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Clifford's Order obtained from Uninorms on Bounded Lattices

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Abstract

Inspired by the work of Clifford on obtaining order from semigroups, many works have proposed different ways of obtaining orders from associative fuzzy logic operations. However, unlike Clifford's relation, these were dependent on the subdomain of its arguments. Recently, it was shown that a property termed *Quasi-Projectivity* (QP) is necessary to obtain an order from Clifford's relation. Further, for the underlying domain [0,1] it was shown that while all t-norms, t-conorms and nullnorms satisfy (QP), giving rise to posets, not all classes of uninorms satisfy (QP). Several constructions of uninorms U exist on bounded lattices, which unlike [0,1] may neither be total nor complete. In this work, we investigate the satisfaction of (QP) for these constructions. This study merits attention since it offers an alternate perspective - that a uninorm U on a lattice L can be seen as a t-norm on the obtained U-poset.

Keywords: Ordered Sets, Bounded lattices, T-norms, Uninorms, F-posets.

1. Introduction

In recent years, many works have appeared that propose order from basic fuzzy logic connectives, see for instance, [1], [2], [3], [4], [5], [6]. Given a bounded lattice $(\mathbb{L}, \leq, 0, 1)$, Karaçal and Kesicioğlu [1] were the first ones to investigate whether the following relation, proposed originally by Clifford [7], would give rise to an order on $\mathbb{L}$

$$x \sqsubseteq y \iff F(\ell, y) = x \text{ , for some } \ell \in \mathbb{L} \text{ ,} \tag{1}$$

when F is a t-norm on $\mathbb{L}$.

While $\sqsubseteq$ gave rise to orders based on both t-norms and t-conorms, to obtain order from a uninorm this relation was further generalised by Ertuğrul *et al.* [2].

Definition 1.1 ([2]). *Let $(\mathbb{L}, \leq, 0, 1)$ be a bounded lattice and U be a uninorm on $\mathbb{L}$ with an identity element $e \in \mathbb{L} \setminus \{0, 1\}$. Define the following relation: For every $x, y \in \mathbb{L}$*

$$x \preceq_U y \Leftrightarrow \begin{cases} \textit{if} \quad x, y \in [0, e] \textit{ and there exists } k \in [0, e] \textit{ such that } U(k, y) = x \textit{ or} , \\ \textit{if} \quad x, y \in [e, 1] \textit{ and there exists } \ell \in [e, 1] \textit{ such that } U(x, \ell) = y \textit{ or} , \\ \textit{if} \quad (x, y) \in \mathbb{L}^2 \setminus \left\{[0, e]^2 \cup [e, 1]^2\right\} \textit{ and } x \leq y \text{ .} \end{cases} \tag{2}$$

The relation $\preceq_U$ is a partial order on $\mathbb{L}$ for any uninorm U.

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1.1. Motivation for this work

While $\preceq_U$ is both a generalisation of $\sqsubseteq$ and gives a partial order on $\mathbb{L}$ for any uninorm U, it is not bereft of some disadvantages, as listed below:

(i) The definition of order is dependent on the way the uninorm U is defined on its subdomains.
(ii) To relate a pair of elements x, y, there are further restrictions on the domain from which the second argument of the U can come from.
(iii) Proposition 1.2 shows that we cannot obtain any richer order-theoretic structure from $\preceq_U$ than the bounded lattice $\mathbb{L}$ that we begin with.

Proposition 1.2 ([2], Proposition 3)**.** *Let $(\mathbb{L}, \leq, 0, 1)$ be a bounded lattice and U be a uninorm on $\mathbb{L}$ with an identity element $e \in \mathbb{L} \setminus \{0, 1\}$. If $x \preceq_U y$ for $x, y \in \mathbb{L}$ then $x \leq y$.*

For instance, if $(\mathbb{L}, \leq)$ is not a chain then the partially ordered set $(\mathbb{L}, \preceq_U)$ is also not a chain. In fact, the poset $(\mathbb{L}, \preceq_U)$ may not even be a lattice, see Example 6.1.

1.2. Contributions of this work

Recently in [8], a property called *Quasi-Projectivity* (**QP**) was shown to be important to obtain an order from Clifford's relation. Further, it was also proven that while all t-norms, t-conorms and nullnorms satisfied (**QP**) giving rise to posets, when the underlying domain is $[0, 1]$, not all classes of uninorms satisfied (**QP**).

Definition 1.3 ([8])**.** *Let $\mathbb{P} \neq \emptyset$. An $F : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ is said to satisfy the **Quasi-Projection** property, if for any $x, y, z \in \mathbb{P}$,*

$$F(x, F(y, z)) = z \Longrightarrow F(y, z) = z \ . \tag{QP}$$

Our study makes the following contributions:

(i) Note that almost all constructions of uninorms U proposed on bounded lattices are done region-wise. By presenting fairly general and unifying results discussing the satisfaction of (**QP**) on these different regions, we show that almost all the existing constructions satisfy (**QP**).
(ii) We show that the orders obtained based on $\sqsubseteq$ and $\preceq$ differ significantly and the relation $\sqsubseteq$ consistently gives rise a richer order-theoretic structures.
(iii) An interesting fall out of this study is the rise of an alternate perspective. Recently there have been a spate of works on constructing t-norms on bounded posets/lattices. Our study shows that a uninorm U on the original lattice $(L, \leq)$ can be seen as a t-norm on the obtained poset $(L, \sqsubseteq_U)$.

1.3. Outline of this submission:

This paper is organised as follows. In Section 2, we provide some relevant order-theoretic definitions and notions. In Section 3, we start with definitions and results of aggregation operators on a bounded lattice and show that all t-norms, t-conorms, t-subnorms and t-superconorms on the bounded lattice $(\mathbb{L}, \leq, 0, 1)$ satisfy (**QP**). In Sections 4 and 5, we discuss the satisfaction of (**QP**) by uninorms defined on bounded lattices, leading to partially ordered sets through the relation (1). In Section 6 we present some interesting perspectives arising from our study, thus both vindicating and highlighting the investigations in this submission. Finally, Section 7 lists a few possible avenues for further exploration.

2. Preliminaries

We begin by presenting some notions from order theory which will be useful in the sequel and for further details refer the readers to the book of Davey and Priestly [9].

Definition 2.1. (i) *Let $\mathbb{L} \neq \emptyset$. A partial order on $\mathbb{L}$ is a binary relation $\leq$ on $\mathbb{L}$ such that, for all $a, b, c \in \mathbb{L}$, the following properties hold:*

- *(Reflexivity): $a \leq a$,*
- *(Antisymmetry): If $a \leq b$ and $b \leq a$, then $a = b$,*
- *(Transitivity): If $a \leq b$ and $b \leq c$, then $a \leq c$.*

Definition 2.2. *Let $(\mathbb{L}, \leq)$ be a poset.*

(i) *An element a in $\mathbb{L}$ is said to be*

- *a **maximal element** if there does not exist any $b \in \mathbb{L}$ such that $a \leq b$.*
- *a **minimal element** if there does not exist any $b \in \mathbb{L}$ such that $b \leq a$.*
- *the **greatest element** (maximum /top element) if for every element b in $\mathbb{L}$ we have that $a \geq b$.*
- *the **least element** (minimum /bottom element) if for every element b in $\mathbb{L}$ we have that $a \leq b$.*

(ii) *A pair of elements $a, b \in \mathbb{L}$ is said to be **comparable**, denoted $a \nparallel b$, if either $a \leq b$ or $b \leq a$. If not, we denote it by $a \parallel b$. Further, we denote the set of all elements incomparable with an $a \in \mathbb{L}$ by $I_a = \{x \in \mathbb{L} | a \parallel x\}$.*

(iii) *$\mathbb{L}$ is said to be*

- *a **chain** or **totally ordered** if for any $a, b \in \mathbb{L}$ either $a \leq b$ or $b \leq a$, i.e., $a \nparallel b$ for every $a, b \in \mathbb{L}$.*
- ***bounded** if there exist elements $\alpha, \beta \in \mathbb{L}$ such that for every $x \in \mathbb{L}$ we have $\alpha \leq x \leq \beta$, in which case, for emphasis, the poset will be denoted as $(\mathbb{L}, \leq, \alpha, \beta)$.*
- *a **join-semilattice** if every pair of elements $x, y \in \mathbb{L}$ has a least upper bound and it will be denoted by $x \vee y$, where the operation $\vee$ is also known as the **join**.*
- *a **meet-semilattice** if every pair of elements $x, y \in \mathbb{L}$ has a greatest lower bound and it will be denoted by $x \wedge y$, where the operation $\wedge$ is also known as the **meet**.*
- *a **lattice** if it is both a meet-semilattice and a join-semilattice. In other words, for any pair of elements in $\mathbb{L}$ both the join and the meet exist.*
- *a **complete lattice** if it is a lattice and for all $Y \subseteq \mathbb{L}$ both the least upper bound of Y - denoted $\sup\ Y$ or $\bigvee Y$, and the greatest lower bound of Y - denoted $\inf\ Y$ or $\bigwedge Y$, exist.*

It is well-known that a lattice can also be seen from an algebraic point of view. For instance, the following perspective is useful in the sequel.

