On the Suitability of the Bandler–Kohout Subproduct as an Inference Mechanism

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***Abstract*—Fuzzy relational inference (FRI) systems form an important part of approximate reasoning schemes using fuzzy sets. The compositional rule of inference (CRI), which was introduced by Zadeh, has attracted the most attention so far. In this paper, we show that the FRI scheme that is based on the Bandler–Kohout (BK) subproduct, along with a suitable realization of the fuzzy rules, possesses all the important properties that are cited in favor of using CRI, viz., equivalent and reasonable conditions for their solvability, their interpolative properties, and the preservation of the indistinguishability that may be inherent in the input fuzzy sets. Moreover, we show that under certain conditions, the equivalence of *first-infer-then-aggregate (FITA)* and *first-aggregate-then-infer (FATI)* inference strategies can be shown for the BK subproduct, much like in the case of CRI. Finally, by addressing the computational complexity that may exist in the BK subproduct, we suggest a hierarchical inferencing scheme. Thus, this paper shows that the BK-subproduct-based FRI is as effective and efficient as the CRI itself.**

***Index Terms*—Bandler–Kohout (BK) subproduct, compositional rule of inference (CRI), correctness and continuity of inference, fuzzy relational equations, fuzzy relational inference (FRI) systems, hierarchical CRI.**

I. Introduction

THE IDEA of linguistic fuzzy models that imitate the human way of thinking was proposed by Zadeh in his pioneering work [1]. Systems that use fuzzy rules and an inference mechanism have been applied in a wide variety of applications, viz., automatic control, decision making, risk analysis, etc.

Let X be a nonempty classical set. Let us recall that a fuzzy set A on X is a mapping from X to the unit interval, i.e., $A : X \to [0, 1]$. Let $\mathcal{F}(X)$ denote the set of all fuzzy sets on X. Given two nonempty classical sets X, Y, a fuzzy IF–THEN rule is usually given as follows:

$$\textbf{IF } x \text{ is } A \quad \textbf{THEN} \quad y \text{ is } B \tag{1}$$

where the antecedent fuzzy set $A \in \mathcal{F}(X)$ and the consequent fuzzy set $B \in \mathcal{F}(Y)$ represent some properties.

Given that a fuzzy observation x is A', where $A' \in \mathcal{F}(X)$, a corresponding output fuzzy set $B' \in \mathcal{F}(Y)$, which means that y is B', is deduced using an *inference mechanism*. Thus, an inference mechanism may, generally, be viewed as an arbitrary mapping from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$ [2], [3].

Many types of inference mechanisms that deal with fuzzy-rule-based systems have been proposed in the literature and used in practical applications. Of the many fuzzy inference schemes, fuzzy relational inferences (FRIs) have received considerable attention both from theoretical researchers and practitioners. Similarity-based reasoning (SBR) [4] and inverse truth-functional modification [5] are two of the representative examples of inference mechanisms that do not use fuzzy relations and that are also well established in the literature. However, it should be mentioned that under certain conditions, an equivalent fuzzy-relation-based description of some of these inference mechanisms can be given (see [6] and [7]).

In this paper, we will focus only on fuzzy-relation-based inference mechanism.

A. Fuzzy Relational Inferences

FRI mechanisms use a fuzzy relation R to model a given fuzzy rule base. Here, a fuzzy IF–THEN rule of the form (1) is represented as a fuzzy relation $R : X \times Y \to [0, 1]$, i.e., $R \in \mathcal{F}(X \times Y)$. Then, given a fact that $x \; is \; A'$, the inferred output $y \; is \; B'$ is obtained as a composition of A' and R, i.e.,

$$B' = A' @ R \tag{2}$$

where $A' \in \mathcal{F}(X)$, $B' \in \mathcal{F}(Y)$, and @ is a fuzzy relational composition[1] that involves fuzzy logic operations.

Up to now, we have discussed the case of a single fuzzy rule only. However, rarely, if ever a single fuzzy rule can be expected to capture the entire knowledge in which the scheme is employed, a fuzzy rule base that consists of multiple fuzzy rules is necessary. Let us consider a fuzzy rule base as follows:

$$\textbf{IF } x \text{ is } A_i \quad \textbf{THEN} \quad y \text{ is } B_i, \quad \text{for } i = 1, \ldots, n \tag{3}$$

where the fuzzy sets $A_i \in \mathcal{F}(X)$ and $B_i \in \mathcal{F}(Y)$ represent some properties.

Clearly, for different representations R of the fuzzy IF–THEN rules in (1) and different compositions @, we obtain different FRI mechanisms, often with varying properties that are applicable in specific contexts. We now present two of the most

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[1] In the literature, we may, very often, find a distinction between *fuzzy relational composition* and *image of a fuzzy set under a fuzzy relation*. The first notion denotes a composition of two binary relations, while the second one denotes our situation when we compose a fuzzy set and a binary fuzzy relation. However, a unary fuzzy relation (i.e., a fuzzy set) on a universe U may be viewed as a binary fuzzy relation on a Cartesian product of an empty set and U. Therefore, we may use the notion *composition* even in our situation and avoid the use of two notions.

commonly employed fuzzy relations R to model a given fuzzy rule base and two of the established FRI mechanisms based on different fuzzy compositions.

B. Distinct Fuzzy Rule Base Models

1) The fuzzy relation $\check{R} \in \mathcal{F}(X \times Y)$

$$\check{R}(x, y) = \bigvee_{i=1}^{n} (A_i(x) * B_i(y)), \quad x \in X,\ y \in Y \tag{4}$$

is the most often used model of fuzzy rules (3) in real-world applications. This is mainly due to the successful applications of this, say, Cartesian product approach, which was described by Mamdani and Assilian [8], and was followed by a huge number of researchers and practitioners (see, e.g., [9] and [10]).

2) Alternatively, to keep the conditional IF–THEN form of fuzzy rules (3), fuzzy relation $\hat{R} \in \mathcal{F}(X \times Y)$

$$\hat{R}(x, y) = \bigwedge_{i=1}^{n} (A_i(x) \to Bi(y)), \quad x \in X,\ y \in Y \tag{5}$$

can be chosen to model the fuzzy rule base. It deals with a mathematically correct extension of a classical implication that is denoted by $\to$.

To stress the difference between both the approaches, let us recall the work of Dubois *et al.* [11], where the authors state that "In the view given by (5), each piece of information (fuzzy rule) is viewed as a constraint. This view naturally leads to a conjunctive way of merging the individual pieces of information since the more information, the more constraints and the less possible values to satisfy them." They have also described the second approach that was proposed by Mamdani and Assilian as follows: "It seems that fuzzy rules modelled by $\check{R}$ are not viewed as constraints, but are considered to be pieces of data. Then, the maximum in (4) expresses accumulation of data."

It should be stressed that both approaches have sound logical foundations but from different viewpoints (see, e.g., [12]–[14]). However, only the approach using $\check{R}$ was widely used in applications, although the implicational approach using $\hat{R}$ is probably as useful as the approach proposed by Mamdani–Assilian (see [15]). Nevertheless, as we show in this paper, the implicational approach using $\hat{R}$ does have an important role to play in the case of Bandler–Kohout (BK) subproducts (see Theorem 3.22). For an extensive study of different fuzzy rules, see [16]–[18].

C. Coherence

Consistency (i.e., nonexistence of contradictory rules) is a crucial issue that needs to be checked when dealing with a fuzzy rule base. In case of the implicational approach (5) to modeling a fuzzy rule base, the situation gets significantly simpler. It was noted by Dubois *et al.* [11] that inconsistent rules lower the largest membership degrees in the resulting fuzzy relation. Departing from this fact, they proposed the concept of so-called coherence for which an existence of $y \in Y$ such that $\hat{R}(x', y) = 1$ for arbitrary $x' \in X$ is required. This condition can be easily checked and ensured (see [11] and [19]).

An analogous issue in case of the Cartesian product approach $\check{R}$ has been suggested [20]. However, the condition has to be redefined, and instead of nonemptiness of the core of $\check{R}$, its convexity up to some predefined value is required. This approach is, unquestionably, more complicated and less preferable.

Generally, the consistency (coherence) is a property of the given fuzzy rule base (its model) and not of the inference mechanism itself. However, as mentioned previously and demonstrated in Section IV, each model of a fuzzy rule base is computationally preferable in combination with a different inference, and thus, these notions cannot be studied independently.

D. Compositional Rule of Inference and BK-Subproduct Inference Mechanisms

As noted previously, depending on the type of composition @, the FRI varies in its properties. Two of the commonly employed fuzzy relational compositions are the $\sup$-$*$ and $\inf$-I compositions (see [21]), which, when employed, lead to the following FRIs.

1) The compositional rule of inference (CRI) that was provided by Zadeh [1] is one of the earliest FRIs. Here, a fuzzy IF–THEN rule of the form (1) is represented as a fuzzy relation $R(x, y) : X \times Y \to [0, 1]$, i.e., $R \in \mathcal{F}(X \times Y)$. Then, given a fact that $x\ is\ A'$, the inferred output B' is obtained as composition of $A'(x)$ and $R(x, y)$, i.e.,

$$B'(y) = \bigvee_{x \in X} (A'(x) * R(x, y)), \quad y \in Y \tag{6}$$

where $*$ is a fuzzy conjunction, which is typically a t-norm (see [22] for more details). We use the following notation to indicate the CRI scheme:

$$B' = A' \circ R. \tag{7}$$

2) Other than the CRI, let us also recall that it was Pedrycz [23] who first proposed an inference scheme based on the BK subproduct, which was proposed by Bandler and Kohout [24]–[26].

