Bandler-Kohout Subproduct with Yager's classes of Fuzzy Implications

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***Abstract*—The Bandler–Kohout Subproduct (BKS) inference mechanism is one of the two established fuzzy relational inference (FRI) mechanisms, the other being Zadeh's Compositional Rule of Inference (CRI). Both these FRIs are known to possess many desirable properties. It can be seen that many of these desirable properties are due to the rich underlying structure, viz., the residuated algebra, from which the employed operations come from. In this work we discuss the BKS relational inference system with the fuzzy implication interpreted as the Yager's classes of implications, which do not form a residuated structure on $[0, 1]$. We show that many of the desirable properties, viz., interpolativity, continuity, robustness that are known for BKS with residuated implications are also available under this framework, thus expanding the choice of operations available to practitioners. Note that, to the best of the authors' knowledge, this is the first attempt at studying the suitability of an FRI where the operations come from a non-residuated structure.**

Keywords*—Bandler-Kohout Subproduct, Fuzzy implications, f-implications, g-implications, Relational Inference, Interpolativity, Continuity and Robustness of inference.

I. Introduction

An Inference mechanism in Approximate Reasoning (AR) can be seen as a function which derives a meaningful output from imprecise inputs. Fuzzy sets have been widely used for this type of purpose. Many kinds of inference mechanism techniques using fuzzy set theory and their logical connectives have been studied in the literature [8], [34], [3], [17]. Among the inference mechanisms proposed in AR using fuzzy logic, (i) Fuzzy Relational Inferences (FRI), (ii) Similarity Based Reasoning (SBR) [5], [6], [17], [20], [31] are very common in the literature. Two of the well-known Fuzzy Relational Inference mechanisms are the Compositional Rule of Inference (CRI) proposed by Zadeh [34], [18], and the Bandler-Kohout Subproduct (BKS) proposed by Pedrycz [25] based on the earlier works of Bandler and Kohout [2], [3], [4]. In this work, our main focus is on FRIs.

A. Measures of 'goodness' of a Fuzzy Inference Mechanism

While dealing with fuzzy inference mechanisms (FIM), the operators employed in it can be picked from a plethora of choices. However, the question that arises is whether an FIM with a particular choice of operators is good. Once again, the 'goodness' of an FIM itself can be measured against different parameters. In the literature, some measures of goodness proposed against which an FIM is compared and contrasted are as follows: (i) interpolativity [22], [30], [7], [13], (ii) continuity [27], [29], [23], (iii) robustness [14], [32] (iv) approximation capability [16] and (v) efficiency, see [11], [10].

Perfilieva and Lehmke [27] studied the continuity and interpolativity of CRI with multiple SISO rules and showed that a fuzzy relation R is a correct model of the given rulebase if and only if it is also a continuous model and thus have shown the equivalence between continuity and interpolativity. The robustness of CRI was dealt with by Klawonn and Castro [14]. Later on Štěpnička and Jayaram [32] have undertaken a similar study for the BKS inference mechanism. In [10], [11], Jayaram has investigated the efficiency of these inference mechanisms.

B. Motivation for the Work

Note that in all the above works the underlying fuzzy logic operations on $[0, 1]$ were obtained from a left-continuous conjunction and hence turned the unit interval into a rich residuated lattice structure possessing very many properties that were employed extensively in the proofs of the results. However, there has been no investigation done so far on the properties retained by the BKS inference when a non-residuated implication is employed.

In this work, we study the above properties of the BKS inference mechanism when the Yager's classes of implication operators [33], i.e, f- and g-implications - which do not give rise to a residuated structure on $[0, 1]$ - are employed. We call these modified BKS-inference as BKS-f and BKS-g inference mechanisms. Note that f- and g-implications satisfy the law of importation (see Definition 2.4) w.r.to the product t-norm and hence an equivalent hierarchical model of inference can be given making it computationally efficient (cf. [10]). However, the other desirable properties have not been rigorously studied - which forms the main motivation for this work.

C. Main contributions of the work

Firstly, we derive some necessary and sufficient conditions for interpolativity for the BKS-f inference mechanism. After defining continuity suitably, we have shown that continuity is equivalent to interpolativity for the BKS-f inference mechanism. Finally, we show that robustness is also available to us under the BKS-f inference mechanism, thus adding more choice of operations under the BKS scheme. A similar study investigating these properties for the BKS-g inference mechanism is also done.

D. Outline Of the Work

Firstly, we present some preliminaries in Section II. Following this, in Section III, we define an extension of the well-

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known Goguen implication and discuss some of its properties, which will prove useful in the sequel. Section IV establishes the context of this work, wherein we present the modified Bandler-Kohout Subproduct (BKS) inference mechanism with Yager's families of fuzzy implications, which we call the BKS-f and BKS-g inference mechanism. Section V gives a brief overview of some relevant previous works which deal with the general solvability of the BKS fuzzy system with f-implications. Section VI contains the core of the work, where we present the results on BKS with f-implications which demonstrate their suitability as an inference mechanism. In Section VII, an anolgous study is done for BKS with g-implications.

II. Preliminaries

To make this paper self-contained, we present definitions of the fuzzy logic operations that are important in the sequel.

A. T-norms, Implications and R-implications

Definition 2.1 ([15], Definition 1.1): A function $T : [0,1]^2 \to [0,1]$ is called a *t-norm* , if it is increasing in both variables, commutative, associative and has 1 as the neutral element.

Definition 2.2 ([1], Definition 1.1.1): A function $I : [0,1]^2 \to [0,1]$ is called a *fuzzy implication* if it satisfies, for all $x, x_1, x_2, y, y_1, y_2 \in [0,1]$, the following conditions:

$$\begin{aligned} &\text{if } x_1 \le x_2, \text{ then } I(x_1,y) \ge I(x_2,y) \ , \\ &\qquad \text{i.e., } I(\,\cdot\,,y) \text{ is decreasing} \ , \\ &\text{if } y_1 \le y_2, \text{ then } I(x,y_1) \le I(x,y_2) \ , \\ &\qquad \text{i.e., } I(x,\,\cdot\,) \text{ is increasing} \ , \\ &I(0,0) = 1 \ , I(1,1) = 1 \ , I(1,0) = 0 \ . \end{aligned}$$

Definition 2.3: [1] A function $I\colon [0,1]^2 \to [0,1]$ is called an *R-implication* if there exists a t-norm T such that

$$I(x,y) = \sup\{t \in [0,1] \mid T(x,t) \le y\} \ , \qquad x,y \in [0,1] \ .$$

Definition 2.4: A fuzzy implication I is said to satisfy

(i) the *left neutrality property*, if
$$I(1,y) = y \ , \qquad y \in [0,1] \ ; \qquad \text{(NP)}$$

(ii) the *T-Law Of Importation*, if for a t-norm T the following holds:
$$I(x, I(y,z)) = I(T(x,y),z) = I(T(y,x),z), \quad x,y,z \in [0,1] \ . \qquad \text{(LI)}$$

B. Yager's Families of Fuzzy Implications

Yager [33] introduced two families of fuzzy implications based on strictly monotonic functions on $[0,1]$.

Definition 2.5 ([1], Definition 3.1.1): Let $f\colon [0,1] \to [0,\infty]$ be a strictly decreasing and continuous function with $f(1) = 0$. The function $I_f\colon [0,1]^2 \to [0,1]$ defined by

$$I_f(x,y) = f^{-1}\left(x \cdot f(y)\right) \ , \qquad x,y \in [0,1] \ , \qquad (1)$$

with the understanding $0 \cdot \infty = 0$, is a fuzzy implication and called an *f-implication*. The function f itself is called an f-generator of the I_f generated as in (1).

We will often write $\longrightarrow_f$ instead of I_f. It is worth mentioning that f is an additive generator of some continuous Archimedean t-norm.

Example 2.6: Table I (cf [1]) lists few of the f-implications along with their generators from which they have been obtained.

Definition 2.7 ([1], Definition 3.2.1): Let $g\colon [0,1] \to [0,\infty]$ be a strictly increasing and continuous function with $g(0) = 0$. The function $I_g\colon [0,1]^2 \to [0,1]$ defined by

$$I_g(x,y) = g^{(-1)}\left(\frac{1}{x} \cdot g(y)\right) , \qquad x,y \in [0,1], \qquad (2)$$

with the understanding $\frac{1}{0} = \infty$ and $\infty \cdot 0 = \infty$, is a fuzzy implication and called a *g-implication*, where the function $g^{(-1)}$ in (2) is the pseudo-inverse of g [1] given by,

$$g^{(-1)}(x) = \begin{cases} g^{-1}(x), & \text{if } x \in [0, g(1)] \ , \\ 1, & \text{if } x \in [g(1), \infty] \ . \end{cases}$$

The function g itself is called a g-generator of the I generated as in (2). We will write $\longrightarrow_g$ instead of I_g.

Example 2.8: Table II (cf [1]) lists few of the g-implications along with their generators from which they have been obtained.

Note that both I_f and I_g satisfy (NP) and also the T-law of importation (LI) with the product t-norm, $T_{\mathbf{P}}(x,y) = xy$.

Proposition 2.9: Let $\mathcal{I}$ be any finite index set and $\longrightarrow$ denote any fuzzy implication. Then

$$x \longrightarrow \bigwedge_{i\in\mathcal{I}}(y_i) = \bigwedge_{i\in\mathcal{I}}(x \longrightarrow y_i) \ , \qquad (3)$$

$$x \longrightarrow \bigvee_{i\in\mathcal{I}}(y_i) \ge \bigvee_{i\in\mathcal{I}}(x \longrightarrow y_i) \ , \qquad (4)$$

$$\bigvee_{i\in\mathcal{I}}(x_i) \longrightarrow y = \bigwedge_{i\in\mathcal{I}}(x_i \longrightarrow y) \ , \qquad (5)$$

$$\bigwedge_{i\in\mathcal{I}}(x_i) \longrightarrow y \ge \bigvee_{i\in\mathcal{I}}(x_i \longrightarrow y) \ . \qquad (6)$$

Proof: Proof follows from the following facts and noting that $\mathcal{I}$ is a finite index set:

- For any non-decreasing function h, $h(\min(x,y)) = \min(h(x),h(y))$ and $h(\max(x,y)) = \max(h(x),h(y))$,
- For any non-increasing function h^*, $h^*(\min(x,y)) = \max(h^*(x),h^*(y))$ and $h^*(\max(x,y)) = \min(h^*(x),h^*(y))$, and
- Any fuzzy implication $\longrightarrow$ is non-increasing in the first variable and non-decreasing in the second variable.

■

Proposition 2.10: The equation (5) is valid for any arbitrary index set $\mathcal{I}$ when $\longrightarrow$ is any f-implication.