Remark 2.3. *From [10] we know that a commutative, idempotent semigroup (L, F) always defines a semilattice with a partial order induced by F. The relation defined as*

$$x \leq_F y \iff F(x, y) = x \tag{3}$$

is a partial order and $(\mathbb{L}, \leq_F)$ is a meet-semilattice. Similarly, the relation

$$x \leq^F y \iff F(x, y) = y \tag{4}$$

is a partial order and $(\mathbb{L}, \leq^F)$ is a join-semilattice.

3. Some associative operations on bounded lattices that give rise to *F*-posets

In this section, we show that all t-norms, t-conorms, t-subnorms and t-superconorms on the bounded lattice $(\mathbb{L}, \leq, 0, 1)$ satisfy (**QP**). Thus the relation defined in (1) does give a partial order on $\mathbb{L}$. We begin this section with results that will be useful in the sequel for discussing the satisfaction of (**QP**).

Definition 3.1 ([11])**.** *Let $(\mathbb{L}, \leq, 0, 1)$ be a bounded lattice and $F : \mathbb{L}^2 \to \mathbb{L}$ a commutative, associative function that is non-decreasing in each variable.*

(i) *F is called a t-norm if the top element 1 is an identity element.*

(ii) *F is called a t-conorm if the bottom element* 0 *is an identity element.*
(iii) *F is called a t-subnorm if* $F(x,y) \leq x \wedge y$ *for all* $x, y \in \mathbb{L}$.
(iv) *F is called a t-superconorm if* $x \vee y \leq F(x,y)$ *for all* $x, y \in \mathbb{L}$.

Definition 3.2 ([8]). *Let F be a binary operation on an* $\mathbb{L} \neq \emptyset$. *F is said to satisfy the* ***Local Left Identity*** *property, if for every* $x \in \mathbb{L}$, *there exists an* $\ell \in \mathbb{L}$ *(possibly depending on x) such that* $F(\ell, x) = x$, *i.e., every element has a local left identity w.r.t. F.*

All t-norms, t-conorms, uninorms and nullnorms satisfy the **Local Left Identity** property.

Theorem 3.3 ([8]). *Let* $\mathbb{L} \neq \emptyset$ *and* $F : \mathbb{L}^2 \to \mathbb{L}$ *be an associative operation. Let the relation* $\sqsubseteq_F$ *on* $\mathbb{L}$ *be defined as in* (1). *The following are equivalent:*

(i) $(\mathbb{L}, \sqsubseteq_F)$ *is a poset.*
(ii) *F has a local left identity and satisfies* (**QP**).

Proposition 3.4. *Let F be a binary operation on the poset* $(\mathbb{L}, \leq)$ *such that one of the following is true:*

(i) $F(\alpha, \beta) \geq \beta$ *for all* $\alpha, \beta \in \mathbb{L}$,
(ii) $F(\alpha, \beta) \leq \beta$ *for all* $\alpha, \beta \in \mathbb{L}$.

Then F satisfies (**QP**).

Proof. This follows directly from [8], Proposition 1. □

The following results immediately follow from Proposition 3.4.

Corollary 3.5. *Let F be a binary operation on the bounded poset* $(\mathbb{L}, \leq, 0, 1)$ *such that it is increasing in the first variable w.r.t.* $\leq$. *If the top or bottom element of* $\mathbb{L}$ *is a left identity of F, then F satisfies* (**QP**).

Corollary 3.6. *Let T be a t-norm and S be a t-conorm on a bounded lattice* $(\mathbb{L}, \leq, 0, 1)$. *If there exists a function* $F : \mathbb{L}^2 \to \mathbb{L}$ *such that one of the following is true:*

$F(\alpha, \beta) \leq T(\alpha, \beta)$ *for all* $(\alpha, \beta) \in \mathbb{L}^2$, ***or***
$S(\alpha, \beta) \leq F(\alpha, \beta)$ *for all* $(\alpha, \beta) \in \mathbb{L}^2$,

then F satisfies (**QP**).

Proposition 3.7. *Let F be a t-subnorm and G be a t-superconorm on a bounded lattice* $(\mathbb{L}, \leq, 0, 1)$. *Then F and G satisfy* (**QP**).

Proof. Since F is a t-subnorm on $\mathbb{L}$, F satisfies (**QP**) by Corollary 3.6. Similarly, we can see that G satisfies (**QP**). □

Since a t-norm is a t-subnorm and a t-conorm is a t-superconorm, we see that t-norms and t-conorms satisfy (**QP**).

Proposition 3.8. *Let T be a t-norm and S be a t-conorm on a bounded lattice* $(\mathbb{L}, \leq, 0, 1)$. *Then* $(\mathbb{L}, \sqsubseteq_T, 0, 1)$ *and* $(\mathbb{L}, \sqsubseteq_S, 1, 0)$ *are bounded T-poset and S-poset, respectively.*

Proposition 3.9 ([10]). *Let* $(\mathbb{L}, \wedge)$ *be meet-semilattice. The order obtained from the meet-semilattice and the* $\wedge$*-poset coincide, i.e.,* $\leq_\wedge = \sqsubseteq_\wedge$.

Proof. Let $x \sqsubseteq_\wedge y$ for some $x, y \in \mathbb{L}$. Then there exist an element ℓ such that

$$
\begin{aligned}
x = \ell \wedge y &\Longrightarrow x = \ell \wedge y = \ell \wedge (y \wedge y), && \lceil\text{by the idempotence of } \wedge \\
&\Longrightarrow \ell \wedge y = (\ell \wedge y) \wedge y, && \lceil\text{by the associativity of } \wedge \\
&\Longrightarrow \ell \wedge y \leq_\wedge y, && \lceil\text{by the definition of order in (3)} \\
&\Longrightarrow x \leq_\wedge y.
\end{aligned}
$$

Suppose $x \leq_\wedge y$ for some $x, y \in \mathbb{L}$, i.e., $x \wedge y = x$. By definition of order in (1), we have $x \sqsubseteq_\wedge y$. □

Thus all t-norms, t-conorms, t-subnorms and t-superconorms on a bounded lattice satisfy (QP), much like their counterparts on $[0,1]$ (see, [8]).

Many constructions of uninorms U exist on bounded lattices, which unlike $[0,1]$, may neither be total nor complete. In fact, even on $[0,1]$, which is a complete bounded lattice, not all uninorms satisfy (**QP**). For instance, representable uninorms on $[0,1]$ do not satisfy (**QP**)(see, [8]).

Hence, it is worthwhile to investigate whether uninorms on bounded lattices $\mathbb{L}$ satisfy (**QP**) - a study which we take up in the sequel.

4. Uninorms on bounded lattices $(\mathbb{L}, \leq, 0, 1)$

Uninorms on bounded lattices have been a hot topic of research in recent years. One reason is perhaps the possibility to deal with lattices that are not linear, allowing the introduction of novel constructions with a number of new properties.

For instance, while for a uninorm U on $[0,1]$, $U(0,1) \in \{0,1\}$, in the case of bounded lattices $U(0,1) = a$ need not be either of the bounds. However, it is still an annihilator of U as shown in Lemma 4.2.

We begin with the definition of a uninorm on a bounded lattice.

Definition 4.1 ([12])**.** *Let $(\mathbb{L}, \leq, 0, 1)$ be a bounded lattice. A commutative, associative function $F : \mathbb{L}^2 \to \mathbb{L}$ that is non-decreasing in each variable is called a uninorm if there is an identity element $e \in \mathbb{L} \setminus \{0,1\}$, i.e., $U(e,x) = x$ for all $x \in \mathbb{L}$.*

Lemma 4.2. *Let U be a uninorm on the bounded lattice $(\mathbb{L}, \leq, 0, 1)$. Then the element $U(0,1) = a$ is an annihilator of U, i.e., $U(a,x) = U(x,a) = a$ for all $x \in \mathbb{L}$.*

Proof. Let U be a uninorm on the bounded lattice $(\mathbb{L}, \leq, 0, 1)$. Then for all $x \in \mathbb{L}$, we have

$$\begin{aligned}
U(1,x) \leq 1 &\Longrightarrow U(0, U(1,x)) \leq U(0,1), && \lceil\text{by the monotonicity of } U \\
&\Longrightarrow U(U(0,1),x) \leq U(0,1), && \lceil\text{by the Associativity of } U \\
&\Longrightarrow U(U(0,1),0) \leq U(U(0,1),x) \leq U(0,1), && \lceil\text{by the monotonicity of } U \\
&\Longrightarrow U(U(0,0),1) \leq U(U(0,1),x) \leq U(0,1), && \\
&\Longrightarrow U(0,1) \leq U(U(0,1),x) \leq U(0,1), && \lceil\text{since } U(0,0) = 0 \\
&\Longrightarrow U(U(0,1),x) = U(0,1).
\end{aligned}$$

□

As mentioned above, uninorms defined on general bounded lattices $\mathbb{L}$ differ from those on $[0,1]$ on many aspects. A second difference is due to the fact that $[0,1]$ is a chain and hence any $e \in]0,1[$ is comparable to every other element in $[0,1]$, which is no more the case in a general lattice $\mathbb{L}$. This has necessitated the introduction of the following subdomain of $\mathbb{L}$ (see Definition 2.2):

$$I_e = \{x \in \mathbb{L} \mid x \parallel e\}.$$

Thus, typically, all the constructions of uninorms on bounded lattices $\mathbb{L}$ are done by considering the different Cartesian products of the following subdomains of $\mathbb{L}$, viz., $[0,e], I_e,]e,1]$.

