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(S, N)- and R-implications: A state-of-the-art survey

Michał Baczyński[a,*], Balasubramaniam Jayaram[b]

[a] Institute of Mathematics, University of Silesia, ul. Bankowa 14, 40-007 Katowice, Poland
[b] Department of Mathematics and Computer Sciences, Sri Sathya Sai University, Prasanthi Nilayam, Andhra Pradesh, India

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Abstract

In this work we give a state-of-the-art review of two of the most established classes of fuzzy implications, viz., (S, N)- and R-implications. Firstly, we discuss their properties, characterizations and representations. Many new results concerning fuzzy negations and (S, N)-implications, notably their characterizations with respect to the identity principle and ordering property, are presented, which give rise to some representation results. Finally, using the presented facts, an almost complete characterization of the intersections that exist among some subfamilies of (S, N)- and R-implications are obtained.

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1. Introduction

Fuzzy implications were introduced and studied in the literature as the generalization of the classical implication that obeys the truth table provided in Table 1. Following are the two main ways of defining an implication in the Boolean lattice $(L, \wedge, \vee, \neg)$:

$$p \to q \equiv \neg p \vee q, \tag{1}$$

$$p \to q \equiv \max\{t \in L \mid p \wedge t \leqslant q\}, \tag{2}$$

where $p, q \in L$ and the relation $\leqslant$ is defined in the usual way, i.e., $p \leqslant q$ iff $p \vee q = q$, for every $p, q \in L$. The implication (1) is usually called the material implication, while (2) is from the intuitionistic logic framework, where the implication is obtained as the residuum of the conjunction, and is often called as the pseudocomplement of p relative to q (see [5]). Interestingly, despite their different formulas, definitions (1) and (2) are equivalent in the Boolean lattice $(L, \wedge, \vee, \neg)$. On the other hand, in the fuzzy logic framework, where the truth values can vary in the unit interval $[0, 1]$, the natural generalizations of the above definitions, viz., (S, N)- and R-implications, are not equivalent. This diversity is more a boon than a bane and has led to some intensive research on fuzzy implications for close to three decades. Quite understandably then, the most established and well-studied classes of fuzzy implications are the above (S, N)- and R-implications (cf. [10,14,15,22]).

* Corresponding author. Tel./fax: +48 32 258 29 76.
E-mail addresses: michal.baczynski@us.edu.pl (M. Baczyński), jbala@ieee.org (B. Jayaram).

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Table 1
Truth table for the classical implication

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

The main goal of this article is to give a state-of-the-art survey of these two families of fuzzy implications by discussing their algebraic properties, characterizations, representations and presenting both existing and some new results connected with their intersections.

The paper is organized as follows. Section 2 gives some preliminary results regarding basic fuzzy logic connectives. Some new results concerning fuzzy negations, especially natural negations obtained from t-norms and t-conorms and the related law of excluded middle have been presented. The definition of fuzzy implication and its basic algebraic properties are introduced and their interdependencies are explored in Section 3. In the next two sections, which should be seen as 'state-of-the-art', we recall the definitions of families of (S, N)- and R-implications listing their main properties and characterization theorems. Subsequently, some representation results are also obtained. In Section 6 a thorough analysis of the intersection between the above two classes is done and an almost complete characterization of this overlap is given based on the results available so far.

2. Preliminaries: basic fuzzy logic connectives

We assume that the reader is familiar with the classical results concerning basic fuzzy logic connectives, but to make this work self-contained, we introduce basic notations used in the text and we briefly mention some of the concepts and results employed in the rest of the work. We start with the notation of the conjugacy (see [23, p. 156]). By Φ we denote the family of all increasing bijections $\varphi\colon [0,1] \to [0,1]$. We say that functions $f, g\colon [0,1]^n \to [0,1]$, where $n \in \mathbb{N}$, are Φ-conjugate, if there exists $\varphi \in \Phi$ such that $g = f_\varphi$, where

$$f_\varphi(x_1, \ldots, x_n) := \varphi^{-1}(f(\varphi(x_1), \ldots, \varphi(x_n))), \quad x_1, \ldots, x_n \in [0,1].$$

Equivalently, g is said to be the Φ-conjugate of f. If A is a non-empty set, then in the family of all real functions from A to $\mathbb{R}$ we can consider the order induced from the standard partial order on $\mathbb{R}$, i.e., if $f_1, f_2\colon A \to \mathbb{R}$, then

$$f_1 \leqslant f_2 \quad :\Longleftrightarrow \quad f_1(x) \leqslant f_2(x) \quad \text{for all } x \in A.$$

If $f_1 \leqslant f_2$ and $f_1 \neq f_2$, then we will consider the strict partial order induced from $\mathbb{R}$ and we will write $f_1 < f_2$.