For a given fuzzy input $A' \in \mathcal{F}(X)$, the fuzzy output $B' \in \mathcal{F}(Y)$ that is obtained by the BK-subproduct inference mechanism is defined as follows:

$$B'(y) = \bigwedge_{x \in X} (A'(x) \to R(x, y)), \quad y \in Y \tag{8}$$

where $\to$ is a residual implication (see Section II for more details), and R is the fuzzy relation that models fuzzy rule (1). We use the following notation to indicate the BK-subproduct scheme:

$$B' = A' \lhd R. \tag{9}$$

We only remark that yet other types of fuzzy relational compositions are studied in the literature, for instance, the $\inf$-S composition, where S is a t-conorm (see [21]).

E. Mathematical Structure for Fuzzy Relational Inference Mechanisms

Based on the previous discussion, fuzzy rules may be viewed as a partial mapping from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$ that assigns $B_i \in \mathcal{F}(Y)$ to $A_i \in \mathcal{F}(X)$ for every $i = 1, \ldots, n$. Then, the inference process itself can be viewed as an extension of this partial mapping to a total one [27]. For better understanding, let us adopt the notation from [28] and consider the following structure:

$$S = (X, Y, \{A_i, B_i\}_{i=1,\ldots,n}, \mathcal{L}, @)$$

where X, Y are nonempty classical sets, $A_i \in \mathcal{F}(X), B_i \in \mathcal{F}(Y)$ for all $i = 1, \ldots, n$ are the antecedent and consequent fuzzy sets in the fuzzy rule base, and $@ : \mathcal{F}(X) \times \mathcal{F}(X \times Y) \to \mathcal{F}(Y)$ is a fuzzy relational composition. For instance, $@$ could be either $\circ$ or $\triangleleft$. Finally, $\mathcal{L}$ is an algebra on the unit interval $[0, 1]$ that provides us with the operations to be employed in the inference process, which is typically a complete residuated lattice (see Section II-A for more details).

Now, by the choice of the fuzzy relation R that models the fuzzy rule base and by the choice of $@$, we define a fuzzy function $f_R^@(A) : \mathcal{F}(X) \to \mathcal{F}(Y)$ such that $f_R^@(A) = A@R$ for an arbitrary $A \in \mathcal{F}(X)$.

F. Studies on the Advantages of CRI and Organization of the Paper

Given a fuzzy rule base, the CRI is the most often and widely used FRI for the following reasons.

1) An important issue in the applicability of an FRI mechanism $@$ in a structure $S = (X, Y, \{A_i, B_i\}_{i=1,\ldots,n}, \mathcal{L}, @)$ is to determine an appropriate fuzzy relation R that models the given fuzzy rules to obtain meaningful conclusions. One of the fundamental properties expected of the corresponding fuzzy adjoint mapping is its *interpolativity*, i.e., $f_R^@(A_i) = B_i$. This pertains to the solvability of fuzzy relation equations. In the case of CRI, necessary and sufficient conditions for the solvability of fuzzy relation equations have been well established for a long time (see, e.g., [29]). The state of the art, as well as analogous results, which concerns the BK subproduct are briefly recalled in Section III-A.
2) Perfilieva and Lehmke [28] and Perfilieva and Novák [30] have dealt with the *continuity* of a fuzzy function f_R° that is adjoint to the CRI mechanism $\circ$ in a structure $S = (X, Y, \{A_i, B_i\}_{i=1,\ldots,n}, \mathcal{L}, \circ)$ and a fuzzy relation R that models fuzzy rules (3). The authors give necessary and sufficient conditions for f_R° to be continuous. They have also shown that the concept of continuity is equivalent to the interpolativity of the function f_R°. We follow their ideas in Section III-B.
3) Klawonn and Castro [31] have proven two important and interesting results about the CRI scheme and the indistinguishability inherent to the fuzzy sets considered. First, the authors show that the indistinguishability induced by the antecedent fuzzy set of the rule cannot be overcome. Second, they have also demonstrated the *robustness* of fuzzy inference systems that employ the CRI mechanism in scenarios where there can be slight discrepancies between the intended input A' and the actual input $\widehat{A}'$, i.e., $f_R^\circ(A') = f_R^\circ(\widehat{A}')$. However, this study was done in the case of a single fuzzy rule, as in (1). We consider multiple fuzzy rules (3) and prove that, in such a case, this property holds when $R = \check{R}$. Then, we continue with an investigation of this property for the BK subproduct. Section III-C describes this issue.
4) In the case of CRI, if the input is a fuzzy singleton, then the output determined by the fuzzy adjoint function f_R° depends only on the relation R that models the given fuzzy rule base, i.e., the inference plays a role only in case of a nonsingleton fuzzy input. A detailed exposition of this topic is provided in Section IV-A.
5) While employing FRIs in a system that consists of multiple fuzzy rules, there are two inference strategies that are usually employed, viz., *first aggregate then infer* (FATI) and *first infer then aggregate* (FITA). In the case of the CRI mechanism, if the fuzzy rules are represented by the fuzzy relation $\check{R}$, then the FATI inference strategy is equivalent to the FITA inference strategy. However, this is not true if we employ $\hat{R}$ instead of $\check{R}$. More details on the FATI and FITA strategies, as well as their subsequent exploration, are provided in Section IV-B.
6) Finally, it should be noted that FRIs, in general, have their drawbacks due to space and time complexities. Many works have concentrated on increasing the efficiency of the inference process. However, so far, all these have been done only for the case of CRI mechanism and, especially, when the fuzzy rules are represented by the fuzzy relation $\check{R}$. The aforementioned properties will be dealt with in a more detailed way in Sections IV-C and D.

G. Motivation for This Paper: Study of the Bandler–Kohout Subproduct and the Relation $\hat{R}$

From the previous section, it is clear that most works have tended to concentrate predominantly on the CRI mechanism. However, two facts emerge from it.

On the one hand, the previous studies on CRI can also be conducted for other FRI mechanisms, and in this paper, we intend to perform a similar investigation into the BK subproduct. On the other hand, also note that some of the advantages available with the CRI mechanism depend, to a large extent, on the fuzzy rules being modeled by the fuzzy relation $\check{R}$, which, as already noted, is appropriate only in the context where the fuzzy rules are viewed as "positive" pieces of information [11], [17]. However, there are situations when the context dictates to view the fuzzy rules in the conditional nature, and the fuzzy relation $\hat{R}$ has to be used to model them [15]. Then, many of the advantages of the CRI are no longer available, viz., the robustness of the CRI mechanism with respect to the indistinguishability of input fuzzy sets in the case of multiple fuzzy rules, the equivalence of FATI and FATI, the many techniques that deal with enhancing the efficiency of the inference procedures.

In this paper, we intend to investigate the following. First, this paper shows that all the properties investigated/touted as an advantage of the CRI mechanism are also available for the BK subproduct and, often, under similar conditions or generality as available on CRI. Second, we also highlight that the conditional form of representation of a fuzzy rule base, in conjunction with the BK subproduct, i.e., $f_{\hat{R}}^{\triangleleft}$ is a strong alternative to $f_{\check{R}}^{\circ}$ in the appropriate contexts.

II. Mathematical Background of Fuzzy Inference Mechanisms

A. Fuzzy Inference Mechanisms

FRI mechanisms are mathematically based on a complete residuated lattice (see, e.g., [12]) that we fix for the whole paper as the basic algebraic structure. Let us only briefly recall that an algebra $\mathcal{L} = (L, \wedge, \vee, *, \rightarrow, 0, 1)$ is a residuated lattice if the following hold.

1) $(L, \wedge, \vee, 0, 1)$ is a lattice with the least and the greatest element.
2) $\mathcal{L} = (L, *, 0, 1)$ is a commutative monoid such that $*$ is isotone in both arguments.
3) The operation $\rightarrow$ is a residuation with respect to $*$, i.e.,

$$a * b \le c, \quad \text{iff } a \rightarrow c \ge b. \tag{10}$$

The following properties [12] are immediately available to us for any $a, b, c \in \mathcal{L}$:

$$a = 1 \rightarrow a \tag{11}$$

$$a \rightarrow c \ge b \rightarrow c, \quad \text{whenever } a \le b \tag{12}$$

$$a \rightarrow b \le a \rightarrow c, \quad \text{whenever } b \le c \tag{13}$$

$$a \rightarrow (b \rightarrow c) = (a * b) \rightarrow c = (b * a) \rightarrow c \tag{14}$$

$$(a \rightarrow b) \rightarrow b \ge a \vee b \tag{15}$$

$$a \rightarrow \bigwedge_{i \in \mathcal{I}} b_i = \bigwedge_{i \in \mathcal{I}} (a \rightarrow b_i) \tag{16}$$

$$\bigvee_{i \in \mathcal{I}} (a_i \rightarrow b) \le \bigwedge_{i \in \mathcal{I}} a_i \rightarrow b \tag{17}$$

$$\bigvee_{i \in \mathcal{I}} a_i \rightarrow b = \bigwedge_{i \in \mathcal{I}} (a_i \rightarrow b) \tag{18}$$

$$(a \rightarrow b) * c \le a \rightarrow (b * c). \tag{19}$$

Let us fix the set $L = [0, 1]$ for the whole paper. Then, $*$ becomes a left-continuous t-norm, and $\rightarrow$ becomes a residual fuzzy implication that is derived from $*$. For more details on these operations, see [12], [22], and [32].