Many construction methods have been proposed by various researchers, viz., [13], [14], [12], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35]. Interestingly, majority of the above construction methods yield a uninorm U that either belongs to the family of $\mathcal{U}_{\mathbf{min}}$ or $\mathcal{U}_{\mathbf{max}}$. However, there do exist construction methods, see for instance, the ones proposed in [24], [35] or **Section 5** of [32], that do not fall into the above classes. In this work, we study the satisfaction of (**QP**) by uninorms that belong to either $\mathcal{U}_{\mathbf{min}}$ or $\mathcal{U}_{\mathbf{max}}$, and also few other constructions given in [24], [35] and [32], thus covering almost all the uninorms proposed on bounded lattices so far.

We now present the definitions and representation theorems for the above mentioned classes.

4.1. The classes $\mathcal{U}_{\mathbf{min}}$ and $\mathcal{U}_{\mathbf{max}}$

We first introduce the $\mathcal{U}_{\mathbf{min}}$ and $\mathcal{U}_{\mathbf{max}}$ classes of uninorms on a bounded lattice $(\mathbb{L}, \leq, 0, 1)$.

Definition 4.3 ([32], Definition 4.1)**.** *Let $(\mathbb{L}, \leq, 0, 1)$ be a bounded lattice and $e \in \mathbb{L}\setminus\{0, 1\}$. The class of all uninorms U on $\mathbb{L}$ with neutral element e satisfying the following condition will be denoted by $\mathcal{U}_{\mathbf{min}}$:*

$$U(x, y) = y, \text{ for all } (x, y) \in]e, 1] \times \mathbb{L}\setminus[e, 1].$$

Similarly, the class of all uninorms U on $\mathbb{L}$ with neutral element e satisfying the following condition will be denoted by $\mathcal{U}_{\mathbf{max}}$:

$$U(x, y) = y, \text{ for all } (x, y) \in [0, e[\times\mathbb{L}\setminus[0, e].$$

A complete representation of such uninorms was given by Zhang *et al.* [32].

Theorem 4.4 ([32], Theorems 4.3 & 4.6)**.** *Let $(\mathbb{L}, \leq, 0, 1)$ be a bounded lattice, $e \in \mathbb{L}\setminus\{0, 1\}$ and U be a binary operation on $\mathbb{L}$. Then*

- *$U \in \mathcal{U}_{\mathbf{min}}$ if and only if there exist a t-conorm S on $[e, 1]$ and a t-subnorm F on $\mathbb{L}\setminus[e, 1]$ such that*

$$U(x, y) = \begin{cases} S(x, y), & if\,(x,y) \in [e, 1]^2, \\ y, & if\,(x,y) \in [e, 1] \times (\mathbb{L}\setminus[e, 1]) \;, \\ x, & if\,(x,y) \in (\mathbb{L}\setminus[e, 1]) \times [e, 1], \\ F(x, y), & if\,(x,y) \in (\mathbb{L}\setminus[e, 1])^2 \;. \end{cases} \tag{5}$$

- *$U \in \mathcal{U}_{\mathbf{max}}$ if and only if there exist a t-norm T on $[0, e]$ and a t-superconorm G on $\mathbb{L}\setminus[0, e]$ such that*

$$U(x, y) = \begin{cases} T(x, y), & if\,(x,y) \in [0, e]^2, \\ y, & if\,(x,y) \in [0, e] \times (\mathbb{L}\setminus[0, e]) \;, \\ x, & if\,(x,y) \in (\mathbb{L}\setminus[0, e]) \times [0, e], \\ G(x, y), & if\,(x,y) \in (\mathbb{L}\setminus[0, e])^2 \;. \end{cases} \tag{6}$$

Example 4.5. *Consider the lattice $L_1 = \{0, x, y, z, e, 1\}$ whose Hasse diagram is given in Figure 1. Consider the t-conorm $S_e = \vee$ on $[e, 1]$ and any t-norm T_e on $[0, e]$. By using Theorem 7 in [34] and Theorem 3.2 in [26] one can define the corresponding uninorms U_S^e and $U_{(2,e)}$ as given in Table 1.*

U_S^e	0	x	y	z	1	e
0	0	0	0	0	0	0
x	0	x	x	x	x	x
y	0	x	y	y	y	y
z	0	x	y	z	1	z
1	0	x	y	1	1	1
e	0	x	y	z	1	e

$U_{(2,e)}$	0	x	y	z	1	e
0	0	0	0	0	0	0
x	0	0	0	x	x	x
y	0	0	0	y	y	y
z	0	x	y	z	1	z
1	0	x	y	1	1	1
e	0	x	y	z	1	e

Table 1: The operations U_S^e and $U_{(2,e)}$ are uninorms on $(L_1, \leq, 0, 1)$.

It is easy to see that both U_S^e and $U_{(2,e)}$ belong to $\mathcal{U}_{\mathbf{min}}$.

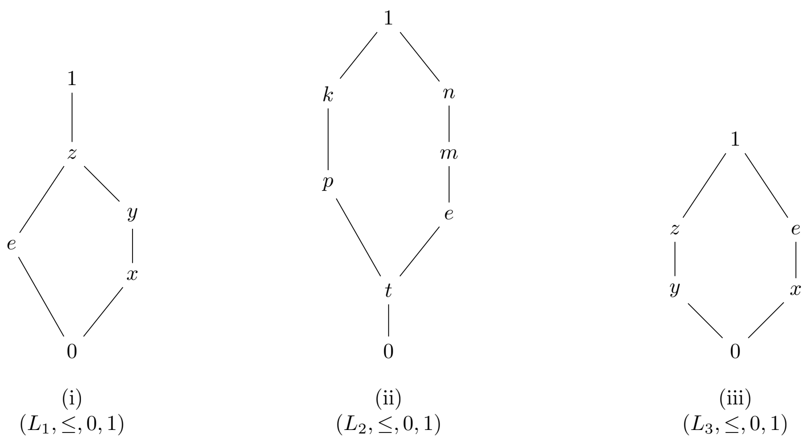


Figure 1: The lattices L_1–L_3 on which the operations given in Tables 1 – 3, respectively, are uninorms.

4.2. Construction of Çaylı [24]

The construction of Çaylı [24] given below is different from the class of $\mathcal{U}_{\mathbf{min}}$ or $\mathcal{U}_{\mathbf{max}}$ as shown after its definition.

Theorem 4.6 ([24], Theorem 5 and 6)**.** *Let $(\mathbb{L}, \leq, 0, 1)$ be a bounded lattice and $e \in \mathbb{L} \setminus \{0, 1\}$. Let T_e be a t-norm on $[0, e]$ and S_e be a t-conorm on $[e, 1]$ such that $T_e(x, y) > 0$ for all $x, y > 0$ and $S_e(x, y) < 1$ for all $x, y < 1$.*

(i) *If $x \parallel y$ for all $x \in I_e$ and $y \in [e, 1[$, then the function ${U_1}^e : \mathbb{L}^2 \to \mathbb{L}$ defined as follows*

$$
{U_1}^e(x, y) = \begin{cases} T_e(x, y), & if\,(x,y) \in [0, e]^2, \\ S_e(x, y), & if\,(x,y) \in [e, 1]^2, \\ x, & if\,(x,y) \in I_e \times [e, 1[\ \cup \ I_e \times]0, e[, \\ y, & if\,(x,y) \in [e, 1[\times I_e \cup]0, e[\times I_e \ , \\ x \vee y, & if\,(x,y) \in {I_e}^2 \cup I_e \times \{1\} \cup \{1\} \times I_e \cup]0, e[\times \{1\} \cup \{1\} \times]0, e[, \\ x \wedge y, & otherwise \end{cases} \tag{7}
$$

is a uninorm on $\mathbb{L}$ with the neutral element e.

(ii) *If $x \parallel y$ for all $x \in I_e$ and $y \in]0, e]$, then the function ${U_2}^e : \mathbb{L}^2 \to \mathbb{L}$ defined as follows*

$$
{U_2}^e(x, y) = \begin{cases} T_e(x, y), & if\,(x,y) \in [0, e]^2, \\ S_e(x, y), & if\,(x,y) \in [e, 1]^2, \\ x, & if\,(x,y) \in I_e \times]0, e] \cup I_e \times]e, 1[, \\ y, & if\,(x,y) \in \ ]0, e] \times I_e \cup \ ]e, 1[\times I_e \ , \\ x \wedge y, & if\,(x,y) \in {I_e}^2 \cup I_e \times \{0\} \cup \{0\} \times I_e \cup]e, 1[\times \{0\} \cup \{0\} \times]e, 1[, \\ x \vee y, & otherwise \end{cases} \tag{8}
$$

is a uninorm on $\mathbb{L}$ with the neutral element e.