Definition 2.1 (*see Fodor and Roubens [14, p. 3]; Klement et al. [21, Definition 11.3]; Gottwald [15, Definition 5.2.1]*). A decreasing function $N\colon [0,1] \to [0,1]$ is called a fuzzy negation if $N(0) = 1$, $N(1) = 0$. A fuzzy negation N is called

(i) strict if it is strictly decreasing and continuous;
(ii) strong if it is an involution, i.e., $N(N(x)) = x$ for all $x \in [0,1]$.

Example 2.2. The classical negation $N_{\mathbf{C}}(x) = 1 - x$ is a strong negation, while $N_{\mathbf{K}}(x) = 1 - x^2$ is only strict, whereas $N_{\mathbf{D1}}$ and $N_{\mathbf{D2}}$—which are the least and greatest fuzzy negations—are non-strong negations:

$$N_{\mathbf{D1}}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x > 0, \end{cases} \qquad N_{\mathbf{D2}}(x) = \begin{cases} 1 & \text{if } x < 1, \\ 0 & \text{if } x = 1. \end{cases}$$

For more examples of fuzzy negations see [14] or [22]. For interesting facts concerning strong negations see [27].

Definition 2.3 (*see Schweizer and Sklar [33] and Klement et al. [21]*). (i) An associative, commutative and increasing operation $T\colon [0,1]^2 \to [0,1]$ is called a t-norm if it has the neutral element equal to 1.

(ii) An associative, commutative and increasing operation $S: [0,1]^2 \to [0,1]$ is called a t-conorm if it has the neutral element equal to 0.

If F is an associative binary operation on $[a, b]$ with the neutral element e, then the power notation $x_F^{[n]}$, where $n \in \mathbb{N}_0$, is defined by

$$x_F^{[n]} := \begin{cases} e & \text{if } n = 0, \\ x & \text{if } n = 1, \\ F(x, x_F^{[n-1]}) & \text{if } n > 1. \end{cases}$$

Definition 2.4 (*Klement et al. [21, Definitions 1.23, 2.9 and 2.13]; Fodor and Roubens [14, Definition 1.7]*). A t-norm T (t-conorm S, respectively) is said to be

(i) continuous if it is continuous in both the arguments;
(ii) border continuous if it is continuous on the boundary of the unit square $[0, 1]^2$, i.e., on the set $[0, 1]^2 \setminus (0, 1)^2$;
(iii) left-continuous if it is left-continuous in each component;
(iv) right-continuous if it is right-continuous in each component;
(v) idempotent if $T(x, x) = x$ ($S(x, x) = x$, respectively) for all $x \in [0, 1]$;
(vi) Archimedean if for every $x, y \in (0, 1)$ there is $n \in \mathbb{N}$ such that $x_T^{[n]} < y$ ($x_S^{[n]} > y$, respectively);
(vii) strict if T (S, respectively) is continuous and strictly monotone, i.e., $T(x, y) < T(x, z)$ ($S(x, y) < S(x, z)$, respectively) whenever $x > 0$ ($x < 1$, respectively) and $y < z$;
(viii) nilpotent if T (S, respectively) is continuous and if each $x \in (0, 1)$ is a nilpotent element, i.e., if for each $x \in (0, 1)$ there exists $n \in \mathbb{N}$ such that $x_T^{[n]} = 0$ ($x_S^{[n]} = 1$, respectively);
(ix) positive if $T(x, y) = 0$ ($S(x, y) = 1$, respectively) implies that either $x = 0$ or $y = 0$ ($x = 1$ or $y = 1$, respectively).

Remark 2.5. (i) For a continuous t-norm T the Archimedean property is given by the simpler condition, that $T(x, x) < x$, for all $x \in (0, 1)$ (see [15, Proposition 5.1.2]).

