimensions:* For an n-input one-output system $(A_1, A_2, \ldots, A_n) \longrightarrow B$ with the cardinality of the base sets X_i of each of the inputs A_i being n_i, we have an n-dimensional matrix having $\prod_{i=1}^{n} n_i$ entries. Also we need to store n-dimensional matrices for every fuzzy if-then rule.

The many works proposing modifications to the classical CRI can be broadly categorized into those that attempt to improve the accuracy of inferencing in CRI and those that intend to enhance the efficiency in its inferencing. Some works relating to the former approach are those of Ying [56], who investigated the "reasonableness" of CRI and proposed a suitable modification of the CRI; and Wangming [52], who was one of the earliest to employ the Sup–T^* composition and has studied the suitability of pairs (T^*, J) – of the t-norm T^* used for composition and the fuzzy implication J that satisfy some appropriate criteria for generalized modus ponens and generalized modus tollens. In [12], necessary and sufficient conditions on operators involved in CRI inferencing are investigated to fit the CRI into the analogical scheme. For works relating to computation of CRI and exact formulas, see those by Fullér *et al.* [24], [25], [23]. For some recent works, see [35]–[37].

In the case when there are more than two antecedents involved in fuzzy inference, Ruan and Kerre [41] have proposed an extension to the classical CRI, wherein, starting from a finite number of fuzzy relations of an arbitrary number of variables but having some variables in common, one can infer fuzzy relations among the variables of interest.

As noted above, because of the multidimensionality of the fuzzy relation (32), inferencing in CRI (33) is difficult to perform. To overcome this difficulty, Demirli and Turksen [17] proposed a *rule breakup* method and showed that rules with two or more independent variables in their premise can be simplified to a number of inferences of rule bases with simple rules (only one variable in their premise). For further modification of this method, see [26]. In the following, we propose a modified but novel form of CRI that alleviates some of the concerns noted above.

C. Hierarchical CRI

In the field of fuzzy control, hierarchical fuzzy systems (HFSs) hold center stage, since, where applicable, they help to a large extent in breaking down the complexity of the system being modeled, in terms of both efficiency and understandability. For a good survey of HFS, see Torra [46] and references therein.

In this section, we propose a new modified form of CRI termed hierarchical CRI owing to the way in which multiple antecedents of a fuzzy rule are operated upon in the inferencing. A distinction should be made between the method proposed in this paper, which uses the given MISO fuzzy rules as is, and typical inferencing in HFS, which is dictated by the hierarchical structure that exists among the modeled system variables. It should be emphasized that the term "hierarchical CRI" as employed here refers to the modifications in the inferencing procedure of CRI and does not impose a hierarchical architecture on the MISO fuzzy rules.

From the law of importation (LI), we see that (32) can be rewritten as, using a t-norm T as the antecedent combiner

$$\begin{aligned} R(x, y; z) &= J[T(A(x), B(y)), C(z)] \\ &= J\{A(x), J[B(y), C(z)]\}. \end{aligned} \tag{38}$$

Taking a cue from this equivalence, we propose a novel way of inferencing called the hierarchical CRI, which is given as shown in (39) at the bottom of the page (using the Sup–T^* composition).

Procedure for Hierarchical CRI

Step 1) Calculate $R' = B \longrightarrow C$.

Step 2) Calculate $\overline{C} = B' \circ R' = B' \circ [B \longrightarrow C]$.

Step 3) Calculate $R'' = A \longrightarrow \overline{C}$.

Step 4) Finally, calculate

$$\begin{aligned} C'' &= A' \circ R'' = A' \circ [A \longrightarrow \overline{C}] \\ &= A' \circ \{A \longrightarrow [B' \circ (B \longrightarrow C)]\}. \end{aligned}$$

We illustrate the gain in efficiency from using the hierarchical CRI through the following example. For simplicity, we have

$$\begin{aligned} C'' &= A' \circ \{A \longrightarrow [B' \circ (B \longrightarrow C)]\} \\ C''(z) &= \mathrm{Sup}_x\, T^*\{A'(x), \{A(x) \longrightarrow [B' \circ (B \longrightarrow C(z))]\}\} \\ &= \mathrm{Sup}_x\, T^*\{A'(x), \{A(x) \longrightarrow [\mathrm{Sup}_y\, T^*\{B'(y),\ (B(y) \longrightarrow C(z))\}]\}\} \end{aligned} \tag{39}$$

Authorized licensed use limited to: IEEE Xplore. Downloaded on January 14, 2009 at 23:40 from IEEE Xplore. Restrictions apply.

considered the Sup–min, i.e., Sup–$T_{\mathbf{M}}$, composition in the example, though it can be substituted by any Sup–T^* composition, for any t-norm T^*. Also we have considered the Łukasiewicz implication $J_{\mathbf{LK}}(x,y) = \min(1, 1-x+y)$, which is both an S- and an R-implication.