Clearly, the uninorms ${U_1}^e$ and ${U_2}^e$ do not belong to the subclass of $\mathcal{U}_{\mathbf{min}}$ or $\mathcal{U}_{\mathbf{max}}$. For instance, ${U_1}^e(1, x) = 1 \vee x = 1 \neq x$ for all $x \in]0, e[$, i.e., ${U_1}^e(x, y) \neq y$, for some $(x, y) \in]e, 1] \times \mathbb{L} \setminus [e, 1]$ and hence ${U_1}^e \notin \mathcal{U}_{\mathbf{min}}$. Further, it is easy to see that ${U_1}^e \notin \mathcal{U}_{\mathbf{max}}$, since ${U_1}^e(0, x) \neq x$ for all $x \in]e, 1[$. Similarly, we can prove that the uninorm ${U_2}^e$ does not belong to the subclasses of $\mathcal{U}_{\mathbf{min}}$ and $\mathcal{U}_{\mathbf{max}}$.

Example 4.7. *Consider the lattice $L_2 = \{0, t, p, e, m, k, n, 1\}$ whose Hasse diagram is given in Figure 1. Consider the t-conorm $S_e = \vee$ on $[e, 1]$ and the t-norm $T_e = \wedge$ on $[0, e]$. By using Theorem 6 in [34] and Theorem 4.6 one can define the corresponding uninorms U_T^e and U_1^e as given in Table 2.*

U_T^e	0	t	p	e	m	k	n	1
0	0	0	p	0	m	k	n	1
t	0	t	p	t	m	k	n	1
p	p	p	p	p	1	k	1	1
e	0	t	p	e	m	k	n	1
m	m	m	1	m	1	1	1	1
k	k	k	k	k	1	k	1	1
n	n	n	1	n	1	1	1	1
1	1	1	1	1	1	1	1	1

U_1^e	0	t	p	e	m	k	n	1
0	0	0	0	0	0	0	0	0
t	0	t	p	t	t	k	t	1
p	0	p	p	p	p	k	p	1
e	0	t	p	e	m	k	n	1
m	0	t	p	m	m	k	n	1
k	0	k	k	k	k	k	k	1
n	0	t	p	n	n	k	n	1
1	0	1	1	1	1	1	1	1

Table 2: The operations U_T^e and U_1^e are uninorms on $(L_2, \leq, 0, 1)$.

4.3. Construction of Hua and Ji [35]

Recently, Hua and Ji [35] proposed a new construction method of uninorms on a bounded lattice which, once again, is also distinct from the subclasses of $\mathcal{U}_{\mathbf{min}}$ and $\mathcal{U}_{\mathbf{max}}$ as shown after its definition.

Theorem 4.8 ([35], Theorem 3.1 and 3.9)**.** *Let $(\mathbb{L}, \leq, 0, 1)$ be a bounded lattice and $e \in \mathbb{L}\setminus\{0,1\}$. Let T_e be a t-norm on $[0,e]$ and S_e be a t-conorm on $[e,1]$.*

(i) *If $x \parallel y$ for all $x \in I_e$ and $y \in [e,1[$ and R is a t-superconorm on $\mathbb{L}$ such that $R(x,y) < 1$ for all $x, y \in [e,1[$, then the function $U_R^e : \mathbb{L}^2 \to \mathbb{L}$ defined as follows*

$$U_R^e(x,y) = \begin{cases} T_e(x,y), & if\,(x,y) \in [0,e]^2, \\ x, & if\,(x,y) \in I_e \times [0,e] \cup I_e \times [e,1[, \\ y, & if\,(x,y) \in [0,e] \times I_e \cup [e,1[\times I_e, \\ R(x,y), & if\,(x,y) \in I_e \times I_e \cup\,]e,1]^2, \\ x \vee y, & if\,(x,y) \in D(e), \\ x \wedge y, & otherwise \end{cases} \tag{9}$$

is a uninorm on $\mathbb{L}$ with the neutral element e, where $D(e) = I_e \times \{1\} \cup \{1\} \times I_e \cup [0,e] \times \{1\} \cup \{1\} \times [0,e] \cup [e,1] \times \{e\} \cup \{e\} \times [e,1]$.

(ii) *If $x \parallel y$ for all $x \in I_e$ and $y \in]0,e]$ and F is a t-subnorm on $\mathbb{L}$ such that $F(x,y) > 0$ for all $x, y \in]0,e]$, then the function $U_F^e : \mathbb{L}^2 \to \mathbb{L}$ defined as follows*

$$U_F^e(x,y) = \begin{cases} S_e(x,y), & if\,(x,y) \in [e,1]^2, \\ x, & if\,(x,y) \in I_e \times]0,e] \cup I_e \times [e,1], \\ y, & if\,(x,y) \in\,]0,e] \times I_e \cup [e,1] \times I_e, \\ F(x,y), & if\,(x,y) \in I_e \times I_e \cup [0,e[^2, \\ x \wedge y, & if\,(x,y) \in E(e), \\ x \vee y, & otherwise \end{cases} \tag{10}$$

is a uninorm on $\mathbb{L}$ with the neutral element e, where $E(e) = I_e \times \{0\} \cup \{0\}_e \cup [e,1] \times \{0\} \cup \{0\} \times [e,1] \cup [0,e] \times \{e\} \cup \{e\} \times [0,e]$.

Clearly, the uninorms U_R^e and U_F^e do not belong to the subclasses of $\mathcal{U}_{\mathbf{min}}$ and $\mathcal{U}_{\mathbf{max}}$. For instance, $U_R^e(1,x) \neq x$ for all $x \in]0,e[$, i.e., $U_R^e(x,y) \neq y$, for some $(x,y) \in]e,1] \times \mathbb{L}\setminus[e,1]$ and hence $U_R^e \notin \mathcal{U}_{\mathbf{min}}$. Further, it is easy to see that $U_R^e \notin \mathcal{U}_{\mathbf{max}}$, since $U_R^e(0,x) \neq x$ for all $x \in]e,1[$. Similarly, we can prove that the uninorm U_F^e does not belong to the subclasses of $\mathcal{U}_{\mathbf{min}}$ and $\mathcal{U}_{\mathbf{max}}$.

Example 4.9. *Consider the lattice $L_3 = \{0, x, y, z, e, 1\}$ whose Hasse diagram is given in Figure 1. By using the construction method in Theorem 4.8, taking $S_e(x,y) = x \vee y$ on $[e,1]$ and $F(x,y) = x \wedge y$ on L_3, we obtain the uninorm $U_F^e : L_3^2 \to L_3$ given in Table 3:*

U_F^e	0	x	y	z	1	e
0	0	0	0	0	0	0
x	0	x	y	z	1	x
y	0	y	y	y	y	y
z	0	z	y	z	z	z
1	0	1	y	z	1	1
e	0	x	y	z	1	e

Table 3: The operation U_F^e is a uninorm on $(L_3, \leq, 0, 1)$.

5. Uninorms on $(\mathbb{L}, \leq, 0, 1)$ and Quasi-Projectivity

We begin this section by presenting some preliminary results (Section 5.1) that allow us to discuss the satisfaction of (**QP**) (Section 5.2) for the constructions and subclasses of uninorms on bounded lattices discussed in Section 4.

5.1. Satisfaction of (**QP**) *by* U *on specific sub-domains of* $\mathbb{L}$

In this section, we discuss the satisfaction of (**QP**) on some specific subdomains when the uninorm U takes a value that is typical of most constructions.

Proposition 5.1 ([12])**.** *Let* $(\mathbb{L}, \leq, 0, 1)$ *be a bounded lattice and* U *be a uninorm on* $\mathbb{L}$ *with an identity element* $e \in \mathbb{L} \setminus \{0, 1\}$*. Then the following are true:*

(i) $U|_{[0,e]^2} : [0, e]^2 \to [0, e]$ *is a t-norm on* $[0, e]$.
(ii) $U|_{[e,1]^2} : [e, 1]^2 \to [e, 1]$ *is a t-conorm on* $[e, 1]$.
(iii) $x \wedge y \leq U(x, y) \leq x \vee y$ *for all* $(x, y) \in A(e)$*, where* $A(e) = [0, e[\times]e, 1] \cup]e, 1] \times [0, e[$.
(iv) $U(x, y) \leq x$ *for all* $(x, y) \in \mathbb{L} \times [0, e]$.
(v) $U(x, y) \leq y$ *for all* $(x, y) \in [0, e] \times \mathbb{L}$.
(vi) $x \leq U(x, y)$ *for all* $(x, y) \in \mathbb{L} \times [e, 1]$.
(vii) $y \leq U(x, y)$ *for all* $(x, y) \in [e, 1] \times \mathbb{L}$.

Proposition 5.2. *Let* $(\mathbb{L}, \leq, 0, 1)$ *be a bounded lattice and* U *be a uninorm on* $\mathbb{L}$ *with an identity element* $e \in \mathbb{L} \setminus \{0, 1\}$*. Then* U *satisfies* (**QP**) *on* $[0, e] \times [0, e] \times \mathbb{L} \cup [e, 1] \times [e, 1] \times \mathbb{L}$ *.*

Proof. Let $x, y \in [0, e], z \in \mathbb{L}$ and $U(x, U(y, z)) = z$. Then by Proposition 5.1, we have

$$\begin{aligned} U(y, z) \leq z &\Longrightarrow U(x, U(y, z)) \leq U(x, z), && \lceil\text{by the monotonicity of } U \\ &\Longrightarrow z = U(x, U(y, z)) \leq U(x, z) \leq z, \\ &\Longrightarrow z = U(x, z), \\ &\Longrightarrow U(y, z) = U(y, U(x, z)), \\ &\Longrightarrow U(y, z) = U(x, U(y, z)), \\ &\Longrightarrow U(y, z) = z. \end{aligned}$$

Hence, U satisfies (**QP**) on $[0, e] \times [0, e] \times \mathbb{L}$.