(ii) If a t-norm T is continuous and Archimedean, then T is nilpotent if and only if there exists some nilpotent element of T, which is equivalent to the existence of some zero divisor of T, i.e., there exist $x, y \in (0, 1)$ such that $T(x, y) = 0$ (see [21, Theorem 2.18]).

(iii) If a t-norm T is strict or nilpotent, then it is Archimedean. Conversely, every continuous and Archimedean t-norm is either strict or nilpotent (see [21, p. 33]).

(iv) A continuous Archimedean t-norm is positive if and only if it is strict (see [14, p. 9]).

(v) By the duality between t-norms and t-conorms, similar properties as above hold for t-conorms with the appropriate changes in either the inequality or the neutral element (cf. [21, Remark 2.20]; [14, Chapter 1]).

Example 2.6 (*see Klement et al. [21]*). Tables 2 and 3 list the basic t-norms and t-conorms with the properties they satisfy. Note that $T_{\mathbf{M}}$, $T_{\mathbf{P}}$ are positive t-norms, while $T_{\mathbf{LK}}$, $T_{\mathbf{D}}$ and $T_{\mathbf{nM}}$ are not. Similarly, $S_{\mathbf{M}}$, $S_{\mathbf{P}}$ are positive t-conorms, while $S_{\mathbf{LK}}$, $S_{\mathbf{D}}$ and $S_{\mathbf{nM}}$ are not.

Table 2
Examples of t-norms and their properties

Name	Formula	Properties
$T_{\mathbf{M}}$: minimum	$\min(x, y)$	Continuous, idempotent
$T_{\mathbf{P}}$: product	xy	Strict
$T_{\mathbf{LK}}$: Łukasiewicz	$\max(x + y - 1, 0)$	Nilpotent
$T_{\mathbf{D}}$: drastic product	$\begin{cases} 0 & \text{if } x, y \in [0, 1) \\ \min(x, y) & \text{otherwise} \end{cases}$	Archimedean, non-continuous
$T_{\mathbf{nM}}$: nilpotent minimum	$\begin{cases} 0 & \text{if } x + y \leqslant 1 \\ \min(x, y) & \text{otherwise} \end{cases}$	Non-Archimedean, left-continuous

Table 3
Examples of t-conorms and their properties

Name	Formula	Properties
$S_{\mathbf{M}}$: maximum	$\max(x, y)$	Continuous, idempotent
$S_{\mathbf{P}}$: probabilistic sum	$x + y - xy$	Strict
$S_{\mathbf{LK}}$: Łukasiewicz	$\min(x + y, 1)$	Nilpotent
$S_{\mathbf{D}}$: drastic sum	$\begin{cases} 1 & \text{if } x, y \in (0, 1] \\ \max(x, y) & \text{otherwise} \end{cases}$	Archimedean, non-continuous
$S_{\mathbf{nM}}$: nilpotent maximum	$\begin{cases} 1 & \text{if } x + y \geqslant 1 \\ \max(x, y) & \text{otherwise} \end{cases}$	Non-Archimedean, right-continuous

Theorem 2.7 (*Klement et al. [21, Theorem 5.11]*). *For a function* $T: [0, 1]^2 \to [0, 1]$ *the following statements are equivalent*:

(i) *T is a continuous* t-*norm*.
(ii) *T is uniquely representable as an ordinal sum of continuous Archimedean* t-*norms, i.e., there exists a uniquely determined* (*finite or countably infinite*) *index set A*, *a family of uniquely determined pairwise disjoint open subintervals* $\{(a_\alpha, e_\alpha)\}_{\alpha\in A}$ *of* $[0, 1]$ *and a family of uniquely determined continuous Archimedean* t-*norms* $(T_\alpha)_{\alpha\in A}$ *such that* $T = (\langle a_\alpha, e_\alpha, T_\alpha\rangle)_{\alpha\in A}$, *i.e.*,

$$T(x, y) = \begin{cases} a_\alpha + (e_\alpha - a_\alpha) \cdot T_\alpha\left(\dfrac{x - a_\alpha}{e_\alpha - a_\alpha}, \dfrac{y - a_\alpha}{e_\alpha - a_\alpha}\right) & \text{if } x, y \in [a_\alpha, e_\alpha], \\ \min(x, y) & \text{otherwise}. \end{cases} \tag{3}$$

One can associate a fuzzy negation to any t-norm or t-conorm as given in the definition below.