In $\mathcal{L}$, we can derive yet another operation that is known as the *biresiduum* and is defined as follows:

$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a), \quad a, b \in \mathcal{L}. \tag{20}$$

The following properties of the biresiduum will be useful in the sequel (see [12]):

$$(a \leftrightarrow b) * (b \leftrightarrow c) \le a \leftrightarrow c \tag{21}$$

$$(a \leftrightarrow b) \wedge (c \leftrightarrow d) \le (a \wedge c) \leftrightarrow (b \wedge d). \tag{22}$$

Finally, by extending an algebraic operation from $\mathcal{L}$ to operations between fuzzy sets, we mean the following:

$$(C \star D)(u, v) = C(u) \star D(v), \quad u \in U, \ v \in V \tag{23}$$

where $\star \in \{\wedge, \vee, *, \rightarrow, \leftrightarrow\}$ and for arbitrary fuzzy sets C, D on arbitrary universes U, V, respectively.

B. Inference Strategies

Now, there are two inference strategies that are called FITA and FATI (see [33]).

In FITA strategy, we first construct individual fuzzy relations $R_i \in \mathcal{F}(X \times Y)$ from each of the n fuzzy rules. Then, the given fuzzy observation $A' \in \mathcal{F}(X)$ is composed with each of these relations R_i by a chosen inference @, and the obtained individual output fuzzy sets $B'_i = A' @ R_i \in \mathcal{F}(Y)$ are then aggregated to form the final output fuzzy set $B' \in \mathcal{F}(Y)$.

In FATI strategy, the individual fuzzy relations $R_i \in \mathcal{F}(X \times Y)$ from each of the n fuzzy rules is aggregated into a single fuzzy relation $R \in \mathcal{F}(X \times Y)$, and this is composed with the given fuzzy observation $A' \in \mathcal{F}(X)$ to obtain the fuzzy output $B' = A' @ R \in \mathcal{F}(Y)$.

III. Desirable Properties of Inference Mechanisms

A. Interpolativity of Fuzzy Inference Systems—Property 1

The interpolativity $f_R^{@}(A_i) = B_i$ is a fundamental property of any inference mechanism. In this case, we say that R is a *correct model* of given fuzzy rules in the given structure S. This leads us to deal with a system of fuzzy relation equations [34], where

$$A_i @ R = B_i, \quad i = 1, \ldots, n \tag{24}$$

is solved with respect to the known $A_i \in \mathcal{F}(X)$ and $B_i \in \mathcal{F}(Y)$ and unknown $R \in \mathcal{F}(X \times Y)$. If R is a solution to (24), then the adjoint fuzzy function fulfills $f_R^{@}(A_i) = B_i$.

In the case of CRI, the previous system of equations reduces to the following:

$$A_i \circ R = B_i, \quad i = 1, \ldots, n. \tag{25}$$

Let us recall some main results that may be found, e.g., in [29] and [34]–[36].

Theorem 3.1: System (25) is solvable if and only if $\hat{R}$ is a solution of the system, and moreover, $\hat{R}$ is the greatest solution of (25).

On the one hand, Theorem 3.1 states the necessary and sufficient condition for the solvability of system (25), and it determines the solution. Moreover, it ensures that the given solution is the greatest one. On the other hand, we still do not know when $\hat{R}$ is the solution, i.e., how the solvability can be ensured.

Theorem 3.2 [36]: Let A_i, for $i = 1, \ldots, n$ be normal.[2] Then, $\check{R}$ is a solution of (25) if and only if the following condition holds for arbitrary $i, j \in \{1, \ldots, n\}$:

$$\bigvee_{x \in X} (A_i(x) * A_j(x)) \le \bigwedge_{y \in Y} (B_i(y) \leftrightarrow B_j(y)). \tag{26}$$

[2]Let us recall that a fuzzy set A on a universe U is called *normal* if there exists an $x \in U$ such that $A(x) = 1$.

Theorem 3.2 specifies a sufficient condition under which the system is solvable, and moreover, it ensures that not only $\hat{R}$ but also $\check{R}$ is a solution of system (25).

It is worth mentioning that condition (26) that appears in Theorem 3.2 is not very convenient in practice. Another sufficient condition for solvability of the systems with a high practical importance was published in [39] and [40].

Theorem 3.3: Let A_i, for $i = 1, \ldots, n$, be normal and fulfill the Ruspini condition

$$\sum_{i=1}^{n} A_i(x) = 1, \quad x \in X. \tag{27}$$

Then, the system (25) is solvable.

Remark 3.4: Besides the case when antecedent fuzzy sets form a fuzzy partition that fulfills the Ruspini condition, the so-called $*$-semipartition [37] also plays an important role.

Let us recall the state of the art that concerns the BK subproduct and the interpolativity issue. In the case of the BK subproduct, the system of equations (24) reduces to the following:

$$A_i \triangleleft R = B_i, \quad i = 1, \ldots, n. \tag{28}$$

Concerning system (28), let us recall the following two basic theorems from Pedrycz [23] and Nosková [38].

Theorem 3.5 [23]: System (28) is solvable if and only if $\check{R}$ is a solution of the system, and moreover, $\check{R}$ is the least solution of system (28).

Theorem 3.6 [38, Th. 2]: Let A_i, for $i = 1, \ldots, n$, be normal. Then, $\hat{R}$ is a solution of (28) if and only if condition (26) holds for arbitrary $i, j \in \{1, \ldots, n\}$.

Again, condition (26) to which Theorem 3.6 refers is, on one hand, very transparent, but not very convenient from a practical point of view. Fortunately, the sufficient condition for solvability of the system, with a high practical importance, which is stated in Theorem 3.3, is valid even for system (28) (see [39] and [40]).

Theorem 3.7: Let A_i, for $i = 1, \ldots, n$, be normal and fulfill the Ruspini condition (27). Then, the system (28) is solvable.

We may easily observe that each of the important results that is well known for the interpolativity issue in case of CRI has its analogy, even for the BK-subproduct case under exactly the same conditions, and therefore, both inferences are equally appropriate from this point of view.

In fact, the availability of such results for the BK subproduct was also one of the motivations to conduct this study of the BK-subproduct inference vís-á-vís the already-known advantageous properties of CRI.

B. *Continuity of Fuzzy Inference Systems—Property 2*

Perfilieva and Lehmke [28] and Perfilieva and Novák [30] have dealt with the continuity of a fuzzy function f_R° that is adjoint to the CRI mechanism and a fuzzy relation that models fuzzy rules (3). They have suitably defined continuity and have shown that it is equivalent to the correctness of the model that is under consideration.

Although the original definition in [28] of a continuous model was given for the particular inference mechanism CRI, i.e., for $@ \equiv \circ$, the particular composition plays no role in the proof of the result (see Theorem 3.11), thereby explaining the nature of the definition. Hence, it can be generalized for an arbitrary fuzzy relational composition.

Definition 3.8: A fuzzy relation $R \in \mathcal{F}(X \times Y)$ is said to be a *continuous* model of fuzzy rules (3) in a structure $S = (X, Y, \{A_i, B_i\}_{i=1,\ldots,n}, \mathcal{L}, @)$ if, for each $i \in I$ and for each $A \in \mathcal{F}(X)$, the following inequality holds:

$$\bigwedge_{y \in Y} (B_i(y) \leftrightarrow (A@R)(y)) \geq \bigwedge_{x \in X} (A_i(x) \leftrightarrow A(x)). \tag{29}$$

Remark 3.9: Inequality (29) can be rewritten in terms of the adjoint fuzzy function $f_R^{@}$ as follows:

$$\bigwedge_{y \in Y} (B_i(y) \leftrightarrow (f_R^{@}(A))(y)) \geq \bigwedge_{x \in X} (A_i(x) \leftrightarrow A(x)). \tag{30}$$

To be precise, the continuity concerns the fuzzy function $f_R^{@}$, i.e., the model of fuzzy rules R as well as the inference mechanism $@$. Therefore, we could, more adequately, talk about continuity of the whole fuzzy inference systems rather than just about the continuity of R. Nevertheless, since Definition 3.8 fixes a structure in which the continuity is introduced, it is mathematically correct, and we adopt the original terminology from [28].

Remark 3.10: Let us explain why (29) expresses the continuity. The closeness between fuzzy sets is measured by the biresiduation operation $\leftrightarrow$, i.e., it is a dual concept to the metric one. Let us consider a continuous Archimedean t-norm $*$ with an additive generator $g : [0, 1] \to [0, +\infty]$. Then, the biresiduum may be written in the form

$$a \leftrightarrow b = g^{-1}(|g(a) - g(b)|) \tag{31}$$

where $g^{-1} : [0, \infty] \to [0, 1]$ is the inverse function and where, in the case of $g(0) = \infty$, we define $g(0) - g(0) = 0$. Now, for an arbitrary nonempty universe X, it is possible to define a metric D_g on $\mathcal{F}(X)$ that is generated by g as follows:

$$D_g(A, B) = \bigvee_{x \in X} |g(A(x)) - g(B(x))|. \tag{32}$$

The following theorem justifies the use of the notion of continuity in Definition 3.8.