Similarly, we can prove that U satisfies (**QP**) on $[e, 1] \times [e, 1] \times \mathbb{L}$. □

Proposition 5.3. *Let* $(\mathbb{L}, \leq, 0, 1)$ *be a bounded lattice and* U *be a conjunctive uninorm on* $\mathbb{L}$ *with an identity element* $e \in \mathbb{L} \setminus \{0, 1\}$*. Let* $Z \subseteq \mathbb{L}^2$ *such that* $U(x, y) = 0$ *for all* $(x, y) \in Z$*. Then* U *satisfies* (**QP**) *on* $Z \times \mathbb{L}$*.*

Proof. Let $U(x, y) = 0$ for all $(x, y) \in Z$ and $U(x, U(y, z)) = z$ for some $(x, y, z) \in Z \times \mathbb{L}$. Since U is a conjunctive uninorm on $\mathbb{L}$, we have $U(U(x, y), z) = U(0, z) = 0 = z$, which implies that $U(y, z) = U(y, 0) = 0 = z$. Hence, U satisfies (**QP**) on $Z \times \mathbb{L}$. □

The proof of the following proposition is along similar lines as above.

Proposition 5.4. *Let* $(\mathbb{L}, \leq, 0, 1)$ *be a bounded lattice and* U *be a disjunctive uninorm on* $\mathbb{L}$ *with an identity element* $e \in \mathbb{L}\setminus\{0,1\}$*. Let* $Z \subseteq \mathbb{L}^2$ *such that* $U(x,y) = 1$ *for all* $(x,y) \in Z$*. Then* U *satisfies* (**QP**) *on* $Z \times \mathbb{L}$*.*

Proposition 5.5. *Let* $(\mathbb{L}, \leq, 0, 1)$ *be a bounded lattice and* U *be a uninorm on* $\mathbb{L}$ *with an identity element* $e \in \mathbb{L}\setminus\{0,1\}$*. Let* Z *be subset of* $\mathbb{L}^2$ *such that* $U(x,y) \in \{x,y\}$ *for any* $(x,y) \in Z$*. Then* U *satisfies* (**QP**) *on* $Z \times \mathbb{L}$*.*

Proof. Let $(x,y) \in Z$ and $z \in \mathbb{L}$ such that $U(x, U(y,z)) = z$. Now, by our assumption, either $U(x,y) = y$ or $U(x,y) = x$. On the one hand, if $U(x,y) = y$ then we have,

$$\begin{aligned} U(x,y) = y &\Longrightarrow U(U(x,y),z)) = U(y,z), \\ &\Longrightarrow U(x, U(y,z)) = U(y,z), \\ &\Longrightarrow z = U(y,z). \end{aligned}$$

On the other hand, if $U(x,y) = x$ then we have,

$$\begin{aligned} U(x,y) = x &\Longrightarrow U(U(x,y),z)) = U(x,z), \\ &\Longrightarrow U(x, U(y,z)) = U(x,z), \\ &\Longrightarrow z = U(x,z), \\ &\Longrightarrow U(y,z) = U(y, U(x,z)), \\ &\Longrightarrow U(y,z) = U(x, U(y,z)), \\ &\Longrightarrow U(y,z) = z. \end{aligned}$$

Hence, U satisfies (**QP**) on $Z \times \mathbb{L}$. □

Corollary 5.6. *Let* $(\mathbb{L}, \leq, 0, 1)$ *be a bounded lattice and* U *be a uninorm on* $\mathbb{L}$ *with an identity element* $e \in \mathbb{L}\setminus\{0,1\}$*. Let* U *be such that either (i)* $U(x,y) = x \wedge y$ *for all* $(x,y) \in A(e)$ *or (ii)* $U(x,y) = x \vee y$ *for all* $(x,y) \in A(e)$*. Then* U *satisfies* (**QP**) *on* $A(e) \times \mathbb{L}$*.*

Proof. Let U be such that either (i) $U(x,y) = x \wedge y$ for all $(x,y) \in A(e)$ or (ii) $U(x,y) = x \vee y$ for all $(x,y) \in A(e)$. Then $U(x,y) \in \{x,y\}$ for all $(x,y) \in A(e)$ and hence, by Proposition 5.5, U satisfies (**QP**) on $A(e) \times \mathbb{L}$. □

5.2. Uninorms and (**QP**)

Based on the results presented above, in this section, we investigate the satisfaction of (**QP**) of the constructions and subclasses of uninorms presented in Section 4.

Proposition 5.7. *If* $U \in \mathcal{U}_{\mathbf{min}}$ *then* U *satisfies* (**QP**)*.*

Proof. Our approach to proving (**QP**) is as follows: we consider a pair x, y to come from any one of the four regions, viz., $[e,1]^2, [e,1] \times (\mathbb{L}\backslash[e,1]), (\mathbb{L}\backslash[e,1]) \times [e,1], (\mathbb{L}\backslash[e,1])^2$ and an admissible $z \in \mathbb{L}$. In each of these regions we show that either the antecedent of (**QP**) does not hold or (**QP**) is satisfied.

Case 1: Let $x, y \in [e,1]$. If $z \in [e,1]$ or $\mathbb{L}\backslash[e,1]$, then by Proposition 5.2, we see that U satisfies (**QP**) on $[e,1]^2 \times \mathbb{L}$.

Case 2: Let $x, y \in \mathbb{L}\backslash[e,1]$. Then by (5), we know that U is closed in $\mathbb{L}\backslash[e,1]$ and that $U(x,y) = F(x,y)$, where F is a t-subnorm on $\mathbb{L}\backslash[e,1]$.

- If $z \in \mathbb{L} \setminus [e,1]$ and $U(x, U(y,z)) = z$ we get $z = U(x, U(y,z)) \leq U(y,z) \leq z$, i.e., $U(y,z) = z$. Hence U satisfies (**QP**) on $(\mathbb{L} \setminus [e,1])^3$.

- If $z \in [e,1]$ and $U(x,U(y,z)) = z$ then by (5), we have $U(x,U(y,z)) = U(x,y) = F(x,y)$. Since F is closed on $\mathbb{L}\backslash[e,1]$, we have $z \in \mathbb{L}\backslash[e,1]$, a contradition to our assumption. Hence U satisfies (**QP**) on $(\mathbb{L}\backslash[e,1])^2 \times \mathbb{L}$.

Case 3: Let $(x,y) \in [e,1] \times (\mathbb{L}\backslash[e,1])$ then by (5), we have $U(x,y) = y$.

- If $z \in [e,1]$ and $U(x,U(y,z)) = z$ then by (5), we have $U(y,z) = y$ and $z = U(x,U(y,z)) = U(x,y) = y$, i.e., $z = y$, which is a contradiction.
- Let $z \in \mathbb{L}\backslash[e,1]$ and $U(x,U(y,z)) = z$. Since U is closed on $\mathbb{L}\backslash[e,1]$ and $x \in [e,1]$ we have $U(x,U(y,z)) = U(y,z) \Longrightarrow U(y,z) = z$. Hence U satisfies (**QP**) on $[e,1] \times (\mathbb{L}\backslash[e,1]) \times \mathbb{L}$.

Case 4: Let $(x,y) \in (\mathbb{L}\backslash[e,1]) \times [e,1]$ then by (5), we have $U(x,y) = x$.

- If $z \in [e,1]$ then by (5), we have $U(x,U(y,z)) = x$, i.e., $U(x,U(y,z)) \neq z$ for all $z \in [e,1]$.
- If $z \in \mathbb{L} \setminus [e,1]$ and $U(x,U(y,z)) = z$ then by (5), we have $U(y,z) = z$. Hence U satisfies (**QP**) on $(\mathbb{L}\backslash[e,1]) \times [e,1] \times \mathbb{L}$.

□

Proposition 5.8. *If $U \in \mathcal{U}_{\mathbf{max}}$ then U satisfies* (**QP**).

Proof. This proof is similar to that of Proposition 5.7 □

Proposition 5.9. *The uninorms ${U_1}^e$ and ${U_2}^e$ defined in Theorem 4.6 satisfy* (**QP**).

Proof. Our approach to proving (**QP**) is as given in the proof of Proposition 5.7, but over the following nine regions, viz., $[0,e]^2, [e,1]^2, I_e^2, [0,e] \times [e,1], [0,e] \times I_e, [e,1] \times [0,e], [e,1] \times I_e, I_e \times [0,e], I_e \times [e,1]$ and for admissible $z \in \mathbb{L}$.