Definition 2.8 (*cf. Nguyen and Walker [31, Definition 5.5.2]; Klement et al. [21, p. 232]*). (i) Let T be a t-norm. A function $N_T: [0, 1] \to [0, 1]$ defined as

$$N_T(x) = \sup\{t \in [0, 1] | T(x, t) = 0\}, \quad x \in [0, 1] \tag{4}$$

is called the natural negation of *T*.

(ii) Let *S* be a t-conorm. A function $N_S: [0, 1] \to [0, 1]$ defined as

$$N_S(x) = \inf\{t \in [0, 1] | S(x, t) = 1\}, \quad x \in [0, 1] \tag{5}$$

is called the natural negation of *S*.

Remark 2.9. (i) It is easy to prove, that both N_T and N_S are fuzzy negations. In the literature N_T is also called the contour line C_0 of *T*, while N_S is called the contour line D_1 of *S* (see [24,26]).

(ii) Since for any t-norm *T* and any t-conorm *S* we have $T(x, 0) = 0$ and $S(x, 1) = 1$ for all $x \in [0, 1]$, the appropriate sets in (4) and (5) are non-empty.

(iii) Notice that if $S(x, y) = 1$ for some $x, y \in [0, 1]$, then $y \geqslant N_S(x)$ and if $T(x, y) = 0$ for some $x, y \in [0, 1]$, then $y \leqslant N_T(x)$.

Example 2.10. Table 4 gives the natural negations of the basic t-norms and t-conorms.

The next result be will useful in the sequel.

Proposition 2.11 (*cf. Maes and De Baets [24, Theorem 1(ii) and Corollary 1]*). *If a* t-*conorm S is right-continuous*, *then*:

(i) *for every* $x, y \in [0, 1]$ *the following equivalence holds*:

$$S(x, y) = 1 \iff N_S(x) \leqslant y; \tag{6}$$

Table 4
Examples of natural negations from t-norms and t-conorms

t-norm T	N_T	t-conorm S	N_S
Positive	$N_{\mathbf{D1}}$	Positive	$N_{\mathbf{D2}}$
$T_{\mathbf{LK}}$	$N_{\mathbf{C}}$	$S_{\mathbf{LK}}$	$N_{\mathbf{C}}$
$T_{\mathbf{D}}$	$N_{\mathbf{D2}}$	$S_{\mathbf{D}}$	$N_{\mathbf{D1}}$
$T_{\mathbf{nM}}$	$N_{\mathbf{C}}$	$S_{\mathbf{nM}}$	$N_{\mathbf{C}}$

(ii) *the infimum in* (5) *is the minimum*, *i.e.*,

$$N_S(x) = \min\{t \in [0,1] | S(x,t) = 1\}, \quad x \in [0,1],$$

where the right side exists for all $x \in [0,1]$;
(iii) N_S is right-continuous.

Proof. (i) Suppose that S is a right-continuous t-conorm and assume firstly that $S(x,y) = 1$ for some $x, y \in [0,1]$, so $y \in \{t \in [0,1] | S(x,t) = 1\}$, and hence $N_S(x) \leqslant y$. On the other side assume that $N_S(x) \leqslant y$ for some $x, y \in [0,1]$. We consider two cases. If $N_S(x) < y$, then there exists some $t' < y$ such that $S(x,t') = 1$, and the monotonicity of S implies that $S(x,y) = 1$. If $N_S(x) = y$ then either $y \in \{t \in [0,1] | S(x,t) = 1\}$ and thus $S(x,y) = 1$, or $y \notin \{t \in [0,1] | S(x,t) = 1\}$. Therefore, there exists a decreasing sequence $(t_i)_{i \in \mathbb{N}}$ such that $t_i > y$ and $S(x,t_i) = 1$ for all $i \in \mathbb{N}$ and $\lim_{i\to\infty} t_i = y$. By the right-continuity of S we get

$$S(x,y) = S\left(x, \lim_{i\to\infty} t_i\right) = \lim_{i\to\infty} S(x,t_i) = \lim_{i\to\infty} 1 = 1,$$

a contradiction.
(ii) From the previous point we know that S and N_S satisfy (6). Because $N_S(x) \leqslant N_S(x)$ for all $x \in [0,1]$, one has $S(x, N_S(x)) = 1$, which means, by the definition of N_S, that the infimum in (5) is the minimum.
(iii) The proof of this point is included in [24, Corollary 1]. □

Now we analyze the law of excluded middle, which in the classical case has the following form: $p \vee \neg p = \top$.