1) Example 2: Let the fuzzy sets A, B, C be as in Example 1. Let J be the Łukasiewicz implication $J_{\mathbf{LK}}(x,y) = \min(1, 1-x+y)$ and the antecedent combiner $\odot$ be the minimum t-norm $T_{\mathbf{M}}(x,y) = \min(x,y)$.

Case 1) Inference with the Classical CRI: Then taking the cylindrical extensions of A and B with respect to $T_{\mathbf{M}}$, we have that

$$T_{\mathbf{M}}(A,B) = \begin{pmatrix} .9 & .6 & .8 \\ .8 & .6 & .8 \\ .7 & .6 & .7 \\ .7 & .6 & .7 \end{pmatrix}.$$

Now, with $x \longrightarrow y = \min(1, 1-x+y)$, $T_{\mathbf{M}}(A,B) \longrightarrow C$ from (32) will be given by $R(A, B : C) = [C(z_1)\ C(z_2)\ C(z_3)]$, where

$$\begin{aligned} T_{\mathbf{M}}(A,B) \longrightarrow C &= T_{\mathbf{M}}(A,B) \longrightarrow [z_1\ z_2\ z_3] \\ &= T_{\mathbf{M}}(A,B) \longrightarrow [.1\ .1\ .2] \end{aligned}$$

and $C(z_i) = T_{\mathbf{M}}(A,B) \longrightarrow z_i$

$$C(z_1) = C(z_2) = \begin{pmatrix} .2 & .5 & .3 \\ .3 & .5 & .3 \\ .4 & .5 & .4 \\ .4 & .5 & .4 \end{pmatrix}$$

$$C(z_3) = \begin{pmatrix} .3 & .6 & .4 \\ .4 & .6 & .4 \\ .5 & .6 & .5 \\ .5 & .6 & .5 \end{pmatrix}.$$

Let $A' = [0\ 0\ 1\ 0]$, $B' = [0\ 1\ 0]$ be the given fuzzy singleton inputs. Then $T_{\mathbf{M}}(A', B')$ is still as given in (36). Taking the Sup–min composition, we have

$$C' = T_{\mathbf{M}}(A', B') \circ [T_{\mathbf{M}}(A,B) \longrightarrow C] = [.5\ .5\ .6]. \quad (40)$$

Case 2) Inference With the Hierarchical CRI: Inferencing with the hierarchical CRI, given an input (A', B'), we have

$$B \longrightarrow C = J_{\mathbf{LK}}(B, C) = \begin{pmatrix} .1 & .1 & .2 \\ .5 & .5 & .6 \\ .3 & .3 & .4 \end{pmatrix}$$

$$\begin{aligned} \overline{C} &= B' \circ [B \longrightarrow C] \\ &= [0\ 1\ 0] \circ [B \longrightarrow C] \\ &= [.5\ .5\ .6] \end{aligned}$$

$$A \longrightarrow \overline{C} = J_{\mathbf{LK}}(A, \overline{C}) = \begin{pmatrix} .6 & .6 & .7 \\ .7 & .7 & .8 \\ .8 & .8 & .9 \\ .8 & .8 & .9 \end{pmatrix}$$

$$\begin{aligned} C'' &= A' \circ [A \longrightarrow \overline{C}] \\ &= [0\ 0\ 1\ 0] \circ [A \longrightarrow \overline{C}] \\ &= [.8\ .8\ .9]. \end{aligned} \quad (41)$$

D. Advantages of Hierarchical CRI

From the above example, it is clear that we can convert a multiple-input system employing CRI inference to a single-input hierarchical system employing CRI. The effect becomes more pronounced when we have more than two input variables. We summarize the advantages of hierarchical CRI in the following.