Theorem 3.11: Let $S = (X, Y, \{A_i, B_i\}_{i=1,\ldots,n}, \mathcal{L}, @)$ be a structure for fuzzy rules (3) such that $\mathcal{L}$ is a residuated lattice on $[0, 1]$, with a continuous Archimedean t-norm $*$ having a continuous additive generator g. A fuzzy relation $R \in \mathcal{F}(X \times Y)$ is a continuous model of the fuzzy rules in the given structure S if and only if

$$D_g(B_i, (A@R)) \leq D_g(A_i, A), \quad i = 1, \ldots, n \tag{33}$$

for each fuzzy set $A \in \mathcal{F}(X)$.

Proof: The proof for $@ \equiv \circ$ may be found in [28]. Its generalization is straightforward. ∎

The main result by Perfilieva and Lehmke [28] and Perfilieva and Novák [30] concerns the relationship of the aforementioned continuity and the interpolativity for the CRI as contained in the following result.

Theorem 3.12 [28, Th. 2]: Let $S = (X, Y, \{A_i, B_i\}_{i=1,\ldots,n}, \mathcal{L}, \circ)$ be a structure for fuzzy rules (3). A fuzzy relation

$R \in \mathcal{F}(X \times Y)$ is a *correct* model of fuzzy rules (3) in the given structure S if and only if it is a continuous model of these rules in S.

In the following, we show that an identical result is valid even for the case of the BK subproduct. Let us start by proving the following lemma that is crucial for further results.

Lemma 3.13: Let $S = (X, Y, \{A_i, B_i\}_{i=1,\dots,n}, \mathcal{L}, \triangleleft)$ be a structure for fuzzy rules (3), and let $R \in \mathcal{F}(X \times Y)$. Then, for any $A \in \mathcal{F}(X)$ and all $i = 1, \dots, n$, as well as $y \in Y$, it is true that

$$B_i(y) \leftrightarrow (A \triangleleft R)(y) \geq \delta_{R,i}(y) * \bigwedge_{x \in X} (A_i(x) \leftrightarrow A(x)) \quad (34)$$

where $\delta_{R,i}(y) = B_i(y) \leftrightarrow (A_i \triangleleft R)(y)$.

Proof: By the transitivity property (21) of $\leftrightarrow$ with respect to $*$, we get

$$\begin{aligned} &B_i(y) \leftrightarrow (A \triangleleft R)(y) \\ &\quad \geq (B_i(y) \leftrightarrow (A_i \triangleleft R)(y)) * ((A_i \triangleleft R)(y) \leftrightarrow (A \triangleleft R)(y)) \end{aligned}$$

where $i \in \{1, \dots, n\}$.

The first multiplicand $B_i(y) \leftrightarrow (A_i \triangleleft R)(y)$ is equal to $\delta_{R,i}(y)$, while the second multiplicand, for an arbitrary $y \in Y$, has the following lower bound:

$$\begin{aligned} &(A \triangleleft R)(y) \leftrightarrow (A_i \triangleleft R)(y) \\ &= \bigwedge_{x \in X} (A(x) \to R(x,y)) \leftrightarrow \bigwedge_{x \in X} (A_i(x) \to R(x,y)) \\ &\geq \bigwedge_{x \in X} ((A(x) \to R(x,y)) \leftrightarrow (A_i(x) \to R(x,y))) \quad \lceil \text{By (22)} \\ &= \bigwedge_{x \in X} (([A(x) \to R(x,y)] \to [A_i(x) \to R(x,y)]) \\ &\quad \wedge ([A_i(x) \to R(x,y)] \to [A(x) \to R(x,y)])) \\ &= \bigwedge_{x \in X} (([A_i(x) * (A(x) \to R(x,y))] \to R(x,y)) \\ &\quad \wedge ([A(x) * (A_i(x) \to R(x,y))] \to R(x,y))) \quad \lceil \text{By (14)} \\ &= \bigwedge_{x \in X} ((A_i(x) \to [(A(x) \to R(x,y)) \to R(x,y)]) \\ &\quad \wedge (A(x) \to [(A_i(x) \to R(x,y)) \to R(x,y)])) \quad \lceil \text{By (14)} \\ &\geq \bigwedge_{x \in X} ([A_i(x) \to A(x)] \wedge [A(x) \to A_i(x)]) \quad \lceil \text{By (15)} \\ &= \bigwedge_{x \in X} (A_i(x) \leftrightarrow A(x)). \end{aligned}$$

Now, (34) follows immediately from the monotonicity of $*$. ■

Due to Lemma 3.13, we may prove the following theorem that is analogous to Theorem 3.12. It, again, shows that the BK subproduct as an inference mechanism has the same property as the CRI.

Theorem 3.14: Let $S = (X, Y, \{A_i, B_i\}_{i=1,\dots,n}, \mathcal{L}, \triangleleft)$ be a structure for fuzzy rules (3). A fuzzy relation $R \in \mathcal{F}(X \times Y)$ is a correct model of fuzzy rules (3) in the given structure S if and only if it is a continuous model of these rules in S.

Proof: Suppose R is a correct model of the fuzzy rules (3) in the given structure. Then, R solves the given system of fuzzy relation equations $A_i \triangleleft R = B_i$ for all $i = 1, \dots, n$, and therefore, $\delta_{R,i}(y) = 1$ for all $i = 1, \dots, n$ and for all $y \in Y$. By (34), R is a continuous model.

Conversely, let R be a continuous model of (3) in the given structure. Then

$$\bigwedge_{y \in Y} (B_i(y) \leftrightarrow (A \triangleleft R)(y)) \geq \bigwedge_{x \in X} (A_i(x) \leftrightarrow A(x)) \quad (35)$$

holds for each $i = 1, \dots, n$ and for arbitrary $A \in \mathcal{F}(X)$. Substituting $A \equiv A_i$ into (35), we obtain

$$\bigwedge_{y \in Y} (B_i(y) \leftrightarrow (A_i \triangleleft R)(y)) \geq 1$$

which implies that $A_i \triangleleft R \equiv B_i$. ■

As in the case of interpolativity, the continuity property is present in both the types of inferences under exactly the same conditions.

C. Robustness of Fuzzy Inference Systems—Property 3

Let X be a classical set, and let $\sim$ be an equivalence relation that is defined on X, i.e., $\sim$ is reflexive, symmetric, and transitive. Immediately, $\sim$ partitions X into equivalence classes. It is well known that an $M \subseteq X$ belongs to this partition if and only if whenever $x \in M$ and $x \sim y$ for some $y \in X$, then $y \in M$. In a sense, the elements of M are indistinguishable and can be represented mathematically as follows:

$$x \in M \text{ and } x \sim y \text{ implies } y \in M.$$

A similar relation between fuzzy equivalence relations and fuzzy sets on X was introduced by Klawonn and Castro [31]. The operation $*$ comes from the residuated lattice $\mathcal{L}$.

Definition 3.15: A fuzzy subset E of the Cartesian product X^2 is called a *fuzzy equivalence* relation on X if the following properties are satisfied for all $x, y, z \in X$:

$$\text{(Reflexivity)} \quad E(x,x) = 1 \quad \text{(ER)}$$

$$\text{(Symmetry)} \quad E(x,y) = E(y,x) \quad \text{(ES)}$$

$$\text{(Transitivity)} \quad E(x,z) \geq E(x,y) * E(y,z). \quad \text{(ET)}$$

Definition 3.16: A fuzzy set $\mu \in \mathcal{F}(X)$ is called *extensional* with respect to a fuzzy equivalence relation E on X if

$$\mu(x) * E(x,y) \leq \mu(y), \quad x, y \in X. \quad (36)$$

If a fuzzy set μ is not extensional with respect to the considered fuzzy equivalence relation E, the smallest fuzzy set is instead considered, which is extensional with respect to E and contains μ.

Definition 3.17: Let $\mu \in \mathcal{F}(X)$, and let E be a fuzzy equivalence relation on X. The fuzzy set

$$\widehat{\mu}(x) = \bigwedge \{\nu \mid \mu \leq \nu \text{ and } \nu \text{ is extensional with respect to } E\}$$

is called the *extensional hull* of μ. Note that by $\mu \leq \nu$, we mean that for all $x \in X$, $\mu(x) \leq \nu(x)$, i.e., we mean ordering in the sense of inclusion and not in the sense of ordering fuzzy quantities.

Proposition 3.18 [31, Proposition 2.9]: Let $\mu \in \mathcal{F}(X)$, and let E be a fuzzy equivalence relation on X. Then, the following hold.

1) $\widehat{\mu}(x) = \bigvee\{\mu(y) * E(x, y) \mid y \in X\}$.
2) $\widehat{\mu}$ is extensional with respect to E.
3) $\widehat{\widehat{\mu}} = \widehat{\mu}$.

The following two important and interesting results about the CRI scheme and the considered indistinguishability that is inherent to the fuzzy sets are proved in [31].