TABLE I. EXAMPLES OF f-IMPLICATIONS

f-generator f	f-implication I_f
$f(x) = -\ln x$	$I_{\mathbf{YG}} = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0, \\ y^x, & \text{if } x > 0 \text{ and } y > 0, \end{cases} \quad x, y \in [0,1].$
$f(x) = 1 - x$	$I_{\mathbf{RC}} = 1 - x + xy, \quad x, y \in [0,1]$.
$f_{\mathbf{c}}(x) = \cos(\frac{\pi}{2}x)$	$I_{f_{\mathbf{c}}}(x,y) = \frac{2}{\pi}\cos^{-1}\left(x \cdot \cos\left(\frac{\pi}{2}y\right)\right), \quad x, y \in [0,1]$.
$f^s(x) = -\ln\left(\frac{s^x - 1}{s-1}\right), \quad s > 0, s \neq 1$.	$I_{f^s}(x,y) = \log_s\left(1 + (s-1)^{1-x}(s^y - 1)^x\right), \quad x, y \in [0,1]$.
$f^{\lambda}(x) = (1-x)^{\lambda}$, where $\lambda \in (0, \infty)$	$I_{f^{\lambda}}(x,y) = 1 - x^{\frac{1}{\lambda}}(1-y), \quad x, y \in [0,1]$.

TABLE II. EXAMPLES OF g-IMPLICATIONS

g-generator g	g-implication I_g
$g(x) = -\ln(1-x)$	$I(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0, \\ 1 - (1-y)^{\frac{1}{x}}, & \text{otherwise}, \end{cases} \quad x, y \in [0,1].$
$g(x) = x$	$I_{\mathbf{G}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ \frac{y}{x}, & \text{if } x > y \end{cases}, \quad x, y \in [0,1]$.
$g(x) = -\frac{1}{\ln x}$	$I_{\mathbf{YG}} = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0, \\ y^x, & \text{if } x > 0 \text{ and } y > 0, \end{cases} \quad x, y \in [0,1].$
$g_{\mathbf{t}}(x) = \tan\left(\frac{\pi}{2}x\right)$	$I_{g_{\mathbf{t}}}(x,y) = \frac{2}{\pi}\tan^{-1}\left(\frac{1}{x} \cdot \tan\left(\frac{\pi}{2}y\right)\right), x, y \in [0,1]$.
$g^s(x) = -\ln\left(\frac{s^{1-x} - 1}{s-1}\right), s > 0, s \neq 1$	$I_{g^s}(x,y) = 1 - \log_s\left(1 + (s-1)^{\frac{x-1}{x}}(s^{1-y} - 1)^{\frac{1}{x}}\right), x, y \in [0,1]$.

Proof:

$$\begin{aligned} \text{L. H. S.} &= \bigvee_{i \in \mathcal{I}}(x_i) \longrightarrow_f y = f^{-1}\left(\bigvee_{i \in \mathcal{I}}(x_i) \cdot f(y)\right) \\ &= f^{-1}\left(\bigvee_{i \in \mathcal{I}}(x_i \cdot f(y))\right) = \bigwedge_{i \in \mathcal{I}} f^{-1}\left((x_i \cdot f(y))\right) \\ &= \bigwedge_{i \in \mathcal{I}}(x_i \longrightarrow_f y) = \text{R. H. S.} \end{aligned}$$

■

III. GOGUEN IMPLICATION AND AN EXTENSION

In this section we recall the definition of the Goguen implication and its bi-implication and also present some of the properties it enjoys being a residual implication. Following this, we propose an extension of the Goguen implication and discuss some of its properties, which play an important role in the sequel in giving crisp expressions to many results and properties.

A. Goguen Implication

Definition 3.1: (i) The Goguen implication, the residual of the product t-norm, $I_{\mathbf{G}} : [0,1]^2 \to [0,1]$ is defined as

$$I_{\mathbf{G}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ \frac{y}{x}, & \text{if } x > y \end{cases}, \qquad x, y \in [0,1] .$$

We denote $I_{\mathbf{G}}$ by $\longrightarrow_{\mathbf{G}}$ for simplicity.

(ii) The bi-implication ([24], Equation 2.24, pg. 27) obtained from $I_{\mathbf{G}}$ is defined and denoted as follows:

$$\begin{aligned} x \longleftrightarrow_{\mathbf{G}} y &= (x \longrightarrow_{\mathbf{G}} y) \wedge (y \longrightarrow_{\mathbf{G}} x) \\ &= \min\left\{1, \frac{x}{y}, \frac{y}{x}\right\}, \quad x, y \in [0,1] . \end{aligned}$$

with the understanding that $\frac{1}{0} = \infty$, $0 \cdot \infty = \infty$ and $\frac{0}{0} = \infty$.

Here we present some important properties possessed by Goguen implication, which will be needed later for proving our results.

Proposition 3.2 ([24], Lemma 2.7): For $a, b, c \in [0,1]$ and " $\longleftrightarrow_{\mathbf{G}}$ " being the *Goguen bi-implication* and '$\cdot$' being the product t-norm, we have,

$$(a \longleftrightarrow_{\mathbf{G}} b) \cdot (b \longleftrightarrow_{\mathbf{G}} c) \leq (a \longleftrightarrow_{\mathbf{G}} c) . \tag{7}$$

Proposition 3.3 ([24], Lemma 2.7): Let $a_i, b_i \in [0,1]$ and $i \in \mathcal{I}$, an index set. Then the following inequalities are true for any $\longleftrightarrow$ coming from a residuated lattice structure,

$$\left(\bigvee_{i \in \mathcal{I}} a_i\right) \longleftrightarrow \left(\bigvee_{i \in \mathcal{I}} b_i\right) \geq \bigwedge_{i \in \mathcal{I}} (a_i \longleftrightarrow b_i) , \tag{8}$$

$$\left(\bigwedge_{i \in \mathcal{I}} a_i\right) \longleftrightarrow \left(\bigwedge_{i \in \mathcal{I}} b_i\right) \geq \bigwedge_{i \in \mathcal{I}} (a_i \longleftrightarrow b_i) . \tag{9}$$

B. Extended Goguen Implication

In this section we modify the Goguen implication, by extending it as a map from $[0,1]^2 \to [0,1]$ to a map on $[0,\infty]^2 \to [0,1]$, leaving the formula unchanged and call it the *Extended Goguen implication*. In the sequel, this function plays an important role in giving crisp expressions to many results and properties and hence we define it here and present some of its important properties.

Definition 3.4: (i) The function $I^*_{\mathbf{G}} : [0,\infty]^2 \to [0,1]$ defined as

$$I^*_{\mathbf{G}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ \frac{y}{x}, & \text{if } x > y \end{cases}, \qquad x, y \in [0,\infty] ,$$

is called the *Extended Goguen* implication. We will also denote $I^*_{\mathbf{G}}$ by $\xrightarrow{*}_{\mathbf{G}}$ for better readability in proofs.

(ii) The bi-implication [24] obtained from $I^*_{\mathbf{G}}$ is defined and denoted as follows:

$$\begin{aligned} x \xleftrightarrow{*}_{\mathbf{G}} y &= (x \xrightarrow{*}_{\mathbf{G}} y) \wedge (y \xrightarrow{*}_{\mathbf{G}} x) \\ &= \min\left\{1, \frac{x}{y}, \frac{y}{x}\right\} \ , \ x, y \in [0, \infty] \ . \end{aligned}$$

with the understanding that $\frac{\alpha}{0} = \infty, \alpha \in [0, \infty]$ and $0 \cdot \infty = \infty$.

C. Some Properties of the Extended Goguen Implication

In this section we present properties of $I^*_{\mathbf{G}}$ which will be needed later for proving our results. The following result shows that (7) is valid even in the the case of *Extended Goguen* implication.

Proposition 3.5: For $a, b, c \in [0, \infty]$ and " $\xleftrightarrow{*}_{\mathbf{G}}$ " being the *Extended Goguen bi-implication* and '$\cdot$' being the product t-norm, we have,

$$\left(a \xleftrightarrow{*}_{\mathbf{G}} b\right) \cdot \left(b \xleftrightarrow{*}_{\mathbf{G}} c\right) \leq \left(a \xleftrightarrow{*}_{\mathbf{G}} c\right) .$$

Proof: We have to prove that for any $a, b, c \in [0, \infty]$

$$\min\left\{1, \frac{a}{b}, \frac{b}{a}\right\} \cdot \min\left\{1, \frac{b}{c}, \frac{c}{b}\right\} \leq \min\left\{1, \frac{a}{c}, \frac{c}{a}\right\} .$$

Let us denote by $\alpha = \min\left\{1, \frac{a}{b}, \frac{b}{a}\right\}, \beta = \min\left\{1, \frac{b}{c}, \frac{c}{b}\right\}$ and $\gamma = \min\left\{1, \frac{a}{c}, \frac{c}{a}\right\}$. Now we have to prove that $\alpha \cdot \beta \leq \gamma$. Note that both α and β contain three terms each. So there will be nine possible values of $\alpha \cdot \beta$. Here we discuss all the cases:

Case-1 $(\alpha = 1, \beta = 1)$:
Now we have,

$$\begin{aligned} \alpha = 1 &\iff a = b, \\ \beta = 1 &\iff b = c \\ &\implies a = b = c. \end{aligned}$$

This implies, $\alpha \cdot \beta = 1 = \min\left\{1, \frac{a}{c}, \frac{c}{a}\right\} = \gamma$.

Case-2 $(\alpha = 1, \beta = \frac{b}{c})$:
Now we have,

$$\begin{aligned} \alpha = 1 &\iff a = b, \\ \beta = \frac{b}{c} &\iff b \leq c \\ &\implies a = b \leq c. \end{aligned}$$

This implies,

$$\begin{aligned} \gamma &= \min\left\{1, \frac{a}{c}, \frac{c}{a}\right\} \\ &= \min\left\{1, \frac{b}{c}, \frac{c}{b}\right\} = \frac{b}{c} = \alpha \cdot \beta. \end{aligned}$$

Case-3 $(\alpha = 1, \beta = \frac{c}{b})$: Same as Case-2.
Case-4 $(\alpha = \frac{a}{b}, \beta = 1)$: Same as Case-2.

Case-5 $(\alpha = \frac{a}{b}, \beta = \frac{b}{c})$:
Now we have,

$$\begin{aligned} \alpha = \frac{a}{b} &\iff a \leq b, \\ \beta = \frac{b}{c} &\iff b \leq c \\ &\implies a \leq b \leq c. \end{aligned}$$

This implies,

$$\gamma = \min\left\{1, \frac{a}{c}, \frac{c}{a}\right\} = \frac{a}{c} = \frac{a}{b} \cdot \frac{b}{c} = \alpha \cdot \beta.$$

Case-6 $(\alpha = \frac{a}{b}, \beta = \frac{c}{b})$:
Now we have,

$$\begin{aligned} &\alpha = \frac{a}{b} \iff a \leq b, \\ &\beta = \frac{c}{b} \iff c \leq b \\ \implies &\alpha \cdot \beta = \frac{a}{b} \cdot \frac{c}{b} \leq \frac{b}{a} \cdot \frac{c}{b} = \frac{c}{a}, \\ &\alpha \cdot \beta = \frac{a}{b} \cdot \frac{c}{b} \leq \frac{a}{b} \cdot \frac{b}{c} = \frac{a}{c} \ \text{ and} \\ &\alpha \cdot \beta \leq 1. \end{aligned}$$

This implies, $\alpha \cdot \beta \leq \min\left\{1, \frac{a}{c}, \frac{c}{a}\right\} = \gamma$.