- *Computational Efficiency:* In the proposed hierarchical system we only need to process two-dimensional matrices at every stage.
- *Storage Efficiency:* It suffices to store the different antecedent fuzzy sets of a fuzzy rule and not their combined multidimensional matrices.
- *Associative Inferencing:* For a given input $(A'_1, A'_2, \ldots, A'_n)$, since we only compose each of the A_is independently at every stage, this relieves us from waiting for all the inputs' A_is to be available for the inference, and the inference can be done associatively.
- *Order Independence:* By the commutativity of the t-norms used in the composition and as the antecedent combiner—T^*, T, respectively—the inputs can be composed in any order, i.e.,

$$\begin{aligned} J[T(x,y,z), w] &= J[T(x, T(y,z)), w] = J[x, J(T(y,z), w)] \\ &= J[x, J(y, J(z,w))]. \end{aligned}$$

Hence it can be applied online, as and when the inputs are given.

Though Example 2 illustrates the computational efficiency of hierarchical CRI, we see that the inference obtained from the classical CRI is different from that obtained from the proposed hierarchical CRI, i.e., $C' \neq C''$. In the following section, we propose some sufficiency conditions under which the outputs of the classical and hierarchical CRI schemes are identical, for the same inputs.

VI. Equivalence Between Classical CRI and Hierarchical CRI

In this section, we investigate the equivalence between (39) and (33). Toward this end, we propose some sufficiency conditions when the inputs to the system are restricted to fuzzy singletons. It is not uncommon to use a t-norm instead of a fuzzy implication to relate antecedents to consequents to obtain the relation R in (32), i.e., the J in (32) is a t-norm T. In such a case, we give both necessary and sufficiency conditions for the equivalence between (39) and (33) when the inputs to the system are restricted to fuzzy singletons. Again, in the case when J is a t-norm, we have also proposed some sufficiency conditions even when the inputs to the system are not fuzzy singletons.

A. Sufficiency Condition for Equivalence Between Classical CRI and Hierarchical CRI

Theorem 16: Let the inputs to the fuzzy system be "singleton" fuzzy sets. Then (39) and (33) are equivalent, i.e., $(A', B') \circ [(A,B) \longrightarrow C] \equiv A' \circ \{A \longrightarrow [B' \circ (B \longrightarrow C)]\}$, when the t-norm T employed for the antecedent combiner and the implication J are such that (LI) holds.

Proof: Let the antecedent combiner be a t-norm T and J be a fuzzy implication such that J satisfies (LI) with T. Also let

Authorized licensed use limited to: IEEE Xplore. Downloaded on January 14, 2009 at 23:40 from IEEE Xplore. Restrictions apply.

us consider the Sup–T^* composition for some t-norm T^*. Let the inputs to the fuzzy rule base be the "singleton" fuzzy sets A' and B' attaining normality at the points $x_0 \in X$, $y_0 \in Y$, respectively. Then we have

$$\begin{aligned}(33) &= C'(z)\\ &= \mathrm{Sup}_{x,y}\, T^*\{T[A'(x), B'(y)], R[A(x), B(y); C(z)]\}\\ &= T^*\{T[A'(x_0), B'(y_0)], J(T[A(x_0), B(y_0)], C(z))\}\\ &= J(T[A(x_0), B(y_0)], C(z)) \qquad (42)\\ (39) &= C''(z)\\ &= \mathrm{Sup}_x\, T^*\{A'(x), R''[A(x), \mathrm{Sup}_y\, T^*\\ &\qquad (B'(y), R'[B(y), C(z)])]\}\\ &= T^*\{A'(x_0), J(A(x_0), T^*[B'(y_0), J(B(y_0), C(z))])\}\\ &= J(A(x_0), T^*[B'(y_0), J(B(y_0), C(z))])\\ &= J(A(x_0), J(B(y_0), C(z))) \qquad (43)\end{aligned}$$

Equation (42) = (43) by (LI). Thus (39) and (33) are equivalent when the t-norm T used for the antecedent combiner and the fuzzy implication J satisfy (LI) and the inputs are fuzzy singletons. ■

1) Example 3: Let the fuzzy sets A, B, C be as defined in Example 1. In Example 1, the antecedent combiner was taken to be the product t-norm $T_{\mathbf{P}}$ and the implication J considered was the Reichenbach implication $J_{\mathbf{RC}}$, which is an S-implication. From Theorem 5 and Table II, we see that the pair $(T_{\mathbf{P}}, J_{\mathbf{RC}})$ does satisfy the conditions given in Theorem 16.

In this example, we infer, using the hierarchical CRI for the identical "singleton" inputs A', B' given in Example 1 and show that the output obtained is the same as that in Example 1, i.e., (37).