Firstly, note that, since U_1^e is a uninorm on $\mathbb{L}$ with $e \in \mathbb{L}\backslash\{0,1\}$, by Proposition 5.2, if $(x,y) \in [0,e]^2 \cup [e,1]^2$ and $z \in \mathbb{L}$ then U_1^e satisfies (**QP**) on $([0,e]^2 \times \mathbb{L}) \cup ([e,1]^2 \times \mathbb{L})$. Further, if $z = 0$ then $U_1^e(y,0) = 0$ by (7) and hence U_1^e trivially satisfies (**QP**) on $\mathbb{L}^2 \times \{0\}$. If $z = 1$ and $U_1^e(x,U_1^e(y,1)) = 1$ then $x \neq 0$ and $y \neq 0$ by (7), which implies that $U_1^e(y,1) = 1$ and hence U_1^e satisfies (**QP**) on $\mathbb{L}^2 \times \{1\}$.

We discuss the rest of the 7 sub-regions for $z \in \mathbb{L}\backslash\{0,1\}$ below.

Case 1: Let $x,y \in I_e$. Then by (7), we have $U_1^e(x,y) = x \vee y$. Let $U_1^e(x,U_1^e(y,z)) = z$ for some $z \in \mathbb{L}$.

- If $z \in [e,1[$. Then, by (7), we have
$$U_1^e(x,U_1^e(y,z)) = U_1^e(x,y) = x \vee y \Longrightarrow z = x \vee y \ .$$
Since, by our assumption $z \in [e,1[$, we see that $x \vee y = z \geq e$ and hence $x < z$ and $y < z$, which contradicts the assumption on the underlying lattice $\mathbb{L}$ in Theorem 4.6, viz., $p \parallel q$ for all $p \in I_e$ and $q \in [e,1[$. Thus, $U_1^e(x,U_1^e(y,z)) \neq z$ for all $z \in [e,1[$.
- If $z \in]0,e[$. Once again, by (7), we have
$$U_1^e(x,U_1^e(y,z)) = x \vee y \Longrightarrow z = x \vee y \leq e \Longrightarrow x < e \text{ and } y < e,$$
which contradicts the fact that $x,y \in I_e$. Thus, $U_1^e(x,U_1^e(y,z)) \neq z$ for all $z \in]0,e[$.
- Let $z \in I_e$ and $U_1^e(x,U_1^e(y,z)) = z$. Then by (7), we have
$$\begin{aligned} U_1^e(x,U_1^e(y,z)) = x \vee y \vee z &\Longrightarrow z = x \vee y \vee z, \\ &\Longrightarrow y \vee z \leq x \vee y \vee z = z, \\ &\Longrightarrow z \leq y \vee z \leq z, \\ &\Longrightarrow y \vee z = z, \\ &\Longrightarrow U_1^e(y,z) = z. \end{aligned}$$
Hence U_1^e satisfies (**QP**) on ${I_e}^2 \times \mathbb{L}\backslash\{0,1\}$.

Case 2: Let $(x, y) \in A(e) = [0, e] \times [e, 1] \cup [e, 1] \times [0, e]$.

- Let $(x, y) \in Z = [0, e] \times [e, 1[\cup]e, 1[\times [0, e] \cup \{(0, 1), (1, 0), (1, e), (e, 1)\}$. Then by (7), we have $U_1^e(x, y) \in \{x, y\}$. Hence, by Proposition 5.5, U_1^e satisfies (**QP**) on $Z \times \mathbb{L}$.
- Let $(x, y) \in]0, e[\times \{1\} \cup \{1\} \times]0, e[$ and $z \in \mathbb{L} \backslash \{0\}$. Then by (7), we have

$$U_1^e(x, U_1^e(y, z)) = 1 \Longrightarrow z = 1 \Longrightarrow U_1^e(y, 1) = 1$$

Hence U_1^e satisfies (**QP**) on $A(e) \times \mathbb{L} \backslash \{0, 1\}$.

Case 3: Let $(x, y) \in X = [0, e] \times I_e \cup [e, 1] \times I_e$.

- Let $(x, y) \in J =]0, e] \times I_e \cup [e, 1[\times I_e$. Once again, by (7), we have $U_1^e(x, y) = y$. Hence, by Proposition 5.5, U_1^e satisfies (**QP**) on $J \times \mathbb{L}$.
- Let $(x, y) \in K = \{0\} \times I_e$. Then by (7), we have $U_1^e(x, y) = 0$. Hence, by Proposition 5.3, U_1^e satisfies (**QP**) on $K \times \mathbb{L}$.
- Let $(x, y) \in \{1\} \times I_e$ and $z \in \mathbb{L} \backslash \{0\}$. Then by (7), we have

$$U_1^e(x, U_1^e(y, z)) = 1 \Longrightarrow z = 1 \Longrightarrow U_1^e(y, 1) = 1$$

Hence U_1^e satisfies (**QP**) on $X \times \mathbb{L} \backslash \{0, 1\}$.

Case 4: The case where $(x, y) \in Y = I_e \times [0, e] \cup I_e \times [e, 1]$ is similar as **Case 3** above and U_1^e satisfies (**QP**) on $Y \times \mathbb{L} \backslash \{0, 1\}$.

Hence, U_1^e satisfies (**QP**) on $\mathbb{L}^3$ and by Theorem 3.3, $(\mathbb{L}, \sqsubseteq_{U_1^e}, 0, e)$ is a bounded U_1^e-poset. Similarly, we can prove that U_2^e satisfies (**QP**) on $\mathbb{L}^3$ and that $(\mathbb{L}, \sqsubseteq_{U_2^e}, 1, e)$ is a bounded U_2^e-poset. □

Proposition 5.10. *The Uninorms U_R^e and U_F^e defined in Theorem 4.8 satisfy* (**QP**).

Proof. Once again, we follow the above approach and consider a pair x, y to come from any one of the nine regions, viz., $[0, e]^2, [e, 1]^2, I_e^2, [0, e] \times [e, 1], [0, e] \times I_e, [e, 1] \times [0, e], [e, 1] \times I_e, I_e \times [0, e], I_e \times [e, 1]$ and for an admissible $z \in \mathbb{L}$.

Since U_R^e is a uninorm on $\mathbb{L}$, by Proposition 5.2, if $(x, y) \in [0, e]^2 \cup [e, 1]^2$ and $z \in \mathbb{L}$ then U_R^e satisfies (**QP**) on $[0, e]^2 \times \mathbb{L} \cup [e, 1]^2 \times \mathbb{L}$.

Case 1: Let $x, y \in I_e$ and $U_R^e(x, U_R^e(y, z)) = z$ for some $z \in \mathbb{L}$.

- If $z \in [0, e]$ then, by (9), we have

$$\begin{aligned} U_R^e(x, U_R^e(y, z)) = U_R^e(x, y) = R(x, y) &\Longrightarrow z = R(x, y), \\ &\Longrightarrow x \vee y \leq R(x, y) = z \leq e, \\ &\qquad \lceil \text{By definition of t-superconorm.} \\ &\Longrightarrow x \leq e \text{ and } y \leq e, \end{aligned}$$

which contradicts the fact $x, y \in I_e$ and hence $U_R^e(x, U_R^e(y, z)) \neq z$ for all $z \in [0, e]$.
- If $z \in [e, 1[$. Now, by (9), we have

$$\begin{aligned} U_R^e(x, U_R^e(y, z)) = U_R^e(x, y) = R(x, y) &\Longrightarrow z = R(x, y), \\ &\Longrightarrow x \vee y \leq R(x, y) = z, \\ &\qquad \lceil \text{By definition of t-superconorm.} \\ &\Longrightarrow x \leq z \text{ and } y \leq z. \end{aligned}$$

Since, by our assumption $z \in [e, 1[$, we see that $x \leq z$ and $y \leq z$, which contradicts the assumption on the underlying lattice $\mathbb{L}$ in Theorem 4.8, viz., $p \parallel q$ for all $p \in I_e$ and $q \in [e, 1[$. Thus $U_R^e(x, U_R^e(y, z)) \neq z$ for all $z \in [e, 1[$.

- If $z = 1$ then $U_R^e(y, 1) = 1$ by (9), and hence U_R^e trivially satisfies (**QP**) on ${I_e}^2 \times \{1\}$.
- Let $z \in I_e$ and $U_R^e(x, U_R^e(y, z)) = z$. Then by (9), we have

$$\begin{aligned} U_R^e(x, U_R^e(y, z)) = R(x, R(y, z)) &\Longrightarrow z = R(x, R(y, z)), \\ &\Longrightarrow x \vee R(y, z) \leq R(x, R(y, z)) = z, \\ &\qquad\qquad \lceil \text{By definition of t-superconorm.} \\ &\Longrightarrow y \vee z \leq R(y, z) \leq z, \\ &\Longrightarrow z \leq R(y, z) \leq z, \\ &\Longrightarrow R(y, z) = z, \\ &\Longrightarrow U_R^e(y, z) = z. \end{aligned}$$

Hence U_R^e satisfies (**QP**) on ${I_e}^2 \times \mathbb{L}$.

Case 2: Let $(x, y) \in A(e) = [0, e] \times [e, 1] \cup [e, 1] \times [0, e]$. Then by (9), we have $U_R^e(x, y) \in \{x, y, x \vee y, x \wedge y\}$. Hence, by Proposition 5.5 and Corollary 5.6, U_R^e satisfies (**QP**) on $A(e) \times \mathbb{L}$.

Case 3: Let $(x, y) \in [0, e] \times I_e \cup [e, 1] \times I_e$.