Case-7 $(\alpha = \frac{b}{a}, \beta = 1)$: Same as Case-4.
Case-8 $(\alpha = \frac{b}{a}, \beta = \frac{b}{c})$: Same as Case-6.
Case-9 $(\alpha = \frac{b}{a}, \beta = \frac{c}{b})$: Same as Case-5.

Combining all the above nine cases we have $\alpha \cdot \beta \leq \gamma$. Hence the proposition. ■

Proposition 3.6: Let $a, b, c, d \in [0, \infty]$ and '$\cdot$' be the product t-norm. Then, the following inequality is true:

$$\left(a \xleftrightarrow{*}_{\mathbf{G}} b\right) \cdot \left(c \xleftrightarrow{*}_{\mathbf{G}} d\right) \leq (a \cdot c) \xleftrightarrow{*}_{\mathbf{G}} (b \cdot d) \ . \quad (10)$$

Proof: The proof is similar to the proof of Proposition 3.5 ■

Proposition 3.7: Let $a_i, b_i \in [0, \infty]$ and $i \in \mathcal{I}$, a finite index set. Then the inequality (8) is true for the extended Goguen bi-implication $\xleftrightarrow{*}_{\mathbf{G}}$.

Proof: We prove this result for $\mathcal{I} = \{1, 2\}$. The result then follows by induction. Now we have to prove the following:

$$\left(\bigvee_{i \in \{1,2\}} a_i\right) \xleftrightarrow{*}_{\mathbf{G}} \left(\bigvee_{i \in \{1,2\}} b_i\right) \geq \bigwedge_{i \in \{1,2\}} \left(a_i \xleftrightarrow{*}_{\mathbf{G}} b_i\right) \ ,$$

this is same as,

$$(a_1 \vee a_2) \xleftrightarrow{*}_{\mathbf{G}} (b_1 \vee b_2) \geq \left(a_1 \xleftrightarrow{*}_{\mathbf{G}} b_1\right) \wedge \left(a_2 \xleftrightarrow{*}_{\mathbf{G}} b_2\right) \ .$$

Now, by using the monotonicity of $\xrightarrow{*}_{\mathbf{G}}$ and its distributivity over $\vee, \wedge$, we have

$$
\begin{aligned}
\text{L.H.S} &= (a_1 \vee a_2) \xleftrightarrow{*}_{\mathbf{G}} (b_1 \vee b_2) \\
&= \Big[(a_1 \vee a_2) \xrightarrow{*}_{\mathbf{G}} (b_1 \vee b_2)\Big] \\
&\qquad\qquad \wedge \Big[(b_1 \vee b_2) \xrightarrow{*}_{\mathbf{G}} (a_1 \vee a_2)\Big] \\
&= \Big[a_1 \xrightarrow{*}_{\mathbf{G}} (b_1 \vee b_2)\Big] \wedge \Big[a_2 \xrightarrow{*}_{\mathbf{G}} (b_1 \vee b_2)\Big] \\
&\quad \wedge \Big[b_1 \xrightarrow{*}_{\mathbf{G}} (a_1 \vee a_2)\Big] \wedge \Big[b_2 \xrightarrow{*}_{\mathbf{G}} (a_1 \vee a_2)\Big] \\
&\geq \Big[(a_1 \xrightarrow{*}_{\mathbf{G}} b_1) \vee (a_1 \xrightarrow{*}_{\mathbf{G}} b_2)\Big] \\
&\quad \wedge \Big[(a_2 \xrightarrow{*}_{\mathbf{G}} b_1) \vee (a_2 \xrightarrow{*}_{\mathbf{G}} b_2)\Big] \\
&\quad\quad \wedge \Big[(b_1 \xrightarrow{*}_{\mathbf{G}} a_1) \vee (b_1 \xrightarrow{*}_{\mathbf{G}} a_2)\Big] \\
&\quad\quad\quad \wedge \Big[(b_2 \xrightarrow{*}_{\mathbf{G}} a_1) \vee (b_2 \xrightarrow{*}_{\mathbf{G}} a_2)\Big] \\
&\geq \Big\{ \Big[(a_1 \xrightarrow{*}_{\mathbf{G}} b_1) \wedge (b_1 \xrightarrow{*}_{\mathbf{G}} a_1)\Big] \\
&\quad \vee \Big[(a_1 \xrightarrow{*}_{\mathbf{G}} b_2) \wedge (b_1 \xrightarrow{*}_{\mathbf{G}} a_2)\Big] \Big\} \\
&\quad\quad \wedge \Big\{ \Big[(a_2 \xrightarrow{*}_{\mathbf{G}} b_1) \wedge (b_2 \xrightarrow{*}_{\mathbf{G}} a_1)\Big] \\
&\quad\quad\quad \vee \Big[(a_2 \xrightarrow{*}_{\mathbf{G}} b_2) \wedge (b_2 \xrightarrow{*}_{\mathbf{G}} a_2)\Big] \Big\} \\
&\quad\quad \Big\lceil \because \text{ we know the fact} \\
&\qquad\qquad (x \vee y) \wedge (w \vee z) \geq (x \wedge w) \vee (y \wedge z) \Big\rfloor \\
&\geq \Big\{ \Big[(a_1 \xrightarrow{*}_{\mathbf{G}} b_1) \wedge (b_1 \xrightarrow{*}_{\mathbf{G}} a_1)\Big] \\
&\quad \wedge \Big[(a_2 \xrightarrow{*}_{\mathbf{G}} b_2) \wedge (b_2 \xrightarrow{*}_{\mathbf{G}} a_2)\Big] \Big\} \\
&\quad\quad \vee \Big\{ \Big[(a_1 \xrightarrow{*}_{\mathbf{G}} b_2) \wedge (b_1 \xrightarrow{*}_{\mathbf{G}} a_2)\Big] \\
&\quad\quad\quad \wedge \Big[(a_2 \xrightarrow{*}_{\mathbf{G}} b_1) \wedge (b_2 \xrightarrow{*}_{\mathbf{G}} a_1)\Big] \Big\} \\
&\quad\quad \Big\lceil \because \text{ we know the fact} \\
&\qquad\qquad (x \vee y) \wedge (w \vee z) \geq (x \wedge w) \vee (y \wedge z) \Big\rfloor \\
&\geq \Big\{ \Big[(a_1 \xrightarrow{*}_{\mathbf{G}} b_1) \wedge (b_1 \xrightarrow{*}_{\mathbf{G}} a_1)\Big] \\
&\quad\quad\quad \wedge \Big[(a_2 \xrightarrow{*}_{\mathbf{G}} b_2) \wedge (b_2 \xrightarrow{*}_{\mathbf{G}} a_2)\Big] \Big\} \\
&= \Big(a_1 \xleftrightarrow{*}_{\mathbf{G}} b_1\Big) \wedge \Big(a_2 \xleftrightarrow{*}_{\mathbf{G}} b_2\Big) \\
&= \text{R.H.S.}
\end{aligned}
$$

Hence proved. ∎

Proposition 3.8: Let $a_i, b_i \in [0, \infty]$ and $i \in \mathcal{I}$, a finite index set. Then the inequality (9) is valid for the extended Goguen bi-implication $\xleftrightarrow{*}_{\mathbf{G}}$.

Proof: We prove this result for $\mathcal{I} = \{1, 2\}$. The result then follows by induction. Now we have to prove the following:

$$
\left(\bigwedge_{i\in\{1,2\}} a_i\right) \xleftrightarrow{*}_{\mathbf{G}} \left(\bigwedge_{i\in\{1,2\}} b_i\right) \geq \bigwedge_{i\in\{1,2\}} \Big(a_i \xleftrightarrow{*}_{\mathbf{G}} b_i\Big) ,
$$

this is same as,

$$
(a_1 \wedge a_2) \xleftrightarrow{*}_{\mathbf{G}} (b_1 \wedge b_2) \geq (a_1 \xleftrightarrow{*}_{\mathbf{G}} b_1) \wedge (a_2 \xleftrightarrow{*}_{\mathbf{G}} b_2).
$$

Now, by using the monotonicity of $\xrightarrow{*}_{\mathbf{G}}$ and its distributivity over $\vee, \wedge$, we have

$$
\begin{aligned}
L.H.S &= (a_1 \wedge a_2) \xleftrightarrow{*}_{\mathbf{G}} (b_1 \wedge b_2) \\
&= \Big[(a_1 \wedge a_2) \xrightarrow{*}_{\mathbf{G}} (b_1 \wedge b_2)\Big] \\
&\qquad \wedge \Big[(b_1 \wedge b_2) \xrightarrow{*}_{\mathbf{G}} (a_1 \wedge a_2)\Big] \\
&= \Big\{\Big[(a_1 \wedge a_2) \xrightarrow{*}_{\mathbf{G}} b_1\Big] \wedge \Big[(a_1 \wedge a_2) \xrightarrow{*}_{\mathbf{G}} b_2\Big]\Big\} \\
&\quad \wedge \Big\{\Big[(b_1 \wedge b_2) \xrightarrow{*}_{\mathbf{G}} a_1\Big] \wedge \Big[(b_1 \wedge b_2) \xrightarrow{*}_{\mathbf{G}} a_2\Big]\Big\} \\
&\geq (a_1 \xrightarrow{*}_{\mathbf{G}} b_1) \wedge (a_2 \xrightarrow{*}_{\mathbf{G}} b_2) \\
&\qquad \wedge (b_1 \xrightarrow{*}_{\mathbf{G}} a_1) \wedge (b_2 \xrightarrow{*}_{\mathbf{G}} a_2) \\
&= \Big[(a_1 \xrightarrow{*}_{\mathbf{G}} b_1) \wedge (b_1 \xrightarrow{*}_{\mathbf{G}} a_1)\Big] \\
&\qquad \wedge \Big[(a_2 \xrightarrow{*}_{\mathbf{G}} b_2) \wedge (b_2 \xrightarrow{*}_{\mathbf{G}} a_2)\Big] \\
&= (a_1 \xleftrightarrow{*}_{\mathbf{G}} b_1) \wedge (a_2 \xleftrightarrow{*}_{\mathbf{G}} b_2).
\end{aligned}
$$

Hence proved. ∎

IV. Fuzzy Inference Mechanism

Let $X \subseteq \mathbb{R}$ be a non-empty classical set. Let $\mathcal{F}(X)$ denote the set of all fuzzy sets on X. Given two non-empty classical sets $X, Y \subseteq \mathbb{R}$, a fuzzy Single Input Single Output (SISO) IF-THEN rule is of the form:

$$
\textbf{IF } \tilde{x} \text{ is } A \textbf{ THEN } \tilde{y} \text{ is } B, \tag{11}
$$

where $\tilde{x}$, $\tilde{y}$ are the linguistic variables and $A \in \mathcal{F}(X), B \in \mathcal{F}(Y)$ are the linguistic values taken by the linguistic variables. A knowledge base consists of a collection of such rules. Hence, we consider a rule base of n SISO rules which is of the form:

$$
\textbf{IF } \tilde{x} \text{ is } A_i \textbf{ THEN } \tilde{y} \text{ is } B_i\ , \tag{12}
$$

where $\tilde{x}$, $\tilde{y}$ and $A_i \in \mathcal{F}(X), B_i \in \mathcal{F}(Y), i = 1, 2, \ldots n$ are as mentioned above.