Inferencing with the hierarchical CRI, given the input (A', B'), we have

$$\begin{aligned}B \longrightarrow C = J_{\mathbf{RC}}(B, C) &= \begin{pmatrix} .1 & .1 & .2\\ .46 & .46 & .52\\ .28 & .28 & .36\end{pmatrix}\\ \overline{C} &= B' \circ [B \longrightarrow C]\\ &= [0\ 1\ 0] \circ [B \longrightarrow C]\\ &= [.46\ .46\ .52].\end{aligned}$$

Now, after some tedious calculations, it can be seen that

$$\begin{aligned}C'' &= A' \circ [A \longrightarrow \overline{C}]\\ &= [0\ 0\ 1\ 0] \circ [A \longrightarrow \overline{C}]\\ &= [.622\ .622\ .664]. \qquad (44)\end{aligned}$$

Quite evidently, the inference obtained from the hierarchical CRI (44) is equal to that obtained from the classical CRI (37), under the conditions of Theorem 16.

Similarly, in Example 2, the fuzzy implication J employed was the Łukasiewicz implication $J_{\mathbf{LK}}$ with minimum t-norm $T_{\mathbf{M}}$ as the antecedent combiner. Now, according to the conditions given in Theorem 16, if we consider the Łukasiewicz t-norm $T_{\mathbf{LK}}$ as the antecedent combiner, it can be easily verified that the outputs obtained from the classical CRI and the hierarchical CRI are indeed identical.

B. Classical CRI With t-Norms Instead of Fuzzy Implications

Even though t-norms do not satisfy all the properties of a fuzzy implication, as given in Definition 8, they are still employed both in *approximate reasoning* and *fuzzy control* to relate antecedents to consequents in fuzzy rules (see, for example, [11] and [19]). Two of the most commonly employed t-norms are Mamdani's *min* ($T_{\mathbf{M}}$) and Larsen's *product* ($T_{\mathbf{P}}$) t-norms.

It is easy to see that if $J = T^*$ is any t-norm, then J satisfies (LI) with a t-norm T, i.e., $T^*(T(x, y), z) \equiv T^*(x, T^*(y, z))$, if and only if $J = T^* \equiv T$. The reverse implication is obtained by associativity and the forward implication can be obtained by taking $z = 1$. Hence Theorem 16 is true in the above case. In fact, when the inputs are singleton fuzzy sets, we can in fact show the following stronger equivalence condition.

Theorem 17: Let the inputs to the fuzzy system be "singleton" fuzzy sets and $J = T^*$ be a t-norm. Then (39) and (33) are equivalent, i.e., $(A', B') \circ [(A, B) \longrightarrow C] \equiv A' \circ \{A \longrightarrow [B' \circ (B \longrightarrow C)]\}$ if and only if the same t-norm T^* is employed for the antecedent combiner.

Proof: Let A', B' be singleton fuzzy sets on the domains X, Y attaining normality at points $x_0 \in X$, $y_0 \in Y$, respectively. Then, for any $z \in Z$, we have with $J = T^*$ and for any Sup–T' composition and for all $x \in X$, $y \in Y$, (39) and (33) become as given in (45) and (46) at the bottom of the page. Now,

$$\begin{aligned}(39) &= C''(z)\\ &= \mathrm{Sup}_x\, T'\{A'(x), \{A(x) \longrightarrow [\mathrm{Sup}_y\, T'\{B'(y),\ (B(y) \longrightarrow C(z))\}]\}\}\\ &= A(x_0) \longrightarrow (B(y_0) \longrightarrow C(z))\\ &= T^*(A(x_0), T^*(B(y_0), C(z))) \qquad (45)\\ (33) &= C'(z)\\ &= \mathrm{Sup}_{x,y}\, T'\{T[A'(x), B'(y)], T[A(x), B(y)] \longrightarrow C(z)\}\\ &= T'\{T[A'(x_0), B'(y_0)], T[A(x_0), B(y_0)] \longrightarrow C(z)\}\\ &= T[A(x_0), B(y_0)] \longrightarrow C(z)\\ &= T^*(T[A(x_0), B(y_0)], C(z)) \qquad (46)\end{aligned}$$

Authorized licensed use limited to: IEEE Xplore. Downloaded on January 14, 2009 at 23:40 from IEEE Xplore. Restrictions apply.