Theorem 3.19 [31, Th. 4.4]: Let $S = (X, Y, \{A, B\}, \mathcal{L}, \circ)$ be a structure for a single fuzzy IF–THEN rule as given in (1). Let E be a fuzzy equivalence relation on X with respect to which A is extensional. Letting $A' \in \mathcal{F}(X)$ be any fuzzy set, then

$$A' \circ \hat{R} = \widehat{A'} \circ \hat{R}$$
$$A' \circ \check{R} = \widehat{A'} \circ \check{R}.$$

The following interpretation of the previous result is given in [31]: *The output obtained from CRI for a given fuzzy rule and an input fuzzy set A' does not change if we substitute A' by its extensional hull $\widehat{A'}$. The indistinguishability inherent in the fuzzy set A cannot be avoided, even if the input fuzzy set A' stands for a crisp value*. Further, *a fuzzified input does not change the outcome of a rule as long as the fuzzy set obtained by the fuzzification is contained in the extensional hull of the original crisp input value.* They finally conclude that *it does not make sense to measure more exactly than the indistinguishability admits.*

In other words, this shows the robustness of the inference in scenarios where there can be slight discrepancies between the intended input and the actual input.

As already observed in [31], it immediately follows that *the indistinguishability induced by the fuzzy set that represents the linguistic expression in the premise of the rule cannot be overcome.*

Even though Theorem 3.19 is proven in [31] for a single fuzzy rule, it can be shown that the result is valid even with n fuzzy rules.

Theorem 3.20: Let $S = (X, Y, \{A_i, B_i\}_{i=1,\ldots,n}, \mathcal{L}, \circ)$ be a structure for fuzzy rules (3). Let E be a fuzzy equivalence relation on X with respect to which A_i is extensional for arbitrary $i = 1, \ldots, n$. Letting $A' \in \mathcal{F}(X)$ be any fuzzy set, then $A' \circ \check{R} = \widehat{A'} \circ \check{R}$.

Proof: The inequality $\widehat{A'} \circ R \geq A' \circ R$ holds for arbitrary fuzzy relation $R \in \mathcal{F}(X \times Y)$.

Thus, it suffices to prove the other inequality for $\check{R}$, i.e., $\widehat{A'} \circ \check{R} \leq A' \circ \check{R}$. Since A_i is extensional with respect to E for arbitrary i, $A_i(x') \geq A_i(x) * E(x, x')$ for any $x, x' \in X$.

For any $x \in X$, we have

$$\widehat{A'}(x) * \bigvee_{i=1}^{n}(A_i(x) * B_i(y))$$
$$= \bigvee_{x' \in X}(A'(x') * E(x, x')) * \bigvee_{i=1}^{n}(A(x) * B(y))$$
$$= \bigvee_{x' \in X}\left([A'(x') * E(x, x')] * \bigvee_{i=1}^{n}(A_i(x) * B_i(y))\right)$$
$$= \bigvee_{i=1}^{n}\bigvee_{x' \in X}([A'(x') * E(x, x')] * (A_i(x) * B_i(y)))$$
$$= \bigvee_{i=1}^{n}\bigvee_{x' \in X}(A'(x') * [E(x, x') * A_i(x)] * B_i(y))$$
$$\leq \bigvee_{i=1}^{n}\bigvee_{x' \in X}(A'(x') * (A_i(x') * B_i(y)))$$

which implies

$$(\widehat{A'} \circ \check{R})(y) \leq (A' \circ \check{R})(y), \quad y \in Y. \qquad \blacksquare$$

It should be emphasized that only the Mamdani–Assilian (i.e., Cartesian product) approach $\check{R}$ generally works in combination with the CRI. We now show the robustness of the BK-subproduct inference mechanism along similar lines as by Klawonn and Castro [31]. Once again, the employed operations come from the residuated lattice $\mathcal{L}$. First, note that if a fuzzy set $\mu \in \mathcal{F}(X)$ is extensional with respect to a fuzzy equivalence relation E on X, then

$$E(x, y) \to \mu(y) \geq \mu(x), \quad x, y \in X. \tag{37}$$

Proposition 3.21: Let $\mu \in \mathcal{F}(X)$, and let E be a fuzzy equivalence relation on X. Then

$$\widehat{\mu}(x) = \bigwedge\{E(x, y) \to \mu(y) \mid y \in X\}. \tag{38}$$

Proof: Let $\tilde{\mu}(y) = \bigwedge\{E(z, y) \to \mu(z) \mid z \in X\}$. We only need to show that $\tilde{\mu} = \widehat{\mu}$. First, note that for any $x \in X$, we have

$$\tilde{\mu}(x) = \bigwedge\{E(z, x) \to \mu(z) \mid z \in X\} \leq E(x, x) \to \mu(x)$$
$$= 1 \to \mu(x) = \mu(x) \leq \widehat{\mu}(x).$$

Let $\nu \in \mathcal{F}(X)$ be extensional with respect to E such that $\nu \geq \mu$, which implies, by definition, that $\widehat{\mu} \leq \nu$. Then, for any $x \in X$, we have

$$\nu(x) \leq E(z, x) \to \nu(z) \quad \text{and} \quad \nu(x) \leq E(z, x) \to \mu(z)$$

for every $z \in X$, and therefore

$$\nu(x) \leq \bigwedge\{E(z, x) \to \mu(z) \mid z \in X\} = \tilde{\mu}(x)$$

i.e., $\tilde{\mu}(x) \geq \widehat{\mu}(x)$, and hence, $\tilde{\mu}(x) = \widehat{\mu}(x)$. $\blacksquare$

Now, we present a result that is analogous to the one given in Theorem 3.19.

Theorem 3.22: Let $S = (X, Y, \{A, B\}, \mathcal{L}, \triangleleft)$ be a structure for fuzzy rule (1). Let E be a fuzzy equivalence relation on X with respect to which A is extensional. Let $A' \in \mathcal{F}(X)$ be any fuzzy set, then

$$A' \triangleleft \hat{R} = \widehat{A'} \triangleleft \hat{R}, \quad A' \triangleleft \check{R} = \widehat{A'} \triangleleft \check{R}.$$

Proof: Let $R \in \mathcal{F}(X \times Y)$ be *any* fuzzy relation. By the definition of $\widehat{A}'$, we have the following inequalities:

$$\begin{aligned} \widehat{A}' \geq A' &\Longrightarrow \widehat{A}' \to R \leq A' \to R \ \lceil \text{By (12)} \\ &\Longrightarrow \widehat{A}' \lhd R \leq A' \lhd R. \end{aligned}$$

Thus, it suffices to prove the other inequality, i.e., $\widehat{A}' \lhd R \geq A' \lhd R$.

Let $R = \hat{R}$, i.e., $R(x,y) = A(x) \to B(y)$ for any $x \in X$, $y \in Y$. Then, by definition, we have

$$(\widehat{A}' \lhd \hat{R})(y) = \bigwedge_{x \in X} (\widehat{A}'(x) \to (A(x) \to B(y))), \quad y \in Y.$$

Since A is extensional with respect to E, $A(x') \geq A(x) * E(x,x')$ for any $x, x' \in X$, and, by (12)

$$A(x') \to B(y) \leq (A(x) * E(x,x')) \to B(y), \quad y \in Y. \tag{39}$$

For any $x \in X$, we have

$$\begin{aligned} &\widehat{A}'(x) \to (A(x) \to B(y)) \\ &= \bigvee_{x' \in X} (A'(x') * E(x,x')) \to (A(x) \to B(y)) \\ &= \bigwedge_{x' \in X} ([A'(x') * E(x,x')] \to (A(x) \to B(y))) \ \lceil \text{By (18)} \\ &= \bigwedge_{x' \in X} (A'(x') \to [E(x,x') \to (A(x) \to B(y))]) \ \lceil \text{By (14)} \\ &= \bigwedge_{x' \in X} (A'(x') \to ((E(x,x') * A(x)) \to B(y))) \ \lceil \text{By (14)} \\ &\geq \bigwedge_{x' \in X} (A'(x') \to (A(x') \to B(y))) \ \lceil \text{By (40)} \end{aligned}$$

which implies

$$\begin{aligned} &(\widehat{A}' \lhd \hat{R})(y) \\ &\quad = \bigwedge_{x \in X} (\widehat{A}'(x) \to (A(x) \to B(y))) \\ &\quad \geq \bigwedge_{x' \in X} (A'(x') \to (A(x') \to B(y))) = (A' \lhd \hat{R})(y) \end{aligned}$$

for any $y \in Y$.

Let $R = \check{R}$, i.e., $R(x,y) = A(x) * B(y)$ for any $x \in X$, $y \in Y$. Then, by definition, we have

$$(\widehat{A}' \lhd \check{R})(y) = \bigwedge_{x \in X} (\widehat{A}'(x) \to (A(x) * B(y))), \quad y \in Y.$$

For any $x \in X$, we have

$$\begin{aligned} &\widehat{A}'(x) \to (A(x) * B(y)) \\ &= \bigvee_{x' \in X} (A'(x') * E(x,x')) \to (A(x) * B(y)) \\ &= \bigwedge_{x' \in X} ((A'(x') * E(x,x')) \to (A(x) * B(y))) \ \lceil \text{By (18)} \\ &= \bigwedge_{x' \in X} (A'(x') \to [E(x,x') \to (A(x) * B(y))]) \ \lceil \text{By (14)} \end{aligned}$$

which, by (13) and (19), is

$$\geq \bigwedge_{x' \in X} (A'(x') \to ((E(x,x') \to A(x)) * B(y)))$$

and, by (16),

$$\begin{aligned} &= \bigwedge_{x' \in X} \left(A'(x') \to \bigwedge_{x \in X} ((E(x,x') \to A(x)) * B(y)) \right) \\ &= \bigwedge_{x' \in X} \left(A'(x') \to \left(\bigwedge_{x \in X} (E(x,x') \to A(x)) * B(y) \right) \right) \\ &= \bigwedge_{x' \in X} (A'(x') \to (\widehat{A}(x') * B(y))) \ \lceil \text{By (38)} \\ &= \bigwedge_{x' \in X} (A'(x') \to (A(x') * B(y))) \lceil \because \widehat{A} = A \end{aligned}$$

which implies that $(\widehat{A}' \lhd \check{R})(y) = (A' \lhd \check{R})(y)$ for any $y \in Y$. ■

The previous result, as already noted in the case of CRI, shows the robustness of the BK-subproduct inference in scenarios where there can be slight discrepancies between the intended input and the actual input, and this reinforces the fact that even in the case of BK subproduct, the indistinguishability that is induced by the fuzzy set that represents the linguistic expression in the premise of the rule cannot be overcome.