As an example, consider the rule

IF *Temperature* is *High* **THEN** *Fanspeed* is *Medium*.

Here *Temperature* and *Fanspeed* are the linguistic variables and *High*, *Medium* are the linguistic values taken by the linguistic variables in a suitable domain. Now given a single SISO rule (11) or a rule base (12) and given any input " $\tilde{x}$ is A' ", the main objective of an inference mechanism is to find B' such that " $\tilde{y}$ is B' ".

A. Bandler-Kohout Subproduct:

Pedrycz [25] proposed an inference mechanism based on the Bandler-Kohout Subproduct composition. For a given SISO rule base (12), the Bandler-Kohout Subproduct (BKS) inference mechanism is denoted as :

$$
B' = A' \triangleleft R\ ,
$$

where $A' \in \mathcal{F}(X)$ is the input, the fuzzy relation $R : X \times Y \to [0,1]$ i.e, $R \in \mathcal{F}(X \times Y)$ represents the rule base, B' is the obtained output and $\lhd$ is the mapping $\lhd\colon \mathcal{F}(X) \times \mathcal{F}(X \times Y) \to \mathcal{F}(Y)$ given as:

$$B'(y) = \bigwedge_{x \in X} [A'(x) \longrightarrow R(x,y)], \qquad y \in Y, \qquad \text{(BKS-}R\text{)}$$

where $\longrightarrow$ is a fuzzy implication. The operator $\lhd$ is also known as the $\inf -I$ composition where I is a fuzzy implication. Usually the fuzzy relation R deemed to have captured the rule base is taken as one of the following:

$$\check{R}(x,y) = \bigvee_{i=1}^{n} (A_i(x) \star B_i(y)), \qquad x \in X, y \in Y\ , \qquad \text{(Conj-}\check{R}\text{)}$$

$$\hat{R}(x,y) = \bigwedge_{i=1}^{n} (A_i(x) \longrightarrow B_i(y)), \qquad x \in X, y \in Y\ , \qquad \text{(Imp-}\hat{R}\text{)}$$

where $\star$ is taken as a t-norm and $\longrightarrow$ is taken as a fuzzy implication. The (Conj-$\check{R}$) form represents a conjunctive form of the rule base, while (Imp-$\hat{R}$) represents a implicative form of the rule base. Regarding the difference in semantics between $\check{R}$ and $\hat{R}$ please see [9].

In [32] various properties like interpolativity, continuity, robustness of the inference mechanism have been investigated for BKS when the underlying operators come from a residuated lattice.

Remark 4.1: Note that a fuzzy relational inference mechanism can be looked at as a fuzzy function $f_R^{@} : \mathcal{F}(X) \to \mathcal{F}(Y)$ where $R \in \mathcal{F}(X \times Y)$ and $@\colon \mathcal{F}(X) \times \mathcal{F}(X \times Y) \to \mathcal{F}(Y)$ is any composition operator. With this notation the BKS inference mechanism becomes $f_R^{\lhd}$, while the Compositional Rule of Inference (CRI) proposed by Zadeh [34] becomes f_R° where $\circ$ is the $\sup -T$ composition operator.

B. Bandler-Kohout Subproduct with Yager's class of fuzzy implications

In this work, we consider the BKS inference mechanism, where the fuzzy implication is one of the Yager's classes of implications. Essentially, we interpret the $\longrightarrow$ in (BKS-R) as an f- or g-implication and denote the modified BKS inference mechanism as $\lhd_f$ and $\lhd_g$, where $\lhd_f = \inf -I_f$ and $\lhd_g = \inf -I_g$ respectively. Specifically,

$$B'(y) = (A' \lhd_f R)(y) = \bigwedge_{x \in X} [A'(x) \longrightarrow_f R(x,y)],\ y \in Y, \qquad \text{(BKS-}f\text{)}$$

$$B'(y) = (A' \lhd_g R)(y) = \bigwedge_{x \in X} [A'(x) \longrightarrow_g R(x,y)],\ y \in Y\ . \qquad \text{(BKS-}g\text{)}$$

The solvability of (BKS-R) is well-known [25], thus making it a valid choice to be employed in an FRI. This immediately leads to the question of solvability of (BKS-f) and (BKS-g), which we show in the following Section V, thus making both these modified inference mechanisms a potential choice for consideration in FRIs.

From Remark 4.1 we see that the above FRIs, viz., (BKS-f) and (BKS-g), can be denoted as $f_R^{\lhd_f}$ and $f_R^{\lhd_g}$, respectively.

V. BKS WITH f-IMPLICATIONS: ITS SOLVABILITY

In [19], [22] the authors have discussed the general solvability of a fuzzy relational equation considering the operators from a residuated lattice structure (e.g, a t-norm and its residual). Later on, Kawaguchi and Miyakoshi [12] have studied the solvability issues in a more general residuated structure, in which the conjunction is non commutative. We present below the relevant definitions and results that are important in the sequel.

Definition 5.1 ([12], Definition 1): A binary conjunction $C\colon [0,1]^2 \to [0,1]$ is an operation satisfying

(i) $C(0,1) = 0 = C(1,0)$ and $C(1,1) = 1$,
(ii) $a \neq 0 \Longrightarrow C(1,a) \neq 0$ and $C(a,1) \neq 0$,
(iii) C is increasing in both the variables.

We often denote C by $\hat{*}$ and use the in-fix notation to be consistent. Note that a t-norm T is a special class of conjunctions C.

Definition 5.2 ([12], Definition 3): The right residual of a conjunction $\hat{*}$ is defined as, for $a, b \in [0,1]$,

$$a \longrightarrow_{\hat{*}} b \equiv \sup\{s \in [0,1] | a \mathbin{\hat{*}} s \leq b\}\ . \tag{13}$$

When $\hat{*} = T$, a t-norm, by the commutativity of T the operation $\longrightarrow_{\hat{*}}$ becomes the R-implication as defined in Definition 2.3.

Let X, Y and Z be non-empty sets, $\mathcal{F}(X \times Y), \mathcal{F}(Y \times Z)$ and $\mathcal{F}(X \times Z)$ the sets of all fuzzy relations on $X \times Y, Y \times Z$ and $X \times Z$, respectively, and $R \in \mathcal{F}(X \times Y), P \in \mathcal{F}(Y \times Z)$ and $Q \in \mathcal{F}(X \times Z)$.

Definition 5.3 ([12], Definition 4): The $\circ_{\hat{*}}$ - composition of R and P, denoted by $R \circ_{\hat{*}} P$, is a fuzzy relation on $X \times Z$ whose grades of membership are defined by, for any $(x,z) \in X \times Z$,

$$(R \circ_{\hat{*}} P)(x,z) \equiv \bigvee_{y \in Y} \big(R(x,y) \mathbin{\hat{*}} P(y,z)\big)\ .$$

Definition 5.4 ([12], Definition 5): The $\lhd_{\hat{*}}$ - composition of R and P, denoted by $R \lhd_{\hat{*}} P$, is a fuzzy relation on $X \times Z$ whose grades of membership are defined by, for any $(x,z) \in X \times Z$,

$$(R \lhd_{\hat{*}} P)(x,z) \equiv \bigwedge_{y \in Y} \big(R(x,y) \longrightarrow_{\hat{*}} P(y,z)\big)\ .$$

The inverse of R, denoted as R^{-1}, is defined as a fuzzy relation on $Y \times X$ whose grades of membership are given by $R^{-1}(y,x) \equiv R(x,y)$ for all $(x,y) \in X \times Y$.

In the following, we recall some results from [12] which are of relevance and necessary to us.

Theorem 5.5 ([12], Theorem 3(I)): Let $R \in \mathcal{F}(X \times Y)$ and $Q \in \mathcal{F}(X \times Z)$ be given, and let

$$\mathcal{X} = \{P | P \in \mathcal{F}(Y \times Z) \text{ and } R \lhd_{\hat{*}} P = Q\}.$$

Then, $\mathcal{X}$ is non-empty iff $R^{-1} \circ_{\hat{*}} Q \in \mathcal{X}$, and in that case $R^{-1} \circ_{\hat{*}} Q$ is the least element in $\mathcal{X}$.

Now if we consider

$$x \mathbin{\hat{*}} y = x \mathbin{\hat{*}_f} y = f^{(-1)}\left(\frac{1}{x} \cdot f(y)\right) , \qquad x, y \in [0,1] ,$$

where f is as in definition 2.5 with the understanding $0 \cdot \infty = 0$ and $\frac{0}{0} = \infty$, then I_f can be seen as the right residual operator corresponding to $\hat{*}_f$ obtained as in (13).

If we let $\hat{*} = \hat{*}_f$ in (13) we see that $\longrightarrow_{\hat{*}} = I_f$ and the composition $\lhd_{\hat{*}}$ is the (BKS-f) composition. Now, one can immediately observe that Theorem 5.5 deals with the solvability of the (BKS-f) composition.

Further, we limit our study to the implicative form of rules with the implication being an f- or g-implication, i.e., the fuzzy relation R representing the rule base is given as:

$$\hat{R}_f(x,y) = \bigwedge_{i=1}^{n} (A_i(x) \longrightarrow_f B_i(y)), \;\; x \in X, y \in Y \; , \tag{Imp-$\hat{R}_f$}$$

and

$$\hat{R}_g(x,y) = \bigwedge_{i=1}^{n} (A_i(x) \longrightarrow_g B_i(y)), \;\; x \in X, y \in Y \; . \tag{Imp-$\hat{R}_g$}$$

In other words, we consider only the following FRIs: $f_{\hat{R}_f}^{\lhd_f}$ and $f_{\hat{R}_g}^{\lhd_g}$.