TABLE VII
RESULTS AT A GLANCE: J—FUZZY IMPLICATION, T—t-NORM, U—UNINORM, F—t-OPERATOR

Name	$J(x,y)$	$L = T/U/F$	$J(x, J(y,z)) = J(L(x,y), z)$
R-implication	J_{T^*}	t-norm T	$\Leftrightarrow T = T^*$
S-implication	$J_{S,N}$	t-norm T	$\Leftrightarrow T$ is the N-dual of S
f-implication	J_f	t-norm T	$\Leftrightarrow T = T_{\mathbf{P}}$
g-implication	J_g	t-norm T	$\Leftrightarrow T = T_{\mathbf{P}}$

from above, we have that (45) = (46) if and only if $J = T^* \equiv T$, i.e., the t-norm T employed for the antecedent combiner and the t-norm T^* relating the antecedent to the consequent are identical. ∎

The following theorem gives a sufficient condition under which (39) and (33) are equivalent, even when the inputs $A', B', \ldots$, are not fuzzy singletons.

Theorem 18: Let T be a left-continuous t-norm. Then (39) and (33) are equivalent, i.e., $(A', B') \circ [(A, B) \longrightarrow C] \equiv A' \circ \{A \longrightarrow [B' \circ (B \longrightarrow C)]\}$, if the same T is employed for the antecedent combiner to relate the antecedents of the rules to their consequents (instead of an implication J) and in the Sup–T composition.

Proof: Let $J = T$ a t-norm; then

$$\begin{aligned}(33) &= \mathrm{Sup}_{x,y}\, T\{T[A'(x), B'(y)], R[A(x), B(y)]; C(z)]\}\\ &= \mathrm{Sup}_{x,y}\, T\{T[A'(x), B'(y)],\ J(T[A(x), B(y)], C(z))\}\\ &= \mathrm{Sup}_{x,y}\, T\{T[A'(x), B'(y)],\ T(T[A(x), B(y)], C(z))\} \quad (47)\\ &= \mathrm{Sup}_{x,y}\, T\{A'(x), B'(y), A(x), B(y), C(z)\}. \quad (48)\end{aligned}$$

We obtain (48) from (47) by the associativity of T.

$$\begin{aligned}(39)\\ &= \mathrm{Sup}_x T\{A'(x), R''[A(x), \mathrm{Sup}_y T(B'(y), R'[B(y), C(z)])]\}\\ &= \mathrm{Sup}_x T\{A'(x), J(A(x), \mathrm{Sup}_y T[B'(y), J(B(y), C(z))])\}\\ &= \mathrm{Sup}_x T\{A'(x), T(A(x), \mathrm{Sup}_y T[B'(y), T(B(y), C(z))])\}\\ &= \mathrm{Sup}_x T\{A'(x), \mathrm{Sup}_y T(A(x), T[B'(y), TB(y)], C(z))])\}\\ &= \mathrm{Sup}_{x,y} T\{A'(x), A(x), B'(y), B(y), C(z)\}. \quad (49)\end{aligned}$$

The last equality follows from the previous step since T is left-continuous and $T(a,\ \mathrm{Sup}_y\ b_y) = \mathrm{Sup}_y\ T(a,\ b_y)]$. Equation (48) $\equiv$ (49) by associativity of t-norms again. Thus (39) and (33) are equivalent if the t-norm used for the "implication" and the antecedent combiner are the same as the t-norm T used in the Sup–T composition. ∎

VII. CONCLUSIONS

The law of importation has not been studied so far in isolation. In this paper, we have given necessary and sufficient conditions under which the classes of fuzzy implications R-, S-, f-, and g-implications satisfy the law of importation (LI). Also we have investigated the general form of law of importation in the more general setting of uninorms and t-operators for all the above classes of fuzzy implications. A summary of the results in this work is given in Table VII for ready reference.

We have also proposed a modification in the CRI inferencing scheme called the hierarchical CRI, which has many advantages, chiefly computational efficiency. We have also shown that the law of importation plays an important role in the equivalence between inferencing with hierarchical CRI and the classical CRI. The hierarchical CRI proposed in this paper has been employed only as an instrument of illustration to the possible applications of the law of importation. The hierarchical CRI method itself merits more focused study and will be taken up in a future work.

By the commutativity of a t-norm, if an implication J has the law of importation with respect to to any t-norm T, then J has the exchange principle, i.e.,

$$J_T(x, J_T(y,z)) = J_T(y, J_T(x,z)), \quad x, y, z \in [0,1].$$

The exchange principle plays a significant role in contrapositivization of fuzzy implications (see [6]).

Recent works investigating classical logic tautologies involving fuzzy implication operators show both their theoretical interests (see [7], [6], [10], [13]–[15], [18], [20], [34], [43], and [48]) and their influence in practical applications (see [8], [9], [28], [45], and [53]). This paper can also be seen in this context.