Once again, as in the case of CRI, we may generalize the result that concerns the indistinguishability of the premises for an arbitrary finite number of rules. Note that in the case of the BK subproduct, the fuzzy relation $\hat{R}$ plays the main role.

Theorem 3.23: Let $S = (X, Y, \{A_i, B_i\}_{i=1,\ldots,n}, \mathcal{L}, \lhd)$ be a structure for fuzzy rules (3). Let E be a fuzzy equivalence relation on X, with respect to which each A_i is extensional, for arbitrary $i = 1, \ldots, n$. Letting $A' \in \mathcal{F}(X)$ be any fuzzy set, then

$$A' \lhd \hat{R} = \widehat{A}' \lhd \hat{R}.$$

Proof: The inequality $\widehat{A}' \lhd R \leq A' \lhd R$ holds for arbitrary fuzzy relation $R \in \mathcal{F}(X \times Y)$; see the proof of Theorem 3.22.

Thus, it suffices to prove the other inequality for $\hat{R}$, i.e., $\widehat{A}' \lhd \hat{R} \geq A' \lhd \hat{R}$. Since each A_i is extensional with respect to E for arbitrary i, $A_i(x') \geq A_i(x) * E(x,x')$ for any $x, x' \in X$, and, by (12), we have

$$A_i(x') \to B_i(y) \leq (A_i(x) * E(x,x')) \to B_i(y), \quad y \in Y. \tag{40}$$

For any $x \in X$, we have

$$\begin{aligned} &\widehat{A}'(x) \to \bigwedge_{i=1}^{n} (A_i(x) \to B_i(y)) \\ &\quad = \bigvee_{x' \in X} (A'(x') * E(x,x')) \to \bigwedge_{i=1}^{n} (A_i(x) \to B_i(y)) \end{aligned}$$

which, by (18), (16), (14), and (40), is

$$= \bigwedge_{x'\in X}\left([A'(x') * E(x,x')] \to \bigwedge_{i=1}^{n}(A_i(x) \to B_i(y))\right)$$
$$= \bigwedge_{i=1}^{n}\bigwedge_{x'\in X}([A'(x') * E(x,x')] \to (A_i(x) \to B_i(y)))$$
$$= \bigwedge_{i=1}^{n}\bigwedge_{x'\in X}(A'(x') \to [E(x,x') \to (A_i(x) \to B_i(y))])$$
$$= \bigwedge_{i=1}^{n}\bigwedge_{x'\in X}(A'(x') \to ((E(x,x') * A_i(x)) \to B_i(y)))$$
$$\geq \bigwedge_{i=1}^{n}\bigwedge_{x'\in X}(A'(x') \to (A_i(x') \to B_i(y)))$$

which implies $(\widehat{A}' \lhd \hat{R})(y) \geq (A' \lhd \hat{R})(y),\ y \in Y$. ■

IV. Computational Aspects of Fuzzy Relational Inferences

In this section, we deal with the computational aspects of CRI and BK-subproduct inferences. First, we show that all the advantages of CRI are also available with the BK subproduct. However, both CRI and BK subproduct—these being FRIs—possess some drawbacks. Recently, Jayaram [41] proposed a modified form of CRI, viz., an hierarchical CRI scheme, to overcome some of these drawbacks. We show that a similar hierarchical inferencing is possible even in the case of BK subproduct and, hence, is a computationally viable alternative for the CRI.

A. Inferencing in the Case of Singleton Fuzzy Inputs—Property 4

The following definition will be useful in this section.

Definition 4.1: A fuzzy set on a nonempty set X, $A : X \to [0,1]$ is said to be a "fuzzy singleton" if there exists an $x_0 \in X$ such that A has the following representation:

$$A(x) = \begin{cases} 1, & \text{if } x = x_0 \\ 0, & \text{if } x \neq x_0. \end{cases} \tag{41}$$

We say that A attains normality at $x_0 \in X$.

It is common in some contexts to deal with fuzzy singleton inputs. For instance, in typical control situations, the input is usually a crisp value that is fuzzified before it is presented to a fuzzy system to obtain the output. There are many fuzzification methods, i.e., procedures to convert a crisp value into a fuzzy set with different shapes and spread based on this value. Most often, the *singleton fuzzifier* that converts a crisp input $x' \in X$ into a singleton $A' \in \mathcal{F}(X)$, which attains normality at x', is used. Note that this seemingly formal conversion is crucial since it allows any FRI to be applied, which, in principle, deals only with fuzzy inputs.

Given a crisp input $x' \in X$, which is fuzzified using a singleton fuzzifier, from a computational point of view, it is highly desirable to deal with such FRI mechanisms whose inferred output is dependent only on the chosen fuzzy relation R that models a given fuzzy rule base, and the inference plays a role only in case of a fuzzy input, i.e., the inferred output $f_R^{@}(A') = B' \in \mathcal{F}(Y)$ is given by $B'(y) = R(x', y)$, for arbitrary $y \in Y$.

This property, which saves computational costs, holds for CRI. From the following equalities, we see that the discussed property is valid even for the BK subproduct. Let the given singleton fuzzy input A' attain normality at some $x' \in X$. Then, the inferred output using the BK subproduct is given by

$$B'(y) = \bigwedge_{x\in X}(A'(x) \to R(x,y))$$
$$= (A'(x') \to R(x',y)) \wedge \bigwedge_{\substack{x\in X\\ x\neq x'}}(A'(x) \to R(x,y))$$
$$= (1 \to R(x',y)) \wedge \bigwedge_{\substack{x\in X\\ x\neq x'}}(0 \to R(x,y))$$
$$= R(x',y) \wedge 1 = R(x',y),\ y \in Y.$$

It should be emphasized that this property is generally not valid for any fuzzy relational composition. Neither the BK superproduct nor the BK square product [24], [42] nor any of the $\inf$-S fuzzy relational compositions retain this essential property. This is one of the reasons why the BK subproduct is, other than the CRI, a privileged composition and gives a clear motivation to study all the properties that are investigated in Section III.

B. Equivalence Between FITA and FATI—Property 5

It is a well-known fact that one of the reasons to use the Cartesian product approach to model a fuzzy rule base is that it, possibly, saves computational efforts. In other words a combination of $\circ$ with $\check{R}$ requires fewer computations than the combination with the fuzzy relation $\hat{R}$. This is due to the following sequence of equalities:

$$B(y) = \bigvee_{x\in X}\left(A'(x) * \bigvee_{i=1}^{n}(A_i(x) * B_i(y))\right)$$
$$= \bigvee_{x\in X}\bigvee_{i=1}^{n}(A'(x) * (A_i(x) * B_i(y)))$$
$$= \bigvee_{i=1}^{n}\bigvee_{x\in X}((A'(x) * A_i(x)) * B_i(y))$$
$$= \bigvee_{i=1}^{n}\left(\bigvee_{x\in X}(A'(x) * A_i(x)) * B_i(y)\right), \quad y \in Y.$$

This means that we do not have to compose all rules to a fuzzy relation; we just find the highest degree of intersection of a given input A' and a particular rule antecedent and multiply it by the corresponding consequent. This approach is, then, applied rule per rule, and the results are composed together by the maximum operation. This reduction in the computational costs is the effect of the equivalence of FITA and FATI inference strategies [2].

Therefore, it may, again, be generally considered even for other inference mechanisms. Due to the following sequence of equalities:

$$
\begin{aligned}
B(y) &= \bigwedge_{x\in X}\left(A'(x) \to \bigwedge_{i=1}^{n}(A_i(x) \to B_i(y))\right)\\
&= \bigwedge_{x\in X}\bigwedge_{i=1}^{n}(A'(x) \to (A_i(x) \to B_i(y)))\\
&= \bigwedge_{i=1}^{n}\bigwedge_{x\in X}((A'(x) * A_i(x)) \to B_i(y))\\
&= \bigwedge_{i=1}^{n}\left(\bigvee_{x\in X}(A'(x) * A_i(x)) \to B_i(y)\right), \quad y \in Y
\end{aligned}
$$

we may state that even in the case of the BK subproduct $\lhd$ and an appropriate model of fuzzy rules (3) that are given by $\hat{R}$, the FATI inference strategy is equivalent to the FITA inference strategy.