VI. BKS with f-Implications: Its Suitability

Firstly, we derive some necessary and sufficient conditions for interpolativity with $R = \hat{R}_f$. After defining continuity suitably, we show that continuity is equivalent to interpolativity. Finally, we show that robustness is also available to us, thus adding more choice of operations under the BKS scheme.

A. Interpolativity of BKS with f-implications

Interpolativity is one of the most fundamental properties of an inference mechanism. A system is said to be *interpolative* if the following is valid: Whenever an antecedent of a rule is given as the input, the corresponding consequent should be the inferred output, i.e.,

$$B_i = A_i \lhd R, \; i = 1, 2 \ldots n., \; A_i \in \mathcal{F}(X), \; R \in \mathcal{F}(X \times Y).$$

Interpolativity pertains to the solvability of the fuzzy relational equations corresponding to the system. See Di Nola [22] for the results pertaining to CRI and Nosková [23], Nosková and Perfilieva [28] for BKS inference. For some recent works on solvability of fuzzy relational equations, especially in the context of fuzzy relational inference, see the works of Perfilieva [26] and an earlier related work of Moser and Navara [21].

Perfilieva and Lehmke [27] studied the continuity and interpolativity of CRI with multiple SISO rules and showed that a fuzzy relation R is a correct model of the given rulebase if and only if it is also a continuous model and thus have shown the equivalence between continuity and interpolativity. Later on Štěpnička and Jayaram [32] have undertaken a similar study for the BKS inference mechanism with R-implications.

The following result gives a necessary and sufficient condition for the interpolativity of the BKS inference mechanism with an f-implication.

Theorem 6.1: Let A_i for $i = 1 \ldots n$ be normal. A necessary and sufficient condition for $\hat{R}_f$ to be a solution to $B_i = A_i \lhd_f R$ is as follows: For any $i, j \in \{1 \ldots n\}$,

$$\bigvee_{x \in X} (A_i(x) \cdot A_j(x)) \leq \bigwedge_{y \in Y} \Big(f\left(B_i(y)\right) \overset{*}{\longleftrightarrow}_{\mathbf{G}} f\left(B_j(y)\right) \Big) \; , \tag{14}$$

where " $\overset{*}{\longleftrightarrow}_{\mathbf{G}}$ " is the extended Goguen bi-implication and f is the generator function of the corresponding f-implication.

Proof: ($\Longrightarrow$):

Let the system have interpolativity. Then we have, for any $y \in Y$, $i \in \mathbb{N}_n$

$$\begin{aligned}
&\left(A_i \lhd_f \hat{R}_f\right)(y) = B_i(y),\\
\Longrightarrow &\bigwedge_{x \in X} \left(A_i(x) \longrightarrow_f \bigwedge_{j=1}^{n} (A_j(x) \longrightarrow_f B_j(y)) \right) = B_i(y),\\
\Longrightarrow &A_i(x) \longrightarrow_f (A_j(x) \longrightarrow_f B_j(y)) \geq B_i(y),\\
&\qquad\qquad\qquad\qquad (\forall j, \forall x),\\
\Longrightarrow &(A_i(x) \cdot A_j(x)) \longrightarrow_f B_j(y) \geq B_i(y),\\
&\qquad\qquad\qquad\qquad (\forall j, \forall x), \; (by \text{ (LI)}),\\
\Longrightarrow &f^{-1}\left(A_i(x) \cdot A_j(x) \cdot f\left(B_j(y)\right)\right) \geq B_i(y),\\
&\qquad\qquad\qquad\qquad (\forall j, \forall x),\\
\Longrightarrow &A_i(x) \cdot A_j(x) \cdot f\left(B_j(y)\right) \leq f\left(B_i(y)\right),\\
&\qquad\qquad\qquad\qquad (\forall j, \forall x),\\
\Longrightarrow &A_i(x) \cdot A_j(x) \leq \frac{f(B_i(y))}{f(B_j(y))},\\
&\qquad\qquad\qquad\qquad (\forall j, \forall x).
\end{aligned}$$

Since i, j are arbitrary, interchanging them in the above inequality, we have,

$$A_j(x) \cdot A_i(x) \leq \frac{f(B_j(y))}{f(B_i(y))} \; .$$

Also trivially we have,

$$A_j(x) \cdot A_i(x) \leq 1 \; .$$

Now from the above inequalities we see,

$$A_i(x)\cdot A_j(x) \le \min\left\{1, \frac{f(B_i(y))}{f(B_j(y))}, \frac{f(B_j(y))}{f(B_i(y))}\right\} \quad (\forall i,j)(\forall x,y),$$

$$\Longrightarrow \bigvee_{x\in X}\left(A_i(x)\cdot A_j(x)\right) \le \bigwedge_{y\in Y}\min\left\{1, \frac{f(B_i(y))}{f(B_j(y))}, \frac{f(B_j(y))}{f(B_i(y))}\right\} \quad (\forall i,j).$$

which is the same as (14).

($\Longleftarrow$): Now let us assume that (14) holds. Firstly, note that the following is always valid:

$$\left(A_i \triangleleft_f \hat{R}_f\right)(y) \le B_i(y)\ , \qquad (\forall i, \forall y). \tag{15}$$

The validity of the above can be seen from the following inequalities:

$$\begin{aligned}
&\left(A_i \triangleleft_f \hat{R}_f\right)(y)\\
&= \bigwedge_{x\in X}\left(A_i(x) \longrightarrow_f \bigwedge_{j=1}^{n}\left(A_j(x)\longrightarrow_f B_j(y)\right)\right)\\
&\le \left(A_i(x_0) \longrightarrow_f \bigwedge_{j=1}^{n}\left(A_j(x_0)\longrightarrow_f B_j(y)\right)\right)\\
&\qquad\qquad\qquad \text{(Assuming } A_i \text{ attains normality at } x_0\text{)}\\
&= \bigwedge_{j=1}^{n}\left(A_j(x_0)\longrightarrow_f B_j(y)\right) \qquad \text{(Using (NP))}\\
&\le A_i(x_0)\longrightarrow_f B_i(y) = B_i(y) \quad \text{(Again Using (NP))}.
\end{aligned}$$

Thus it only remains to show that

$$\left(A_i \triangleleft_f \hat{R}_f\right)(y) \ge B_i(y), \qquad (\forall i, \forall y). \tag{16}$$

We have from (14),

$$\begin{aligned}
&A_i(x)\cdot A_j(x) \le \min\left\{1, \frac{f(B_i(y))}{f(B_j(y))}, \frac{f(B_j(y))}{f(B_i(y))}\right\} \quad (\forall i,j)(\forall x,y),\\
&\Longrightarrow A_i(x)\cdot A_j(x) \le \frac{f(B_i(y))}{f(B_j(y))}, \quad (\forall i,j,\forall x,y),\\
&\Longrightarrow A_i(x)\cdot A_j(x)\cdot f\left(B_j(y)\right) \le f\left(B_i(y)\right), \quad (\forall j, \forall x),\\
&\Longrightarrow f^{-1}\left(A_i(x)\cdot A_j(x)\cdot f\left(B_j(y)\right)\right) \ge B_i(y), \quad (\forall i,j,\forall x,y),\\
&\Longrightarrow \left(A_i(x)\cdot A_j(x)\right)\longrightarrow_f B_j(y) \ge B_i(y), \quad (\forall i,j,\forall x,y),\\
&\Longrightarrow A_i(x)\longrightarrow_f \left(A_j(x)\longrightarrow_f B_j(y)\right) \ge B_i(y), \quad (\forall i,j,\forall x,y),\ (\textit{by}\ \text{(LI)}),\\
&\Longrightarrow \bigwedge_{j=1}^{n}\left(A_i(x)\longrightarrow_f \left(A_j(x)\longrightarrow_f B_j(y)\right)\right) \ge B_i(y), \quad (\forall i,\forall x,y)\\
&\Longrightarrow \left(A_i(x)\longrightarrow_f \bigwedge_{j=1}^{n} big(A_j(x)\longrightarrow_f B_j(y))\right) \ge B_i(y), \quad (\forall i,\forall x,y),\ (\textit{by}\ (3)),\\
&\Longrightarrow \bigwedge_{x\in X}\left(A_i(x)\longrightarrow_f \bigwedge_{j=1}^{n}\left(A_j(x)\longrightarrow_f B_j(y)\right)\right) \ge B_i(y), \quad (\forall i,\forall y),\\
&\Longrightarrow \left(A_i \triangleleft_f \hat{R}_f\right)(y) \ge B_i(y)\ , \qquad (\forall i,\forall y)\ .
\end{aligned}$$

Now from (15) and (16) it follows that

$$\left(A_i \triangleleft_f \hat{R}_f\right)(y) = B_i(y).$$

This completes the proof. ■

B. *Continuity of BKS with f-implications*

In [27], [29] Perfilieva *et al*. discussed the *continuity* of a CRI inference mechanism, once again when the underlying operators were from a residuated lattice. Further, the authors have defined the *correctness* of a model in terms of its *interpolativity*. Later on Štěpnička and Jayaram [32] have dealt with the continuity of the BKS inference mechanism. Since we are dealing with operations that come from a non-residuated lattice structure we define continuity suitably and show that, once again, continuity is equivalent to the correctness of the model.

Definition 6.2: A fuzzy relation $R\in\mathcal{F}(X\times Y)$ is said to be a **continuous** model of fuzzy rules (12) in a BKS inference mechanism with f-implications, if for each $i\in I$ and for each

$A \in \mathcal{F}(X)$, the following inequality holds:

$$\bigwedge_{y\in Y}\left[f\left(B_i(y)\right) \overset{*}{\longleftrightarrow}_{\mathbf{G}} f\left(\left(A \lhd_f R\right)(y)\right)\right] \geq \bigwedge_{x\in X}\left[A_i(x) \overset{*}{\longleftrightarrow}_{\mathbf{G}} A(x)\right] \tag{17}$$

where 'f' is the generator function of the corresponding f-implication.

Remark 6.3: (i) Note that in the Definition 6.2 above, the bi-implication on the right side of the inequality $\overset{*}{\longleftrightarrow}_{\mathbf{G}}$ is equivalent to $\longleftrightarrow_{\mathbf{G}}$, since $A_i(x), A(x) \in [0,1]$. However, for notational consistency, we have retained the above form.

(ii) Note that if we consider f-implications with $f(0)=1$ then (17) reduces to the following where $\longleftrightarrow_{\mathbf{G}}$ is the Goguen bi-implication:

$$\bigwedge_{y\in Y}\left[f\left(B_i(y)\right) \longleftrightarrow_{\mathbf{G}} f\left(\left(A \lhd_f R\right)(y)\right)\right] \geq \bigwedge_{x\in X}\left[A_i(x) \longleftrightarrow_{\mathbf{G}} A(x)\right] \ . \tag{18}$$

(iii) Further, from [15], Example 11.7(ii), we see that $\longleftrightarrow_{\mathbf{G}}$ can be represented as

$$x \longleftrightarrow_{\mathbf{G}} y = t^{(-1)}(|t(x)-t(y)|) \ ,$$

where $t:[0,1] \longrightarrow [0,\infty]$ is any additive generator of the product t-norm and hence we have $t(0)=\infty$.