Definition 2.12. Let S be a t-conorm and N a fuzzy negation. We say that the pair (S, N) satisfies the law of excluded middle if

$$S(N(x), x) = 1, \quad x \in [0,1]. \tag{LEM}$$

A graphical interpretation of the law of excluded middle (LEM) is the following: the graph of the negation N demarcates the region on the unit square $[0,1]^2$ above which $S = 1$. It is possible that there are a few more points below the graph of N on whom S assumes the value 1. For example, consider the Łukasiewicz t-conorm $S_{\mathbf{LK}}$ and the strict negation $N_{\mathbf{K}}(x) = 1 - x^2$. Then $S_{\mathbf{LK}}(N_{\mathbf{K}}(0.5), 0.5) = S_{\mathbf{LK}}(0.75, 0.5) = 1$. Also notice that $S_{\mathbf{LK}}(0.5, 0.5) = 1$.
Now the following result is easy to see.

Lemma 2.13. *Let S be a* t-*conorm and N a fuzzy negation. If the pair* (S, N) *satisfies* (LEM), *then*

(i) $N \geqslant N_S$;
(ii) $N_S \circ N(x) \leqslant x$, *for all* $x \in [0,1]$.

Proof. (i) On the contrary, if for some $x_0 \in [0,1]$ we have $N(x_0) < N_S(x_0)$, then $S(N(x_0), x_0) < 1$ by the definition of N_S.
(ii) From Definition 2.8 we have

$$N_S(N(x)) = \inf\{t \in [0,1] | S(N(x), t) = 1\}, \quad x \in [0,1].$$

Now, since $S(N(x), x) = 1$ we have $x \geqslant N_S(N(x))$ for all $x \in [0,1]$. □

Example 2.14. Any t-conorm satisfies (LEM) with the greatest fuzzy negation $N_{\mathbf{D2}}$. Indeed, for any t-conorm S and $x \in [0, 1]$ we have

$$S(N_{\mathbf{D2}}(x), x) = \begin{cases} S(1, x) & \text{if } x < 1 \\ S(0, x) & \text{if } x = 1 \end{cases} = \begin{cases} 1 & \text{if } x < 1 \\ x & \text{if } x = 1 \end{cases} = 1. \tag{7}$$

From the previous result and Table 4 it follows, that if S is a positive t-conorm, then it satisfies (LEM) only with $N_{\mathbf{D2}}$.

Example 2.15. That the conditions in Lemma 2.13 are only necessary and not sufficient follow from the following example. Consider the non-right-continuous nilpotent maximum t-conorm

$$S_{\mathbf{nM}^*}(x, y) = \begin{cases} 1 & \text{if } x + y > 1, \\ \max(x, y) & \text{otherwise,} \end{cases} \qquad x, y \in [0, 1].$$

Then its natural negation is the classical negation, i.e., $N_S(x) = N_{\mathbf{C}}(x) = 1 - x$ and $N_S \circ N_{\mathbf{C}}(x) = x$ for all $x \in [0, 1]$. But the pair $(S_{\mathbf{nM}^*}, N_{\mathbf{C}})$ does not satisfy (LEM). Indeed, for $x = 0.5$ we get

$$S_{\mathbf{nM}^*}(N_S(0.5), 0.5) = S_{\mathbf{nM}^*}(0.5, 0.5) = 0.5.$$

Interestingly, for the right-continuous t-conorms the condition (i) from Lemma 2.13 is both necessary and sufficient.

Proposition 2.16. *For a right-continuous* t-*conorm S and a fuzzy negation N the following statements are equivalent*:

(i) *The pair* (S, N) *satisfies* (LEM).
(ii) $N \geqslant N_S$.

Proof. From Lemma 2.13 it is enough to show that (ii) $\Longrightarrow$ (i). Assume, that $N(x) \geqslant N_S(x)$ for all $x \in [0, 1]$. By virtue of (6) we get, that $S(x, N(x)) = 1$ for all $x \in [0, 1]$, so the pair (S, N) satisfies (LEM). □

In the class of continuous functions we get the following important fact.