ACKNOWLEDGMENT

The author wishes to express his gratitude to Prof. C. J. M. Rao for his most valuable suggestions and remarks and the reviewers for their insightful comments, especially for the elegant proof suggested by one of them for Theorem 10.

REFERENCES

[1] M. Baczyński, "On a class of distributive fuzzy implications," *Int. J. Uncertain., Fuzz. Knowl.-Based Syst.*, vol. 9, no. 2, pp. 229–238, 2001.

[2] M. Baczyński, "Residual implications revisited: Notes on the Smets-Magrez theorem," *Fuzzy Sets Syst.*, vol. 145, pp. 267–277, 2004.

[3] M. Baczyński and B. Jayaram, "On some properties of the new class of Yager's fuzzy implications," in *Proc. FSTA 2006 8th Int. Conf. Fuzzy Set Theory Applicat.*, Liptovský Ján, Slovak Republic, Feb. 2006, pp. 20–21.

[4] M. Baczyński and B. Jayaram, "On the characterisations of (S, N)-implications from continuous negations," in *Proc. IPMU 2006 11th Int. Conf. Inf. Process. Manag. Uncertain. Knowl.-Based Syst.*, Paris, France, Jul. 2006, pp. 436–443.

[5] B. D. Baets, "Idempotent uninorms," *Eur. J. Ope. Res.*, vol. 118, pp. 631–642, 1999.

[6] J. Balasubramaniam, "Contrapositive symmetrization of fuzzy implications—Revisited," *Fuzzy Sets Syst.*, vol. 157, pp. 2291–2310, 2006.

[7] J. Balasubramaniam, "Yager's new class of implications J_f and some classical tautologies," *Inf. Sci.*, vol. 177, pp. 930–946, 2007.

[8] J. Balasubramaniam and C. J. M. Rao, Nikos and E. Mastorakis, Eds., "A lossless rule reduction technique for a class of fuzzy systems," in *Proc. 3rd WSEAS Int. Conf. Fuzzy Sets Fuzzy Systems—Recent Adv. Simul., Comput. Meth. Soft Comput.*, Interlaken, Switzerland, Feb. 2002, pp. 228–233.

[9] J. Balasubramaniam and C. J. M. Rao, "$\mathbb{R}$-implication operators and rule reduction in Mamdani-type fuzzy systems," in *Proc. Joint Conf. Inf. Sci.*, H. J. Caulfield, S. -H. Chen, H. -D. Cheng, R. J. Duro, V. Honavar, E. E. Kerre, M. Lu, M. G. Romay, T. K. Shih, D. Ventura, P. P. Wang, and Y. Yang, Eds.——, "$\mathbb{R}$-implication operators and rule reduction in Mamdani-type fuzzy systems," *Proc. Assoc. Intell. Mach. Joint Conf. Inf. Sci.*, pp. 82–84, 2002.

Authorized licensed use limited to: IEEE Xplore. Downloaded on January 14, 2009 at 23:40 from IEEE Xplore. Restrictions apply.

[10] J. Balasubramaniam and C. J. M. Rao, "On the distributivity of implication operators over $\mathbb{T}$ and $\mathbb{S}$ norms," *IEEE Trans. Fuzzy Syst.*, vol. 12, pp. 194–198, 2004.

[11] B. Bouchon-Meunier and V. Kreinovich, Axiomatic description of implication leads to a classical formula with logical modifiers: In particular, Mamdani's choice of "AND" as implication is not so weird after all Comput. Sci. Dept., Univ. of Texas at El Paso, Tech. Rep. UTEP-CS-96-23, 1996.

[12] B. Bouchon-Meunier, R. Mesiar, C. Marsala, and M. Rifqi, "Compositional rule of inference as an analogical scheme," *Fuzzy Sets Syst.*, vol. 138, no. 1, pp. 53–65, 2003.

[13] W. Combs, "Author's reply," *IEEE Trans. Fuzzy Syst.*, vol. 7, p. 371, 1999.

[14] W. Combs, "Author's reply," *IEEE Trans. Fuzzy Syst.*, vol. 7, pp. 478–479, 1999.

[15] W. Combs and J. Andrews, "Combinatorial rule explosion eliminated by a fuzzy rule configuration," *IEEE Trans. Fuzzy Syst.*, vol. 6, pp. 1–11, 1998.