- Let $(x, y) \in Z = [0, e] \times I_e \cup [e, 1[\times I_e$. Then by (9), we have $U_R^e(x, y) = y$. Hence, by Proposition 5.5, U_R^e satisfies (**QP**) on $Z \times \mathbb{L}$.
- Let $(x, y) \in \{1\} \times I_e$. Once again, by (9), we have $U_R^e(x, y) = 1$. Hence, by Proposition 5.4, U_R^e satisfies (**QP**) on $\{1\} \times I_e \times \mathbb{L}$.

Case 4: The case where $(x, y) \in I_e \times [0, e] \cup I_e \times [e, 1]$ is similar as **Case 3** above and U_R^e satisfies (**QP**).

Hence, U_R^e satisfies (**QP**) on $\mathbb{L}^3$ and by Theorem 3.3, $(\mathbb{L}, \sqsubseteq_{U_R^e}, 1, e)$ is a bounded U_R^e-poset. Similarly, we can prove that U_F^e satisfies (**QP**) on $\mathbb{L}^3$ and $(\mathbb{L}, \sqsubseteq_{U_F^e}, 0, e)$ is a bounded U_F^e-poset. □

6. Impact of the above study: Some Perspectives

In this section, we present some interesting perspectives arising from our study, thus both vindicating and highlighting the investigations in this submission.

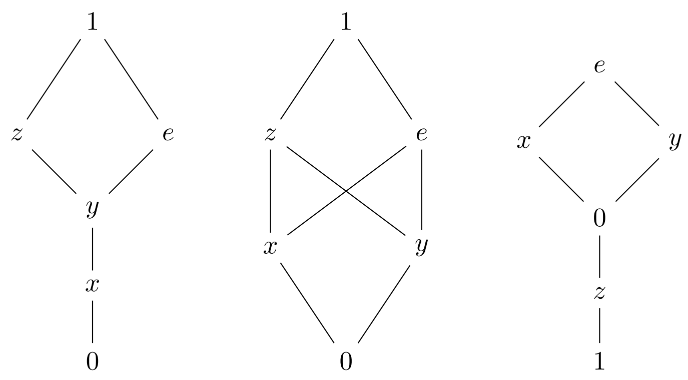
6.1. Posets obtained from $\sqsubseteq_U$ vs $\preceq_U$: A comparison

As was mentioned earlier, due to Proposition 1.2, whenever $x \preceq_U y$ then $x \leq y$, i.e., the best order theoretic structure we can hope to get is the lattice $\mathbb{L}$ we started with. In fact, often the poset obtained from $\preceq_U$ is not even a lattice, as shown below.

Example 6.1. *Let $(L = \{0, x, y, e, z, 1\}, \leq, 0, 1)$ be the bounded lattice depicted by the Hasse diagram in Fig 2. Define the operation U in Table 4 as follows:*

U	0	x	y	e	z	1
0	0	0	0	0	z	1
x	0	0	0	x	z	1
y	0	0	0	y	z	1
e	0	x	y	e	z	1
z	z	z	z	z	1	1
1	1	1	1	1	1	1

Table 4: The operation U is a uninorm on $(L, \leq, 0, 1)$ and a t-norm on $(L, \sqsubseteq_U, 1, e)$.



$(L, \leq, 0, 1)$ $\quad$ $(L, \preceq_U, 0, 1)$ $\quad$ $(L, \sqsubseteq_U, 1, e)$

Figure 2: The lattice $(L, \leq, 0, 1)$ and the U-posets obtained from the relations (2) and (1) for the operation given in Table 4.

From Fig 2, it can be verified that the poset $(L, \preceq_U, 0, 1)$ w.r.t. the order given in (2) is not a lattice. Note that neither $x \vee y$ nor $z \wedge e$ exist. However, the U-poset obtained from the order given in (1) is a lattice, as can be seen from Fig. 2.

As the following easy to prove result shows, the orders obtained from (1) and (2) are equal only if U is a t-norm.

Theorem 6.2. *Let* $(\mathbb{L}, \leq, 0, 1)$ *be a bounded lattice and* U *be an associative, commutative and monotone operation on* $\mathbb{L}$ *with an identity element* $e \in \mathbb{L}$*. Then* $\sqsubseteq_U = \preceq_U$ *if and only if* $e = 1$*.*

In fact, the obtained U-poset is always different from the original lattice $(\mathbb{L}, \leq, 0, 1)$ since 1 is no more the top element and the annihilator $U(0, 1) = a$ need not be either 0 or 1, see Lemma 4.2.

Theorem 6.3. *Let* $(\mathbb{L}, \leq, 0, 1)$ *be a bounded lattice and* U *be a uninorm on* $\mathbb{L}$ *with an identity element* $e \in \mathbb{L} \setminus \{0, 1\}$*. If* U *satisfies* (**QP**) *then* $(\mathbb{L}, \sqsubseteq_U, f, e)$ *is a bounded* U*-poset with top element* e *and bottom element* $f = U(1, 0)$*.*

Proof. Let U be a uninorm satisfying (**QP**). Then by Theorem 3.3, the relation $\sqsubseteq_U$ defined in (1) gives rise to a U-poset $(\mathbb{L}, \sqsubseteq_U)$. Since U is a uninorm with identity e, we have $U(U(0, 1), x) = U(0, 1)$ by Lemma 4.2 and $U(x, e) = x$ for all $x \in \mathbb{L}$. Hence, by the definition of order in (1), we have $U(0, 1) \sqsubseteq_U x \sqsubseteq_U e$ for all $x \in \mathbb{L}$, i.e., $U(0, 1)$ and e are the bottom and top elements w.r.t. $\sqsubseteq_U$, respectively. ☐

The above result also accounts for the case where a uninorm U can also be a nullnorm on $\mathbb{L}$, i.e., have an annihilator $a \in \mathbb{L} \setminus \{0, 1\}$, see Example 6.4 below.

Example 6.4. *Consider the lattice* $L = \{0, n, a, p, e, s, m, 1\}$ *whose Hasse diagram is given in Figure 3 (i). Consider the t-conorm* $S = \vee$ *on* $[0, a]$*. By using Theorem 3 in [36], one can define the corresponding function* F*, as given in Table 5, which is both a uninorm and a nullnorm.*

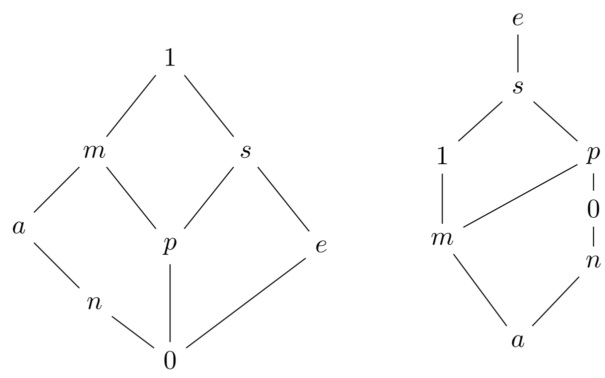
For the uninorms listed in Examples 4.5, 4.7, 4.9, while the posets obtained from $\preceq_U$ coincide with the original lattice considered, i.e., $(\mathbb{L}, \leq, 0, 1) \approx (\mathbb{L}, \preceq_U, 0, 1)$, Fig. 4 which presents the U-posets obtained from $\sqsubseteq_U$, illustrates the variety and richness in the obtained U-posets.

Note that while the original lattice $(L_1, \leq)$ is only bounded, it is neither modular (consider the sublattice $\{0, e, x, y, z\}$) nor a chain. In contrast, among the obtained U-posets, $(L_1, \sqsubseteq_{U_{(2,e)}})$ is modular, while $(L_1, \sqsubseteq_{U_S^e}$) is a chain and hence, is distributive.

Similarly, the underlying lattice $(L_2, \leq)$ is only bounded - it is neither modular (consider the sublattice $\{t, m, n, k, 1\}$) nor a chain. However, the obtained U-poset $(L_2, \sqsubseteq_{U_1^e})$ is a chain and hence, is distributive. On the other hand, the obtained U-poset $(L_2, \sqsubseteq_{U_T^e})$ is only bounded lattice.

F	0	n	a	p	e	s	m	1
0	0	n	a	0	0	0	a	a
n	n	n	a	n	n	n	a	a
a	a	a	a	a	a	a	a	a
p	0	n	a	p	p	p	m	m
e	0	n	a	p	e	s	m	1
s	0	n	a	p	s	s	m	1
m	a	a	a	m	m	m	m	m
1	a	a	a	m	1	1	m	1

Table 5: The operation F is both the uninorm and the nullnorm on $(L, \leq, 0, 1)$.



(i) $(L, \leq, 0, 1)$ (ii) $(L, \sqsubseteq_F, a, e)$

Figure 3: The lattice $(L, \leq, 0, 1)$ and the F-poset obtained from the operation given in Table 5 which is both a uninorm and a nullnorm.

Once again, the original lattice $(L_3, \leq)$ is only bounded - it is neither modular (consider the sublattice $\{0, e, y, z, 1\}$) nor a chain. However, the obtained U-poset $(L_3, \sqsubseteq_{U_F^e})$ is, in fact, a chain and hence, is distributive.

As was shown in the above examples, not only do we get a bounded U-poset from the uninorms, we can also obtain lattices and in some cases even special lattices, like linear, modular or distributive.