The difference lies in the propriety of the chosen fuzzy relation that models fuzzy rules (3) with respect to a chosen inference mechanism. In the case of CRI, there is no other choice but $\check{R}$ if we want to reduce the computational efforts by an equivalent FITA strategy. Whereas in case of the BK subproduct, the same holds for $\hat{R}$. In other words, there should be other reasons that lead to the use of $\check{R}$ than the computational costs because if this is the only reason, we may still reduce the computational efforts by using the BK subproduct as an inference mechanism, while still keeping the conditional nature of rules (3) by the choice of $\hat{R}$. Then, the correctness of the model (i.e., fundamental interpolation condition) should be ensured by the conditions that are given in Theorem 3.6. On the other hand, this is also the case when $\check{R}$ is employed where the same conditions have to be imposed (see Theorem 3.2).

Remark 4.2: Note that the expression $\bigvee_{x\in X}(A(x) * A'(x))$, with $*$ being a t-norm, is, in fact, one of the earliest measures that was proposed by Zadeh [43] to determine the similarity between two fuzzy sets $A, A' \in \mathcal{F}(X)$. Moreover, typically, antecedent fuzzy sets form some partition (e.g., the Ruspini partition [44]), and the input fuzzy set is of a limited support so that there are such i for which the similarity between the input A' and the antecedent A_i, given by the earlier measure, is zero, and the computation gets simplified further.

C. *Drawbacks of a Fuzzy Relational Inference*

So far, we have considered only single-input–single-output (SISO) fuzzy rules. However, in practice, often, such situations are encountered that demand that inputs from multiple sources/dimensions be considered, and hence, the need arises to deal with multiple-input–single-output (MISO) fuzzy rules of the following type. For the sake of notational simplicity and ease of understanding, we deal only with a two-input–one-output MISO fuzzy rules, which can be extended in an obvious way to more than two input dimensions

$$\textbf{IF } x \text{ is } A_i \textbf{ AND } y \text{ is } B_i \textbf{ THEN } z \text{ is } C_i \tag{42}$$

where $A_i \in \mathcal{F}(X)$, $B_i \in \mathcal{F}(Y)$, and $C_i \in \mathcal{F}(Z)$, respectively.

Note that in the case of both MISO and SISO rules, the input fuzzy set(s) can be seen to be a fuzzy set on either a single domain or a Cartesian product of the domains; hence, all the results presented so far, although discussed in the framework of SISO rules, are valid even when dealing with MISO fuzzy rules.

FRI schemes have their drawbacks also because of the computational and space complexities that are involved (see, e.g., [45]–[47]). These are compounded greatly, especially while dealing with MISO fuzzy rules. Since both CRI and the BK subproduct belong to the class of FRIs, they are not immune to these drawbacks.

The complexity of an inference algorithm stems mainly from two factors.

1) *The process of inference itself*: The fuzzy inferencing schemes are generally resource-consuming (in terms of both memory and time). Many of the inference schemes discretize the underlying domains, and hence, the process becomes computationally intensive.
2) *The structure, complexity, and the number of rules*: Depending on the shape of the underlying fuzzy sets, the number of parameters stored and processed varies. Similarly, the manner in which multiple antecedents are combined affects the processing complexity. Also, an increase in the number of rules only exacerbates the problem. As the number of input variables and/or input fuzzy sets increases, there is a combinatorial explosion of rules in multiple-fuzzy-rule-based systems.

We illustrate these factors through the following example.

Example 4.3: Let $A = [0.9\ \ 0.8\ \ 0.7\ \ 0.7]$, $B = [1\ \ 0.6\ \ 0.8]$, and $C = [0.1\ \ 0.1\ \ 0.2]$ denote fuzzy sets that are defined, respectively, on the following classical sets:

$$X = \{x_1, x_2, x_3, x_4\}, \quad Y = \{y_1, y_2, y_3\}, \quad \text{and}$$
$$Z = \{z_1, z_2, z_3\}.$$

Let $S = (X, Y, Z, \{A, B, C\}, \mathcal{L}, \lhd)$ be the structure considered for the single fuzzy rule

$$\textbf{IF } x \text{ is } \mathcal{A} \textbf{ AND } y \text{ is } \mathcal{B} \textbf{ THEN } z \text{ is } \mathcal{C}.$$

Let $\mathcal{L}$ be the Łukasiewicz complete residuated lattice, i.e., $\mathcal{L} = ([0,1], \wedge, \vee, \otimes, \to_\otimes, 0, 1)$, where $\otimes$ stands for the Łukasiewicz t-norm $x \otimes y = \max(0, x + y - 1)$, and $\to_\otimes$ stands for the Łukasiewicz implication $x \to_\otimes y = \min(1, 1 - x + y)$.

Now, taking the Cartesian product of A and B with respect to $\otimes$, we have

$$A \otimes B = \begin{pmatrix} 0.9 & 0.5 & 0.7\\ 0.8 & 0.4 & 0.6\\ 0.7 & 0.3 & 0.5\\ 0.7 & 0.3 & 0.5 \end{pmatrix}.$$

Now, we have $\hat{R}(A, B; C) = [\hat{R}(z_1)\ \ \hat{R}(z_2)\ \ \hat{R}(z_3)]$, where $(A \otimes B) \to_\otimes C = (A \otimes B) \to_\otimes [0.1\ \ 0.1\ \ 0.2]$, and $\hat{R}(z_i) =$

$(A \otimes B) \rightarrow_{\otimes} C(z_i)$. Thus

$$\hat{R}(z_1) = \hat{R}(z_2) = \begin{pmatrix} 0.2 & 0.6 & 0.4 \\ 0.3 & 0.7 & 0.5 \\ 0.4 & 0.8 & 0.6 \\ 0.4 & 0.8 & 0.6 \end{pmatrix}$$

$$\hat{R}(z_3) = \begin{pmatrix} 0.3 & 0.7 & 0.5 \\ 0.4 & 0.8 & 0.6 \\ 0.5 & 0.9 & 0.7 \\ 0.5 & 0.9 & 0.7 \end{pmatrix}.$$

Let $A' = [0.7\,0.6\,0.5\,0.5]$, $B' = [0.8\,0.5\,0.7]$ be the given fuzzy (nonsingleton) inputs. Then

$$A' \otimes B' = \begin{pmatrix} 0.5 & 0.1 & 0.4 \\ 0.4 & 0.1 & 0.3 \\ 0.3 & 0 & 0.2 \\ 0.3 & 0 & 0.2 \end{pmatrix}.$$

The output that is obtained from the BK subproduct is given by

$$C' = (A' \otimes B') \lhd ((A \otimes B) \rightarrow_{\otimes} C) = [0.7 \;\; 0.7 \;\; 0.8]. \quad (43)$$

Remark 4.4: With the help of Example 4.3, the following observations can be made.

1) *Computational complexity*: Although the computational complexity depends largely on the choice of operators employed, let us consider the following general case of a p-input–1-output system where the ith fuzzy rule is modeled by the fuzzy relation $R_i \in \mathcal{F}(X_1, \ldots, X_p, Y)$, where $R_i = ((A_i^1 * \cdots * A_i^p) \rightarrow B_i)$. Let the universe of discourse X_j be discretized into p_j points for each $j = 1, \ldots, p$. Then, the complexity of a single inference is proportional to $\mathcal{O}(\prod_{j=1}^{p} p_j)$. If $p_j = m$, then it is $\mathcal{O}(m^p)$.
2) *Space complexity:* Again, for a p-input–1-output system, we have a p-dimensional matrix that has $\prod_{j=1}^{p} p_j$ entries. Hence, we need to store p-dimensional matrices for every fuzzy IF–THEN rule.
3) *Run-time space requirements:* For example, consider inferencing with the BK-subproduct inference scheme [see (8)] in the case of a two-input fuzzy IF–THEN rule. Let the universes of discourse X_1, X_2, and Y be discretized into $m, k,$ and l points, respectively. Then, the memory requirements of the algorithm are as follows (see also [47]): $m \cdot k \cdot l$ for $(A * B) \rightarrow C$, $m \cdot k$ for combining the given facts $A' * B'$ and l for the consequent, where $*$ denotes the t-norm used for the Cartesian product. Overall, it is $m \cdot k \cdot l + m \cdot k + l$. In the case where $m = k = l$, the memory requirements of the algorithm become $m^3 + m^2 + m$. Generalizing this, in the case of p inputs, the memory requirements of the algorithm are $\mathcal{O}(m^{(p+1)})$.

There are many works that have proposed modifications to the classical CRI in an attempt to enhance the efficiency in its inferencing (see, for example, the works of Fullér and coauthors [48]–[50] and Moser and Navara [51]–[53]). In the case when there are more than two antecedents involved in fuzzy inference, Ruan and Kerre [54] proposed an extension to the classical CRI, wherein starting from a finite number of fuzzy relations of an arbitrary number of variables, but having some variables in common, fuzzy relations can be inferred among the variables of interest. Demirli and Türksen [47] proposed a *rule breakup* method and showed that rules with two or more independent variables in their premise can be simplified to a number of inferences of rule bases with simple rules (only one variable in their premise). For further modification of this method, see [55].

However, to the best of the authors' knowledge, no such works exist for the BK subproduct. In the following, we propose a modified form of BK subproduct that alleviates some of the concerns noted previously, along the lines of the hierarchical CRI that was proposed by Jayaram [41].