[16] C. Cornelis, M. D. Cock, and E. Kerre, *Efficient Approximate Reasoning with Positive and Negative Information*. Heidelberg, Germany: Springer-Verlag, 2004, vol. 3214, KES 2004, LNAI, pp. 779–785.

[17] K. Demirli and I. B. Turksen, "Rule break up with compositional rule of inference," in *Proc. FUZZ-IEEE'92*, San Diego, 1992, pp. 949–956.

[18] S. Dick and A. Kandel, "Comments on 'Combinatorial rule explosion eliminated by a fuzzy rule configuration'," *IEEE Trans. Fuzzy Syst.*, vol. 7, pp. 478–479, 1999.

[19] D. Driankov, H. Hellendoorn, and M. Reinfrank, *An Introduction to Fuzzy Control*. Berlin, Germany: Springer-Verlag, 1993.

[20] J. Fodor, "Contrapositive symmetry of fuzzy implications," *Fuzzy Sets Syst.*, vol. 69, pp. 141–156, 1995.

[21] J. Fodor and M. Roubens, *Fuzzy Preference Modeling and Multicriteria Decision Support*. Dordrecht: Kluwer, 1994.

[22] J. Fodor, R. Yager, and A. Rybalov, "Structure of uninorms," *Int. J. Uncertain., Fuzz. Knowl.-Based Syst.*, vol. 5, no. 4, pp. 411–427, 1997.

[23] R. Fullér, *Fuzzy Reasoning and Fuzzy Optimization*. Turku, Finland: TUCS, 1998.

[24] R. Fullér and B. Werners, B. Riecan and M. Duchon, Eds., "The compositional rule of inference with several relations," in *Proc. Int. Conf. Fuzzy Sets Applicat.*, Liptovsky Mikulas, Czechoslovakia, Feb. 1992, pp. 39–44.

[25] R. Fullér and H.-J. Zimmermann, "On computation of the compositional rule of inference under triangular norms," *Fuzzy Sets Syst.*, vol. 51, pp. 267–275, 1992.

[26] M. Golob, "Decomposition of a fuzzy controller based on the inference break-up method," *Intell. Data Anal.*, vol. 3, no. 2, pp. 127–137, 1999.

[27] S. Gottwald, *A Treatise on Many-Valued Logics*. New York: Taylor & Francis, 2001.

[28] C. Igel and K. Temme, "The chaining syllogism in fuzzy logic," *IEEE Trans. Fuzzy Syst.*, vol. 12, pp. 849–853, 2004.

[29] E. P. Klement, R. Mesiar, and E. Pap, *Triangular Norms*. Dordrecht, The Netherlands: Kluwer, 2000.

[30] J. Martin, G. Mayor, and J. Torrens, "On locally internal monotonic operators," *Fuzzy Sets Syst.*, vol. 137, pp. 27–42, 2003.

[31] M. Mas, G. Mayor, and J. Torrens, "t-operators," *Int. J. Uncertain., Fuzz. Knowl.-Based Syst.*, vol. 7, no. 1, pp. 31–50, 1999.

[32] M. Mas, M. Monserrat, and J. Torrens, "On left and right uninorms," *Int. J. Uncertain., Fuzz. Knowl.-Based Syst.*, vol. 9, pp. 491–508, 2001.

[33] M. Mas, M. Monserrat, and J. Torrens, "QL versus D-implications," *Kybernetika*, vol. 42, no. 3, pp. 351–366, 2006.

[34] J. M. Mendel and Q. Liang, "Comments on "combinatorial rule explosion eliminated by a fuzzy rule configuration"," *IEEE Trans. Fuzzy Syst.*, vol. 7, pp. 369–371, 1999.

[35] B. Moser and M. Navara, "Conditionally firing rules extend the possibilities of fuzzy controllers," in *Proc. CIMCA '99*, Amsterdam, The Netherlands, 1999, pp. 242–245.

[36] B. Moser and M. Navara, "Which triangular norms are convenient for fuzzy controllers?," in *Proc. EUSFLAT-ESTYLF Joint Conf. '99*, Palma, Mallorca, Spain, 1999, pp. 75–78.

[37] B. Moser and M. Navara, "Fuzzy controllers with conditionally firing rules," *IEEE Trans. Fuzzy Syst.*, vol. 10, pp. 340–348, 2002.

[38] J. Pavelka, "On fuzzy logic II," *Zeitschr. f. Math. Logik und Grundlagen d. Math*, vol. 25, pp. 119–134, 1979.