These observations lead to some rather natural questions, as listed below:

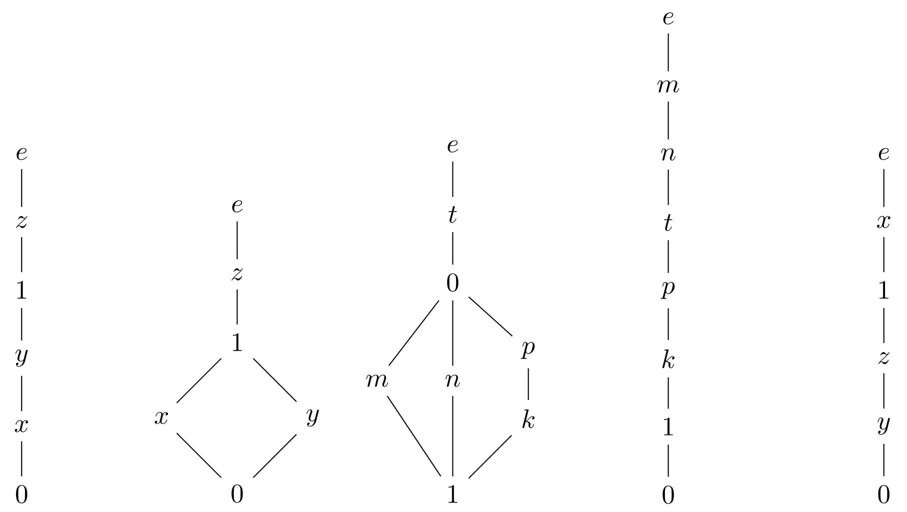
Problem 6.5. (i) *Do all classes of uninorms on bounded lattices $\mathbb{L}$ give rise to a U-Lattice?*
(ii) *Under what conditions on U do we get special lattices, like linear, modular or distributive?*

We have only some partial answers to the above questions. For instance, we show that all idempotent uninorms on a bounded lattice give rise to a U-semilattice by noting that both the Clifford's relation from U and the relation obtained as in (3) coincide, see Lemma 6.6. Further, we see that all internal uninorms on a bounded lattice $\mathbb{L}$ make it a chain $(\mathbb{L}, \sqsubseteq_U)$ and, hence, $(\mathbb{L}, \sqsubseteq_U)$ is also distributive.

Lemma 6.6. *If U is an idempotent uninorm on $\mathbb{L}$ with an identity element $e \in \mathbb{L} \setminus \{0, 1\}$ then $(\mathbb{L}, \sqsubseteq_U) = (\mathbb{L}, \leq_U)$.*

Proof. This follows directly from, Proposition 3.9. □

An internal uninorm U on $\mathbb{L}$ [37, 38] is such that $U(x, y) \in \{x, y\}$ for all $x, y \in \mathbb{L}$. Clearly, such a uninorm is also idempotent.



(i) $(L_1, \sqsubseteq_{U_S^e}, 0, e)$ (ii) $(L_1, \sqsubseteq_{U_{(2,e)}}, 0, e)$ (iii) $(L_2, \sqsubseteq_{U_T^e}, 1, e)$ (iv) $(L_2, \sqsubseteq_{U_1^e}, 0, e)$ (iv) $(L_3, \sqsubseteq_{U_F^e}, 0, e)$

Figure 4: The U-lattices $(L_i, \sqsubseteq_{U_i}, 0, e)$ obtained from the operations U given in Tables 1 – 3.

Theorem 6.7. *Let* $(\mathbb{L}, \leq, 0, 1)$ *be a bounded lattice and* U *be a internal uninorm on* $\mathbb{L}$ *with an identity element* $e \in \mathbb{L} \setminus \{0, 1\}$. *Then* $(\mathbb{L}, \sqsubseteq_U, U(0, 1), e)$ *is a bounded chain.*

Proof. This follows directly from Lemma 6.6, Theorem 6.3 and Remark 2.3. □

6.2. Uninorms as t-norms on the obtained U-posets

It is well known that a uninorm is a generalisation of a t-norm and a t-conorm. Algebraically speaking, both a uninorm and a t-norm on a bounded lattice $(\mathbb{L}, \leq, 0, 1)$ are commutative ordered monoids. However, additionally a t-norm is also an integral monoid, i.e., the identity element is also the top element of the bounded lattice, whereas by construction the identity $e \in \mathbb{L} \setminus \{0, 1\}$.

Interestingly, as shown above, if a uninorm U satisfies (**QP**) then the poset obtained from $\sqsubseteq_U$ is again a bounded poset with e as the top element and $f = U(0, 1)$ as the bottom element (see Theorem 6.3). Further, the given uninorm U remains order preserving on the poset $(\mathbb{L}, \sqsubseteq_U, f, e)$ due to its commutativity and associativity, see Theorem 6.8). Thus, on this obtained bounded U-poset $(\mathbb{L}, \sqsubseteq_U, f, e)$ a uninorm is also an integral commutative ordered monoid, which implies that U can be considered as a t-norm on $(\mathbb{L}, \sqsubseteq_U, f, e)$.

Theorem 6.8. *Let* $(\mathbb{L}, \leq, 0, 1)$ *be a bounded lattice and* U *be a uninorm on* $\mathbb{L}$ *with an identity element* $e \in \mathbb{L} \setminus \{0, 1\}$ *satisfying* (**QP**). *Then* U *becomes a t-norm on the generated bounded* U*-poset* $(\mathbb{L}, \sqsubseteq_U, f, e)$ *with identity* e *and zero element* $f = U(0, 1)$.

Proof. Since U satisfies (**QP**), from Theorem 6.3, we see that $(\mathbb{L}, \sqsubseteq_U, f, e)$ is a bounded poset with identity e and zero element $f = U(0, 1)$.

To show that U is a t-norm on the generated bounded U-poset $(\mathbb{L}, \sqsubseteq_U, f, e)$ it is sufficient to show that U is increasing w.r.t $\sqsubseteq_U$. Let $x, y \in \mathbb{L}$ such that $x \sqsubseteq_U y$, then by the definition of order in (1) there exists an element $\ell \in \mathbb{L}$ such that $U(\ell, y) = x$. For all $z \in \mathbb{L}$, we have

$$\begin{aligned} U(\ell, y) = x &\Longrightarrow U(U(\ell, y), z)) = U(x, z), \\ &\Longrightarrow U(\ell, U(y, z)) = U(x, z), \qquad \lceil \text{by Associativity.} \\ &\Longrightarrow U(x, z) \sqsubseteq_U U(y, z), \end{aligned}$$

i.e., U is increasing in the first variable w.r.t. $\sqsubseteq_U$. By commutativity of U, it also follows that U is increasing in the second variable w.r.t. $\sqsubseteq_U$. Hence, U is a t-norm on the bounded U-poset $(\mathbb{L}, \sqsubseteq_U, U(0,1), e)$. □

The following corollaries immediately follow from Proposition 5.7, 5.9, 5.10 and Theorem 6.8.

Corollary 6.9. *Let $U \in \mathcal{U}_{\mathbf{min}}$. Then the operation U is a t-norm on $(\mathbb{L}, \sqsubseteq_U, 0, e)$.*

Corollary 6.10. *Let $U \in \mathcal{U}_{\mathbf{max}}$. Then the operation U is a t-norm on $(\mathbb{L}, \sqsubseteq_U, 1, e)$.*

Corollary 6.11. *Let U_1^e and U_2^e be the uninorms as defined in Theorem 4.6. Then the operations U_1^e and U_2^e are t-norms on $(\mathbb{L}, \sqsubseteq_{U_1^e}, 0, e)$ and $(\mathbb{L}, \sqsubseteq_{U_2^e}, 1, e)$, respectively.*

Corollary 6.12. *Let U_R^e and U_F^e be the uninorms as defined in Theorem 4.8. Then the operations U_R^e and U_F^e are t-norms on $(\mathbb{L}, \sqsubseteq_{U_R^e}, 1, e)$ and $(\mathbb{L}, \sqsubseteq_{U_F^e}, 0, e)$, respectively.*

Recently, there have been a spate of works on constructing t-norms on given bounded posets. The discussion in this section offers an alternative by showing that uninorms satisfying (**QP**) allow us to construct posets on which they become t-norms.

7. Concluding Remarks

In this work, our main focus was on investigating the satisfaction of the Quasi-Projection (**QP**) property of uninorms on bounded lattices, since such uninorms give rise to orders from the domain-independent Clifford's relation. Thanks to the characterisation results of [32], most of the existing constructions of uninorms could be classified under the $\mathcal{U}_{\min}, \mathcal{U}_{\max}$ classes, and our results show that all these uninorms satisfy (**QP**). Some interesting outcomes of these investigations are the richer posets that we obtain with the Clifford's order and the ability to look at these uninorms on such obtained posets as t-norms.

There remain many interesting questions to be addressed, for instance, see Problem 6.5. Yet another poser that is worth discussing is the role of the underlying lattice in the satisfaction of (**QP**). For instance, in spite of $[0, 1]$ being a complete chain, there exist uninorms on $[0, 1]$ that do not satisfy (**QP**). We believe these and other such related questions can give some insightful relationships between the underlying order and the associative operations typically considered in fuzzy set theory.

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