Still considering f-generators with $f(0)=1$, let us define a $D_X : \mathcal{F}(X)\times\mathcal{F}(X) \to \mathbb{R}$ and a $D_Y : \mathcal{F}(Y)\times\mathcal{F}(Y) \to \mathbb{R}$ as follows:

$$D_X\left(A_1, A_2\right) = \bigvee_{x\in X}\left(\left|t\left(A_1(x)\right) - t\left(A_2(x)\right)\right|\right) \ , \quad A_1, A_2 \in \mathcal{F}(X) \ ,$$

$$D_Y\left(B_1, B_2\right) = \bigvee_{y\in Y}\left(\left|\left(t\circ f\right)\left(B_1(y)\right) - \left(t\circ f\right)\left(B_2(y)\right)\right|\right) \quad B_1, B_2 \in \mathcal{F}(Y) \ .$$

It can be easily shown that $\left(\mathcal{F}(X), D_X\right)$ and $\left(\mathcal{F}(Y), D_Y\right)$ are metric spaces.

The following equivalences, along the lines of the proof of Theorem 1 in [27], demonstrate why (18) can be considered as an expression capturing the continuity:

$$\begin{aligned}
&\bigwedge_{y\in Y}\left[f\left(B_i(y)\right) \longleftrightarrow_{\mathbf{G}} f\left(\left(A \lhd_f R\right)(y)\right)\right] \geq \bigwedge_{x\in X}\left[A_i(x) \longleftrightarrow_{\mathbf{G}} A(x)\right] \\
\Longleftrightarrow &\bigwedge_{y\in Y} t^{-1}\left(\left|t\left(f\left(B_i(y)\right)\right) - t\left(f\left(\left(A \lhd_f R\right)(y)\right)\right)\right|\right) \geq \bigwedge_{x\in X} t^{-1}\left(\left|t\left(A_i(x)\right) - t\left(A_i(x)\right)\right|\right) \\
\Longleftrightarrow &\bigvee_{y\in Y}\left(\left|t(f(B_i(y))) - t\left(f\left(\left(A \lhd_f R\right)(y)\right)\right)\right|\right) \leq \bigvee_{x\in X}\left(\left|t(A_i(x)) - t(A_i(x))\right|\right) \\
\Longleftrightarrow &\bigvee_{y\in Y}\left(\left|\left(t\circ f\right)\left(B_i(y)\right) - \left(t\circ f\right)\left(\left(A \lhd_f R\right)(y)\right)\right|\right) \leq \bigvee_{x\in X}\left(\left|t\left(A_i(x)\right) - t\left(A_i(x)\right)\right|\right) \\
\Longleftrightarrow &D_Y\left(B_i, \left(A \lhd_f R\right)\right) \leq D_X\left(A_i, A\right) \ .
\end{aligned}$$

From the classical definition of continuity, we see that for any given $\epsilon > 0$, we have a $\delta > 0$ such that whenever $D_X\left(A_i, A\right) < \delta$ for any $i \in I$, we have that $D_Y\left(B_i, \left(A \lhd_f R\right)\right) < \epsilon$. Clearly, in our case $\delta = \epsilon$ is one possibility.

Theorem 6.4: Let us consider a BKS inference mechanism with f-implications. The fuzzy relation $R \in \mathcal{F}(X \times Y)$ over finite non-empty sets X and Y is a correct model of fuzzy rules (12) if and only if it is a continuous model of these rules.

Proof: Let R be a continuous model of the fuzzy if-then rules (12). By Definition 6.2, the inequality (17) is valid for all $i = 1, 2, \ldots n$ and an arbitrary $A \in \mathcal{F}(X)$. Now putting $A = A_i$ in (17), we have by the strictness of f,

$$\begin{aligned}
&\bigwedge_{y\in Y}\left[f\left(B_i(y)\right) \overset{*}{\longleftrightarrow}_{\mathbf{G}} f\left(\left(A_i \lhd_f R\right)(y)\right)\right] \geq 1, && (\forall i), \\
\Longrightarrow\ & f\left(B_i(y)\right) \overset{*}{\longleftrightarrow}_{\mathbf{G}} f\left(\left(A_i \lhd_f R\right)(y)\right) = 1, && (\forall i, y), \\
\Longrightarrow\ & f\left(B_i(y)\right) = f\left(\left(A_i \lhd_f R\right)(y)\right), && (\forall i, y), \\
\Longrightarrow\ & \left(A_i \lhd_f R\right)(y) = B_i(y), && (\forall i, y).
\end{aligned}$$

Thus we have interpolativity starting from continuity.

Now let us assume that the model has interpolativity. Towards proving (17), for arbitrary $y \in Y$, note that the following is true for any $i = 1, 2, \ldots n$:

$$
\begin{aligned}
& f\left((A \triangleleft_f R)(y)\right) \xleftrightarrow{*}_{\mathbf{G}} f\left(B_i(y)\right) \\
& = f\left((A \triangleleft_f R)(y)\right) \xleftrightarrow{*}_{\mathbf{G}} f\left((A_i \triangleleft_f R)(y)\right), \\
& \qquad\qquad\qquad\qquad (\text{Since } (A_i \triangleleft_f R)(y) = B_i(y)) \\
& = f\left(\bigwedge_{x \in X} [A(x) \longrightarrow_f R(x,y)]\right) \\
& \qquad\qquad \xleftrightarrow{*}_{\mathbf{G}} f\left(\bigwedge_{x \in X} [A_i(x) \longrightarrow_f R(x,y)]\right)
\end{aligned}
$$

$$
\begin{aligned}
& = f\left\{\bigwedge_{x \in X} f^{-1}\left[A(x) \cdot f\left(R(x,y)\right)\right]\right\} \\
& \qquad\qquad \xleftrightarrow{*}_{\mathbf{G}} f\left\{\bigwedge_{x \in X} f^{-1}\left[A_i(x) \cdot f\left(R(x,y)\right)\right]\right\} \\
& = \bigvee_{x \in X}\left[A(x) \cdot f\left(R(x,y)\right)\right] \xleftrightarrow{*}_{\mathbf{G}} \bigvee_{x \in X}\left[A_i(x) \cdot f\left(R(x,y)\right)\right] \\
& \geq \bigwedge_{x \in X}\left[A(x) \cdot f\left(R(x,y)\right) \xleftrightarrow{*}_{\mathbf{G}} A_i(x) \cdot f\left(R(x,y)\right)\right] \\
& \qquad\qquad\qquad\qquad (\text{Using Proposition 3.7}) \\
& \geq \bigwedge_{x \in X} \\
& \left(\left[A(x) \xleftrightarrow{*}_{\mathbf{G}} A_i(x)\right] \cdot \left[f(R(x,y)) \xleftrightarrow{*}_{\mathbf{G}} f(R(x,y))\right]\right) \\
& \qquad\qquad\qquad\qquad (\text{Using}(10)) \\
& = \bigwedge_{x \in X}\left[A(x) \xleftrightarrow{*}_{\mathbf{G}} A_i(x)\right],
\end{aligned}
$$

from which we obtain (17). ■

The following result shows that if we consider f-generators with $f(0) = 1$ then the finiteness of the sets X, Y can be dispensed with and the above result still remains valid.

Theorem 6.5: Let us consider a BKS inference mechanism with f-implications with $f(0) = 1$. The fuzzy relation $R \in \mathcal{F}(X \times Y)$, where X, Y are any continuous non-empty domains, is a correct model of fuzzy rules (12) if and only if it is a continuous model of these rules.

Proof: Note that since $f(0) = 1$, the continuity equation (17) reduces to (18). Since $\longleftrightarrow_{\mathbf{G}}$ satisfies (7) and (8) even for an infinite index set $\mathcal{I}$, the proof follows immediately along the lines of the proof of Theorem 6.4. ■

C. Robustness of BKS with $f-$implications

Similar to the equivalence relation in classical set theory, similarity relation or fuzzy equivalence relation has been proposed. Similarity relations have been used to characterize the inherent indistinguishability in a fuzzy system [14].

Definition 6.6 ([14], Definition 2.5): A fuzzy equivalence relation $E : X \times X \to [0,1]$ with respect to the t-norm $\star$ on X is a fuzzy relation over $X \times X$ which satisfies the following for all $x, y, z \in X$:

- $E(x,x) = 1$. (Reflexivity)
- $E(x,y) = E(y,x)$. (Symmetry)
- $E(x,y) \star E(y,z) \leq E(x,z)$. ($\star$ -Transitivity)

We denote a fuzzy equivalence relation by $(E, \star)$.

Definition 6.7 ([14], Definition 2.7): A fuzzy set $A \in \mathcal{F}(X)$ is called extensional with respect to a fuzzy equivalence relation $(E, \star)$ on X if,

$$A(x) \star E(x,y) \leq A(y), \qquad x, y \in X.$$

Definition 6.8 ([14], Definition 2.8): Let $A \in \mathcal{F}(X)$ and $(E, \star)$ be a fuzzy equivalence relation on X. The fuzzy set,

$$\hat{A} = \bigwedge\{C : A \leq C \text{ and } C \text{ is extensional w.r.to } (E, \star)\},$$

is called the extensional hull of A. By $A \leq C$ we mean that for all $x \in X, A(x) \leq C(x)$, i.e, ordering in the sense of inclusion.

Proposition 6.9 ([14], Proposition 2.9): Let $A \in \mathcal{F}(X)$ and $(E, \star)$ be a fuzzy equivalence relation on X. Then

$$\hat{A}(x) = \bigvee\{A(y) \star E(x,y) \mid y \in X\}.$$

Definition 6.10: Let $A \in \mathcal{F}(X)$ and $(E, \star)$ be a fuzzy equivalence relation on X. A fuzzy relational inference mechanism $f_R^{@}$ is said to be **robust** w.r.to $(E, \star)$ if

$$f_R^{@}(A) = f_R^{@}(\hat{A}). \tag{19}$$

Robustness of an FRI $f_R^{@}$ deals with how variations in the intended input affect the conclusions. It is different from continuity in that, we expect that even when the actual input fuzzy set is not equal to the intended fuzzy set but both are *equivalent* - in a certain predefined sense based on the equality relations on the underlying set - the output fuzzy set should be equal to the corresponding intended output. In other words, the FRI $f_R^{@}$ respects the order and equivalence present in the underlying universe of discourse.