D. Hierarchical BK Subproduct—Property 6

Jayaram [41] proposed a hierarchical variant of the CRI where observation on particular axes are taken independently and hierarchically, and the overall output is deduced after all observations were used in this step-by-step chain procedure. This contradicts the usual case where a Cartesian product of all observations is computed, and such a product serves as the only fuzzy input with a vector variable. We follow this idea with the BK subproduct as well.

Procedure for Hierarchical BK Subproduct

Step 1 FOR $i = 1$ TO n DO
- i) Calculate $R_i' \in \mathcal{F}(Y \times Z)$: $R_i' = B_i \rightarrow C_i$.
- ii) Calculate $C_i' \in \mathcal{F}(Z)$: $C_i' = B \lhd R_i'$.
- iii) Calculate $R_i'' \in \mathcal{F}(X \times Z)$: $R_i'' = A_i \rightarrow C_i'$.
- iv) Calculate $C_i'' \in \mathcal{F}(Z)$: $C_i'' = A \lhd R_i''$.

Step 2 AGGREGATE ALL C_i'' BY MINIMUM.

Although the algorithm for the hierarchical CRI is provided in [41] only for the case of inferencing with a single MISO rule, we remark that it can be extended in a straightforward manner to the case of multiple MISO rules, as is done here. However, for the sake of simplicity, the following example, which demonstrates the reduction in computational efforts and memory savings, is given in the context of a single MISO rule.

Example 4.5: Let the fuzzy sets A, B, C, A', B' be as in Example 4.3, with the same structure S for the given fuzzy rule. Inferencing with the hierarchical BK subproduct, given the input (A', B'), we have the following:

Step 1 (i) $$B \rightarrow_{\otimes} C = \begin{pmatrix} 0.1 & 0.1 & 0.2 \\ 0.5 & 0.5 & 0.6 \\ 0.3 & 0.3 & 0.4 \end{pmatrix}$$

Step 1 (ii) $$\begin{aligned} C' &= B' \lhd (B \rightarrow_{\otimes} C) \\ &= [0.8\,0.5\,0.7] \lhd (B \rightarrow_{\otimes} C) \\ &= [0.3 \;\; 0.3 \;\; 0.4] \end{aligned}$$

Step 1 (iii) $$A \rightarrow_{\otimes} C' = \begin{pmatrix} 0.4 & 0.4 & 0.5 \\ 0.5 & 0.5 & 0.6 \\ 0.6 & 0.6 & 0.7 \\ 0.6 & 0.6 & 0.7 \end{pmatrix}$$

Step 1 (iv) $$\begin{aligned} C'' &= A' \lhd (A \rightarrow_{\otimes} C') \\ &= [0.7\,0.6\,0.5\,0.5] \lhd (A \rightarrow_{\otimes} C') \\ &= [0.7 \;\; 0.7 \;\; 0.8]. \end{aligned} \quad (45)$$

Remark 4.6: From the previous example, it is clear that we can convert a multi-input system to a single-input hierarchical system by employing the BK-subproduct inference.The effect becomes more pronounced when we have more than two input variables. From Example 4.5, it can be noted that the most memory-intensive step in the inference is the calculation of the "current" output fuzzy set (steps 1 (ii) and 1 (iv)). Once again, considering the case of a p-input fuzzy rule, if the input universe of discourse $X_j, j = 1, 2, \dots, p$ is discretized into p_j points, and the output universe of discourse Z is discretized into l points, then the memory requirements of this step and, hence, of the algorithm itself, can be easily deduced to be $p^* \cdot l + l + p^*$, where $p^* = \max_{j=1}^{p} p_j$. In the case $m = p^* = l$, we have that the overall memory requirements are $2m + m^2$. It should also be emphasized that the memory requirements are independent of the number of input variables, as can be expected in any hierarchical setting.

Example 4.5 not only illustrates the computational efficiency of hierarchical BK-subproduct inference, but it also shows that the inference obtained from the original BK subproduct is identical to the one that is obtained from the proposed hierarchical BK subproduct, i.e., (43) $= C' = C'' =$ (45). The following result shows that this equivalence is always guaranteed under the structure S that is considered in this paper.

Theorem 4.7: Let $S = (X, Y, Z, \{A_i, B_i, C_i\}_{i=1,\dots,n}, \mathcal{L}, \lhd)$ be a structure for fuzzy rules as given in (42), and let $\hat{R} \in \mathcal{F}(X \times Y \times Z)$ be given by

$$\bigwedge_{i=1}^{n} ((A_i(x) * B_i(y)) \to C_i(z)), \quad x \in X, y \in Y, z \in Z.$$

Then, for any $A \in \mathcal{F}(X)$ and $B \in \mathcal{F}(Y)$, all $i = 1, \dots, n$, it is true that

$$(A * B) \lhd \hat{R} \equiv \bigwedge_{i=1}^{n} (A \lhd (A_i \to (B \lhd (B_i \to C_i)))).$$

Proof: For an arbitrary $z \in Z$, we have

$$\begin{aligned}
&[(A * B) \lhd \hat{R}](z) \\
&= \bigwedge_{x,y} \left((A(x) * B(y)) \to \bigwedge_{i=1}^{n} ((A_i(x) * B_i(y)) \to C_i(z)) \right) \\
&= \bigwedge_{i=1}^{n} \bigwedge_{x,y} ((A(x) * B(y)) \to ((A_i(x) * B_i(y)) \to C_i(z)))
\end{aligned}$$

and by triple use of (14), we get

$$\begin{aligned}
&= \bigwedge_{i=1}^{n} \bigwedge_{x,y} ((A(x) * A_i(x) * B(y) * B_i(y)) \to C_i(z)) \\
&= \bigwedge_{i=1}^{n} \bigwedge_{x} \left((A(x) * A_i(x)) \to \bigwedge_{y} ((B(y) * B_i(y)) \to C_i(z)) \right)
\end{aligned}$$

which equals to

$$\bigwedge_{i=1}^{n} \bigwedge_{x} \left(A(x) \to \left(A_i(x) \to \bigwedge_{y} (B(y) \to (B_i(y) \to C_i(z))) \right) \right)$$

and, therefore

$$[(A * B) \lhd \hat{R}](z) = \bigwedge_{i=1}^{n} (A \lhd (A_i \to (B \lhd (B_i \to C_i)))).$$

■

Theorem 4.7 proves the equivalence of outputs that are obtained from the proposed algorithm of the hierarchical BK inference mechanism and the original BK subproduct with 2-D inputs. Indeed, it may be systematically extended into a case of inputs of an arbitrary finite dimension.

Now, we may state the following corollary of Theorems 3.6 and 4.7, which claims that if condition (26) certifies the solvability of (28), then even the proposed hierarchical BK-subproduct inference procedure keeps the fundamental interpolation condition fulfilled.

Corollary 4.8: Let all the assumptions of Theorem 4.7 be valid. Furthermore, let

$$\bigvee_{i=1}^{n} (A_i(x) * A_j(x) * B_i(y) * B_j(y)) \le \bigwedge_{i=1}^{n} (C_i(z) \leftrightarrow C_j(z))$$

hold for arbitrary $x \in X, y \in Y, z \in Z$, as well as for arbitrary $i, j \in \{1, \dots, n\}$. Then

$$\bigwedge_{i=1}^{n} (A_i \lhd (A_i \to (B_i \lhd (B_i \to C_i)))) \equiv C_i.$$

V. Conclusion

In this paper, after recalling some of the properties that are usually cited in favor of using the CRI, which was introduced by Zadeh [1], viz., equivalent and reasonable conditions for their solvability, their interpolative properties, and the preservation of the indistinguishability that may be inherent in the input fuzzy sets, we have shown that the BK subproduct that was introduced in [24] possesses all the aforementioned properties and, hence, is equally suitable to be considered when reasoning with a system of fuzzy rules. Toward this end, some new but equivalent results on indistinguishability operations have also been presented.

Moreover, we show that under certain conditions, the equivalence of FITA and FATI can be shown for the BK subproduct, much like in the case of CRI. After citing some of the main drawbacks of FRIs, we propose an hierarchical inferencing scheme that alleviates many of these drawbacks in the BK-subproduct inference. This method is amply illustrated with numerical examples. Finally, we have also shown that if the structure for the considered fuzzy rules is chosen appropriately, then the outputs that are obtained from the hierarchical BK subproduct and the original BK subproduct are identical, thus addressing the issues related to computational complexity.

Based on this paper, it can be concluded that the BK subproduct is as advantageous as the classical CRI that was proposed by Zadeh and, hence, can be employed alternatively in applications. The main difference lies in the fuzzy relation that models a fuzzy rule base, which is combined with a particular inference mechanism. It is shown that some computational advantages of the very popular Mamdani–Assilian approach are valid only in the case of the CRI inference mechanism, while if we use the BK subproduct, then many of the advantages of using $\check{R}$ are lost, and the implicational approach that employs the fuzzy relation $\hat{R}$ assumes this privilege.

Therefore, we conclude that there is another added value to the existence of two inference schemes with the same appropriate properties—the possibility to freely choose between two approaches of modeling a fuzzy rule base. Up to now, the approach denoted by $\hat{R}$ that employs genuine implication was considered to be a disadvantage because of its computational complexity, although for some problems, it is much more suitable [15]. This investigation shows that from the computational point of view, there is neither a preferable model of a fuzzy rule base nor a preferable inference mechanism; there are only preferable combinations of them.

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