[39] W. Pedrycz, Fuzzy control and fuzzy systems Dept. of Math., Delft Univ. of Technology, 1982, Tech. Rep. 82 – 14.

[40] W. Pedrycz, "On generalised fuzzy relational equations and their applications," *J. Math. Anal. Appl.*, vol. 107, pp. 520–536, 1985.

[41] D. Ruan and E. E. Kerre, "On the extension of the compositional rule of inference," *Int. J. Intell. Syst.*, vol. 8, pp. 807–817, 1993.

[42] D. Ruiz and J. Torrens, "Distributive idempotent uninorms," *Int. J. Uncertain., Fuzz. Knowl.-Based Syst.*, vol. 11, no. 4, pp. 413–428, 2003.

[43] D. Ruiz and J. Torrens, "Distributive strong implications from uninorms," in *Proc. AGOP-2005*, 2005, pp. 103–108.

[44] B. Schweizer and A. Sklar, *Probabilistic Metric Spaces*. New York: North-Holland, 1983.

[45] B. A. Sokhansanj, G. H. Rodrigue, and J. P. Fitch, "Applying URC fuzzy logic to model complex biological systems in the language of biologists," in *Proc. 2nd Int. Conf. Systems Biology (ICSB 2001)*, Pasadena, CA, Nov. 2001.

[46] V. Torra, "A review of the construction of hierarchical fuzzy systems," *Int. J. Intell. Syst.*, vol. 17, no. 5, pp. 531–543, 2002.

[47] E. Trillas, "Sobre funciones de negación en la teoría de conjuntos difusos," *Stochastica*, vol. 3, pp. 47–84, 1979.

[48] E. Trillas and C. Alsina, "On the law $[p \wedge q \longrightarrow r] \equiv [(p \longrightarrow q) \vee (p \longrightarrow r)]$ in fuzzy logic," *IEEE Trans. Fuzzy Syst.*, vol. 10, pp. 84–88, 2002.

[49] E. Trillas, C. del Campo, and S. Cubillo, "When QM-operators are implication functions and conditional fuzzy relations," *Int. J. Intell. Syst.*, vol. 15, pp. 647–655, 2000.

[50] E. Trillas and L. Valverde, "On implication and indistinguishability in the setting of fuzzy logic," in *Management Decision Support Systems Using Fuzzy Sets and Possibility Theory*. Rhineland, Germany: Verlag TÜV, 1985, pp. 198–212.

[51] I. Türksen, V. Kreinovich, and R. Yager, "A new class of fuzzy implications. Axioms of fuzzy implication revisited," *Fuzzy Sets Syst.*, vol. 100, pp. 267–272, 1998.

[52] W. Wangming, "Fuzzy reasoning and fuzzy relational equations," *Fuzzy Sets Syst.*, vol. 20, no. 1, pp. 1–129, 1986.

[53] J. Weinschenk, W. Combs, and R. J. Marks, II, "Avoidance of rule explosion by mapping fuzzy sytems to a union rule configuration," in *Proc. FUZZ-IEEE'03*, St. Louis, MO, May 2003, pp. 43–48.

[54] R. Yager, "On some new classes of implication operators and their role in approximate reasoning," *Inf. Sci.*, vol. 163, pp. 193–216, 2004.

[55] R. Yager and A. Rybalov, "Uninorm aggregation operators," *Fuzzy Sets Syst.*, vol. 80, pp. 111–120, 1996.

[56] M. Ying, "Reasonableness of the compositional rule of fuzzy inference," *Fuzzy Sets Syst.*, vol. 36, no. 2, pp. 305–310, 1990.

[57] L. A. Zadeh, "Outline of a new approach to the analysis of complex systems and decision processes," *IEEE Trans. Syst., Man, Cybern.*, vol. 3, pp. 28–44, 1973.

Balasubramaniam Jayaram (S'02–M'04) received the M.Sc. and Ph.D. degrees in mathematics from Sri Sathya Sai Institute of Higher Learning, India, in 1999 and 2004, respectively.

He is currently a Lecturer in the Department of Mathematics and Computer Sciences, Sri Sathya Sai Institute of Higher Learning. His research interests are in fuzzy aggregation operators, chiefly fuzzy implication operators, and approximate reasoning. He has published more than 15 papers in refereed international journals and conferences. He is a member of many scientific societies and a regular reviewer of many reputed international journals and conferences.

Authorized licensed use limited to: IEEE Xplore. Downloaded on January 14, 2009 at 23:40 from IEEE Xplore. Restrictions apply.