The robustness of CRI was dealt with by Klawonn and Castro [14]. Later on Štěpnička and Jayaram [32] have undertaken a similar study for BKS inference mechanism with R-implications. Both the above works show that, when combined with appropriate models of fuzzy rules, CRI and BKS are robust inference mechanisms. In the following, we show a similar result which ensures the robustness of BKS-f inference mechanism.

Theorem 6.11: Let $(E, \cdot)$ be a fuzzy equivalence relation on X, where '$\cdot$' is the product t-norm. Let us consider a rule base of the form (12) such that every A_i, $i = 1, 2, \ldots n$ is extensional w.r.to $(E, \cdot)$. Then $f_{\hat{R}_f}^{\triangleleft_f}$ is robust w.r.to $(E, \cdot)$, i.e., $f_{\hat{R}_f}^{\triangleleft_f}$ satisfies (19) for any fuzzy set $A' \in \mathcal{F}(X)$, i.e.,

$$A' \triangleleft_f \hat{R}_f = \hat{A}' \triangleleft_f \hat{R}_f.$$

Proof: By definition of $\hat{A}'$ we have the following:

$$
\begin{aligned}
\hat{A}' \geq A' & \Longrightarrow \hat{A}' \longrightarrow_f \hat{R}_f \leq A' \longrightarrow_f \hat{R}_f \\
& \Longrightarrow \hat{A}' \triangleleft_f \hat{R}_f \leq A' \triangleleft_f \hat{R}_f.
\end{aligned}
$$

Since $\hat{R}_f$ is given by (Imp-$\hat{R}_f$), we have

$$\begin{aligned}
&\left(\hat{A}' \lhd_f \hat{R}_f\right)(y) \\
&= \bigwedge_{x\in X}\left[\hat{A}'(x) \longrightarrow_f \bigwedge_{i=1}^{n}\left(A_i(x) \longrightarrow_f B_i(y)\right)\right], \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad y \in Y.
\end{aligned}$$

Since every A_i is extensional with respect to E, for any $x, x' \in X$ and for any $i = 1, 2, \ldots n$, we have,

$$\begin{aligned}
&A_i(x') \geq A_i(x) \cdot E(x, x') \\
&\Longrightarrow A_i(x') \longrightarrow_f B_i(y) \leq [A_i(x) \cdot E(x, x')] \longrightarrow_f B_i(y), \\
&\qquad\qquad\qquad\qquad\qquad\qquad y \in Y. \qquad (20)
\end{aligned}$$

Now for any $x \in X$,

$$\begin{aligned}
&\hat{A}'(x) \longrightarrow_f \bigwedge_{i=1}^{n}\left(A_i(x) \longrightarrow_f B_i(y)\right) \\
&= \left(\bigvee_{x'\in X}[A'(x') \cdot E(x, x')]\right) \\
&\qquad\qquad \longrightarrow_f \bigwedge_{i=1}^{n}\left(A_i(x) \longrightarrow_f B_i(y)\right), \\
&\qquad\qquad\qquad\qquad \text{(Using Proposition 6.9)} \\
&= \bigwedge_{x'\in X} \\
&\qquad \left([A'(x') \cdot E(x, x')] \longrightarrow_f \bigwedge_{i=1}^{n}\left(A_i(x) \longrightarrow_f B_i(y)\right)\right), \\
&\qquad\qquad\qquad\qquad \text{(Using Proposition 2.10)} \\
&= \bigwedge_{i=1}^{n}\bigwedge_{x'\in X}\left([A'(x') \cdot E(x, x')] \longrightarrow_f \left(A_i(x) \longrightarrow_f B_i(y)\right)\right), \\
&\qquad\qquad\qquad\qquad \text{(Using (3))} \\
&= \bigwedge_{i=1}^{n}\bigwedge_{x'\in X} \\
&\qquad \left(A'(x') \longrightarrow_f [E(x, x') \longrightarrow_f (A_i(x) \longrightarrow_f B_i(y))]\right), \\
&\qquad\qquad\qquad\qquad \text{(By (LI))} \\
&= \bigwedge_{i=1}^{n}\bigwedge_{x'\in X}\left(A'(x') \longrightarrow_f [(E(x, x') \cdot A_i(x)) \longrightarrow_f B_i(y)]\right), \\
&\qquad\qquad\qquad\qquad \text{(By (LI))} \\
&\geq \bigwedge_{i=1}^{n}\bigwedge_{x'\in X}\left(A'(x') \longrightarrow_f [A_i(x') \longrightarrow_f B_i(y)]\right), \\
&\qquad\qquad\qquad\qquad \text{(Using (20))} \\
&= (A' \lhd_f \hat{R}_f)(y)\ .
\end{aligned}$$

Thus $\hat{A}' \lhd_f \hat{R}_f \geq A' \lhd_f \hat{R}_f$ and the result follows. ■

The above study clearly demonstrates that, as in the case of BKS and CRI, the (BKS-f) inference mechanism also possesses all the desirable properties like, interpolativity, continuity and robustness.

VII. BKS With g-Implications : Its Suitability

In this section we consider the BKS inference mechanism with g-implications and do a similar analysis of what has been done for BKS with f-implications. We will denote a g-implication by "$\longrightarrow_g$" where 'g' is the corresponding generator function.

A. *Interpolativity of BKS with g-implications*

Theorem 7.1: Let A_i for $i = 1, 2, \ldots n$ be normal. A necessary and sufficient condition for $\hat{R}_g$ to be a solution to $B_i = A_i \lhd_g R$ is as follows: For any $i, j \in \{1, 2, \ldots n\}$,

$$\bigvee_{x\in X}\left(A_i(x) \cdot A_j(x)\right) \leq \bigwedge_{y\in Y}\left(g(B_i(y)) \overset{*}{\longleftrightarrow}_{\mathbf{G}} g(B_j(y))\right)\ . \tag{21}$$

where 'g' is the generator function of the corresponding g-implication.

Proof: ($\Longrightarrow$): Let the system have interpolativity. Then, for any $y \in Y$, $i \in \mathbb{N}_n = \{1, 2, \ldots n\}$

$$\begin{aligned}
&\left(A_i \lhd_g \hat{R}_g\right)(y) = B_i(y) \\
&\Longrightarrow \bigwedge_{x\in X}\left(A_i(x) \longrightarrow_g \bigwedge_{j=1}^{n}\left(A_j(x) \longrightarrow_g B_j(y)\right)\right) = B_i(y), \\
&\Longrightarrow A_i(x) \longrightarrow_g \left(A_j(x) \longrightarrow_g B_j(y)\right) \geq B_i(y), \\
&\qquad\qquad\qquad\qquad\qquad\qquad (\forall j, \forall x), \\
&\Longrightarrow \left(A_i(x) \cdot (A_j(x)\right) \longrightarrow_g B_j(y) \geq B_i(y), \\
&\qquad\qquad\qquad\qquad\qquad (\forall j, \forall x)\ (by\ \text{(LI)}), \\
&\Longrightarrow g^{(-1)}\left(\frac{1}{A_i(x) \cdot A_j(x)} \cdot g\left(B_j(y)\right)\right) \geq B_i(y), \\
&\qquad\qquad\qquad\qquad\qquad\qquad (\forall j, \forall x)\ , \\
&\Longrightarrow g^{-1}\left(\min\left\{\frac{1}{A_i(x) \cdot A_j(x)} \cdot g\left(B_j(y)\right), g(1)\right\}\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \geq B_i(y), \\
&\qquad\qquad\qquad\qquad\qquad\qquad (\forall j, \forall x), \\
&\Longrightarrow \min\left\{\frac{1}{A_i(x) \cdot A_j(x)} \cdot g\left(B_j(y)\right), g(1)\right\} \geq g\left(B_i(y)\right), \\
&\qquad\qquad\qquad\qquad\qquad\qquad (\forall j, \forall x), \\
&\Longrightarrow \frac{1}{A_i(x) \cdot A_j(x)} \cdot g\left(B_j(y)\right)) \geq g\left(B_i(y)\right), \\
&\qquad\qquad\qquad\qquad\qquad\qquad (\forall j, \forall x), \\
&\Longrightarrow A_i(x) \cdot A_j(x) \leq \frac{g\left(B_j(y)\right)}{g\left(B_i(y)\right)}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad (\forall j, \forall x).
\end{aligned}$$

Since i, j are arbitrary, interchanging them in the above inequality, we have,

$$A_j(x) \cdot A_i(x) \leq \frac{g(B_i(y))}{g(B_j(y))}\ .$$

We also trivially have,

$$A_j(x) \cdot A_i(x) \leq 1 \ .$$

Now from the above inequalities we see,

$$A_i(x) \cdot A_j(x) \leq \min\left\{1, \frac{g(B_i(y))}{g(B_j(y))}, \frac{g(B_j(y))}{g(B_i(y))}\right\} \qquad (\forall i, j)(\forall x, y),$$

$$\Longrightarrow \bigvee_{x \in X} (A_i(x) \cdot A_j(x)) \tag{22}$$

$$\leq \bigwedge_{y \in Y} \min\left\{1, \frac{g(B_i(y))}{g(B_j(y))}, \frac{g(B_j(y))}{g(B_i(y))}\right\} \qquad (\forall i, j).$$

which is same as (21).

($\Longleftarrow$): Now let us assume that (21) holds. Then,

$$A_i(x) \cdot A_j(x) \leq \min\left\{1, \frac{g(B_i(y))}{g(B_j(y))}, \frac{g(B_j(y))}{g(B_i(y))}\right\} \qquad (\forall i, j)(\forall x, y),$$

$$\Longrightarrow A_i(x) \cdot A_j(x) \leq \frac{g(B_j(y))}{g(B_i(y))}, \qquad (\forall i, j, \forall x, y),$$

$$\Longrightarrow \frac{1}{A_i(x) \cdot A_j(x)} \cdot g\left(B_j(y)\right)) \geq g\left(B_i(y)\right), \qquad (\forall i, j, \forall x, y),$$

$$\Longrightarrow \min\left\{\frac{1}{A_i(x) \cdot A_j(x)} \cdot g\left(B_j(y)\right), g(1)\right\} \geq g\left(B_i(y)\right), \qquad (\forall i, j, \forall x, y),$$

$$\Longrightarrow g^{-1}\left(\min\left\{\frac{1}{A_i(x) \cdot A_j(x)} \cdot g\left(B_j(y)\right), g(1)\right\}\right) \geq B_i(y), \qquad (\forall i, j, \forall x, y),$$

$$\Longrightarrow g^{(-1)}\left(\frac{1}{A_i(x) \cdot A_j(x)} \cdot g\left(B_j(y)\right)\right) \geq B_i(y), \qquad (\forall i, j, \forall x, y) \ ,$$

$$\Longrightarrow (A_i(x) \cdot (A_j(x)) \longrightarrow_g B_j(y) \geq B_i(y), \qquad (\forall i, j, \forall x, y),$$

$$\Longrightarrow A_i(x) \longrightarrow_g (A_j(x) \longrightarrow_g B_j(y)) \geq B_i(y), \qquad (\forall i, j, \forall x, y), (by\ \text{(LI)}),$$

$$\Longrightarrow \bigwedge_{x \in X}\left(A_i(x) \longrightarrow_g \bigwedge_{j=1}^{n} (A_j(x) \longrightarrow_g B_j(y))\right) \geq B_i(y), \qquad (\forall i, \forall y),$$

$$\Longrightarrow \left(A_i \lhd_g \hat{R}_g\right)(y) \geq B_i(y), \qquad (\forall i, \forall y).$$

So we have the following:

$$\left(A_i \lhd_g \hat{R}_g\right)(y) \geq B_i(y), \qquad (\forall i, \forall y). \tag{23}$$

Once again, the following inequality is always true, the proof of which is very much along the lines as that given for (15):

$$\left(A_i \lhd_g \hat{R}_g\right)(y) \leq B_i(y), \qquad (\forall i, \forall y). \tag{24}$$

Now from (23) and (24) it follows that

$$\left(A_i \lhd_g \hat{R}_g\right)(y) = B_i(y) \ , \ \text{ for all } y \in Y.$$

■

B. Continuity of BKS with g-implications

Definition 7.2: A fuzzy relation $R \in \mathcal{F}(X \times Y)$ is said to be a **continuous** model of fuzzy rules (12) in a BKS inference mechanism with g-implications, if for each $i \in I$ and for each $A \in \mathcal{F}(X)$, the following inequality holds:

$$\bigwedge_{y \in Y}\left[g\left(B_i(y)\right) \stackrel{*}{\longleftrightarrow}_{\mathbf{G}} g\left((A \lhd_g R)(y)\right)\right] \geq \bigwedge_{x \in X}\left[A_i(x) \stackrel{*}{\longleftrightarrow}_{\mathbf{G}} A(x)\right], \tag{25}$$

where 'g' is the generator function of the corresponding g-implication.

Remark 7.3: (i) Note that if we consider g-implications with $g(1) = 1$ then (25) reduces to the following where $\longleftrightarrow_{\mathbf{G}}$ is the Goguen bi-implication:

$$\bigwedge_{y \in Y}\left[g\left(B_i(y)\right) \longleftrightarrow_{\mathbf{G}} g\left((A \lhd_g R)(y)\right)\right] \geq \bigwedge_{x \in X}\left[A_i(x) \longleftrightarrow_{\mathbf{G}} A(x)\right] \ . \tag{26}$$

(ii) A similar explanation as in Remark 6.3 can be given as to why (26) can be considered as an expression capturing the continuity.

Theorem 7.4: Let us consider a BKS inference mechanism with g-implications over finite non-empty sets X and Y . The fuzzy relation $R \in \mathcal{F}(X \times Y)$ is a correct model of fuzzy rules (12) if and only if it is a continuous model of these rules.

Proof: Let R be a continuous model of the fuzzy if-then rules (12). By Definition 7.2 we have,

$$\bigwedge_{y \in Y}\left[g\left(B_i(y)\right) \stackrel{*}{\longleftrightarrow}_{\mathbf{G}} g\left(\left(A \lhd_g R\right)(y)\right)\right] \geq \bigwedge_{x \in X}\left[A_i(x) \stackrel{*}{\longleftrightarrow}_{\mathbf{G}} A(x)\right] \ ,$$

for all $i = 1, 2, \ldots n$ and an arbitrary $A \in \mathcal{F}(X)$. Letting $A = A_i$ in the above inequality, we get

$$\bigwedge_{y \in Y}\left[g\left(B_i(y)\right) \stackrel{*}{\longleftrightarrow}_{\mathbf{G}} g\left(\left(A \lhd_g R\right)(y)\right)\right] \geq 1, \qquad (\forall i),$$

$$\Longrightarrow g\left(B_i(y)\right) \stackrel{*}{\longleftrightarrow}_{\mathbf{G}} g\left(\left(A_i \lhd_g R\right)(y)\right) = 1, \qquad (\forall i, y),$$

$$\Longrightarrow g\left(B_i(y)\right) = g\left(\left(A_i \lhd_g R\right)(y)\right), \qquad (\forall i, y),$$

$$\Longrightarrow \left(A_i \lhd_g R\right)(y) = B_i(y), \qquad (\forall i, y).$$

So we have interpolativity starting from continuity.

Now let us assume that the model has interpolativity, i.e., $(A_i \lhd_g R)(y) = B_i(y),\ \forall i, y$. Towards proving (25), let us consider the following inequalities:

$$
\begin{aligned}
& g\left(B_i(y)\right) \overset{*}{\longleftrightarrow}_{\mathbf{G}} g\left(\left(A \lhd_g R\right)(y)\right) \\
& = g\left(\left(A \lhd_g R\right)(y)\right) \overset{*}{\longleftrightarrow}_{\mathbf{G}} g\left(\left(A_i \lhd_g R\right)(y)\right)\ , \\
& \qquad\qquad\qquad\qquad (\text{Since} (A_i \lhd_g R)(y) = B_i(y)) \\
& = g\left(\bigwedge_{x \in X} [A(x) \longrightarrow_g R(x,y)]\right) \\
& \qquad\qquad \overset{*}{\longleftrightarrow}_{\mathbf{G}} g\left(\bigwedge_{x \in X} [A_i(x) \longrightarrow_g R(x,y)]\right) \\
& = g\left\{\bigwedge_{x \in X} g^{(-1)}\left[\frac{1}{A(x)} \cdot g(R(x,y))\right]\right\} \\
& \qquad \overset{*}{\longleftrightarrow}_{\mathbf{G}} g\left\{\bigwedge_{x \in X} g^{(-1)}\left[\frac{1}{A_i(x)} \cdot g(R(x,y))\right]\right\} \\
& = g\left\{\bigwedge_{x \in X} g^{-1}\left[\min\left\{\frac{1}{A(x)} \cdot g(R(x,y)), g(1)\right\}\right]\right\} \\
& \overset{*}{\longleftrightarrow}_{\mathbf{G}} g\left\{\bigwedge_{x \in X} g^{-1}\left[\min\left\{\frac{1}{A_i(x)} \cdot g(R(x,y)), g(1)\right\}\right]\right\} \\
& = \bigwedge_{x \in X}\left[\min\left\{\frac{1}{A(x)} \cdot g(R(x,y)), g(1)\right\}\right] \\
& \qquad \overset{*}{\longleftrightarrow}_{\mathbf{G}} \bigwedge_{x \in X}\left[\min\left\{\frac{1}{A_i(x)} \cdot g(R(x,y)), g(1)\right\}\right] \\
& \geq \bigwedge_{x \in X}\Big[\min\left\{\frac{1}{A(x)} \cdot g(R(x,y)), g(1)\right\} \\
& \qquad \overset{*}{\longleftrightarrow}_{\mathbf{G}} \min\left\{\frac{1}{A_i(x)} \cdot g(R(x,y)), g(1)\right\}\Big] \\
& \qquad\qquad\qquad\qquad (\text{By Proposition } 3.8) \\
& \geq \bigwedge_{x \in X}\Big[\left\{\frac{1}{A(x)} \cdot g(R(x,y)) \overset{*}{\longleftrightarrow}_{\mathbf{G}} \frac{1}{A_i(x)} \cdot g(R(x,y))\right\} \\
& \qquad\qquad \wedge \left\{g(1) \overset{*}{\longleftrightarrow}_{\mathbf{G}} g(1)\right\}\Big] \\
& \qquad\qquad\qquad\qquad (\text{By Proposition } 3.8) \\
& = \bigwedge_{x \in X}\left[\frac{1}{A(x)} \cdot g(R(x,y)) \overset{*}{\longleftrightarrow}_{\mathbf{G}} \frac{1}{A_i(x)} \cdot g(R(x,y))\right] \\
& \geq \bigwedge_{x \in X}\left[\frac{1}{A(x)} \overset{*}{\longleftrightarrow}_{\mathbf{G}} \frac{1}{A_i(x)}\right] \cdot \\
& \qquad\qquad\qquad \left[g(R(x,y)) \overset{*}{\longleftrightarrow}_{\mathbf{G}} g(R(x,y))\right], \\
& \qquad\qquad\qquad\qquad\qquad (\text{Using}(10)) \\
& = \bigwedge_{x \in X}\left[\frac{1}{A(x)} \overset{*}{\longleftrightarrow}_{\mathbf{G}} \frac{1}{A_i(x)}\right] = \bigwedge_{x \in X}\left[A_i(x) \overset{*}{\longleftrightarrow}_{\mathbf{G}} A(x)\right]\ ,
\end{aligned}
$$

from whence we have (25). ■

The following result shows that if we consider g-generators with $g(1) = 1$ then the finiteness of the sets X, Y can be dispensed with and the above result still remains valid.

Theorem 7.5: Let us consider a BKS inference mechanism with g-implications with $g(1) = 1$. The fuzzy relation $R \in \mathcal{F}(X \times Y)$, where X, Y are any continuous non-empty domains, is a correct model of fuzzy rules (12) if and only if it is a continuous model of these rules.

Proof: Note that since $g(1) = 1$, the continuity equation (25) reduces to (26). Since $\longleftrightarrow_{\mathbf{G}}$ satisfies (7) and (8) even for an infinite index set $\mathcal{I}$, the proof follows immediately along the lines of the proof of Theorem 7.4. ■

C. Robustness of BKS with g-implications

The following result shows that BKS with g-implications are also robust inference mechanisms. The proof is along the same lines as that of the proof of Theorem 6.11 and hence is not presented here.

Theorem 7.6: Let $(E, \cdot)$ be a fuzzy equivalence relation on a finite non-empty set X, where '$\cdot$' is the product t-norm. Let us consider a rule base of the form (12) such that every $A_i,\ i = 1, 2, \ldots n$ is extensional w.r.to $(E, \cdot)$. Then $f_{\hat{R}_g}^{\lhd_g}$ is robust w.r.to $(E, \cdot)$, i.e., $f_{\hat{R}_g}^{\lhd_g}$ satisfies (19) for any fuzzy set $A' \in \mathcal{F}(X)$, i.e.,

$$A' \lhd_g \hat{R}_g = \hat{A}' \lhd_g \hat{R}_g\ .$$

VIII. Concluding Remarks

Fuzzy Relational Inferences (FRIs) are one of the earliest inference mechanisms to be studied. However, all of the known works have concentrated on FRIs that employ operations that come from a residuated lattice structure. In this work, we have shown that the properties like interpolativity, continuity and robustness which are available for the BKS inference mechanism with residuated implications are also available when we employ the Yager's classes of fuzzy implications, which do not come from a residuated structure. We believe our results in this work will open up more options for the practitioners to choose from.

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