Proposition 2.17 (*cf. Fodor and Roubens [14, Theorem 1.8]*). *For a continuous* t-*conorm S and a continuous fuzzy negation N the following statements are equivalent*:

(i) *The pair* (S, N) *satisfies* (LEM).
(ii) *S is a nilpotent* t-*conorm, i.e., S is* Φ-*conjugate with the Łukasiewicz* t-*conorm* $S_{\mathbf{LK}}$, *i.e., there exists* $\varphi \in \Phi$, *which is uniquely determined, such that S has the representation*

$$S(x, y) = \varphi^{-1}(\min(\varphi(x) + \varphi(y), 1)), \quad x, y \in [0, 1]$$

and

$$N(x) \geqslant N_S(x) = \varphi^{-1}(1 - \varphi(x)), \quad x \in [0, 1].$$

Proof. (i) $\Longrightarrow$ (ii) Assume, that S is a continuous t-conorm, N a continuous fuzzy negation and the pair (S, N) satisfies (LEM). Since S is continuous, it is uniquely representable as an ordinal sum of continuous Archimedean t-conorms (see [21, Corollaries 5.12 and 5.5]). The negation N is continuous, so Theorem 3.4 from [22] implies that it has exactly one fixed point $e \in (0, 1)$ for which $S(e, e) = S(N(e), e) = 1$. Therefore, for all $x \geqslant e$ we have $S(x, x) = 1$. Thus S is not idempotent or strict. This fact follows also from Example 2.14, since they are both positive t-conorms.

Assume now that S is not Archimedean, i.e., from the above observation it is not nilpotent. Since it is continuous, there exists $x_0 \in (0, 1)$ such that $S(x_0, x_0) = x_0$ (cf. Remark 2.5(i)). Hence there exists $a \in [x_0, e] \subset (0, 1)$ such that the summand $\langle a, 1, S_1 \rangle$ of S is such that S_1 a nilpotent t-conorm, and it is well known that S_1 is Φ-conjugate with the Łukasiewicz t-conorm $S_{\mathbf{LK}}$ for some unique $\phi \in \Phi$ (cf. [21, Corollary 5.7 and Remark 5.8]). One can calculate that in this case the natural negation of S has the following form:

$$N_S(x) = \begin{cases} 1 & \text{if } 0 \leqslant x \leqslant a, \\ a + (1-a)\phi^{-1}\left(1 - \phi\left(\frac{x-a}{1-a}\right)\right) & \text{if } a < x < 1, \\ 0 & \text{if } x = 1. \end{cases} \tag{8}$$

Indeed, firstly we show that $N_S(x) = 1$ for $x \in [0, a]$. If we assume that $a \geqslant N_S(a)$, then by the monotonicity of S and Proposition 2.16 we get

$$a = S(a, a) \geqslant S(N_S(a), a) = 1,$$

contradictory to the assumption $a < 1$. Assume now that $a < N_S(a)$. The representation of an ordinal sum implies that $S(a, a) = a$ and $S(a, 1) = 1$, so by the continuity of S there exists $y_0 > a$, such that $N_S(a) = S(a, y_0)$, hence

$$N_S(a) = S(a, y_0) = S(S(a, a), y_0) = S(a, S(a, y_0)) = S(a, N_S(a)) = 1.$$

Since N_S is a negation, it is decreasing, so $N_S(x) = 1$ for every $x \in [0, a]$. Next see that from the structure of the ordinal sum and the formula for $S_{\mathbf{LK}}$ we get, in particular, that

$$S(x, y) = \begin{cases} a + (1-a)\min\left(\phi^{-1}\left(\phi\left(\frac{x-a}{1-a}\right) + \phi\left(\frac{y-a}{1-a}\right)\right), 1\right) & \text{if } x, y \in [a, 1], \\ \max(x, y) & \text{if } x \in [a, 1], y \in [0, a), \end{cases}$$

from which we obtain (8) for $x \in (a, 1]$. Let us observe now that N_S is not continuous for $x = 1$, because $\lim_{x \to 1^-} N_S(x) = a < 1$. From Lemma 2.13(i) we get that $N \geqslant N_S$, but one can easily check that there does not exist any continuous fuzzy negation which is greater than or equal to N_S, a contradiction.

It shows that S is Archimedean, so it has to be nilpotent, and in this case $a = 0$. Therefore S is Φ-conjugate with the Łukasiewicz t-conorm for some unique $\varphi \in \Phi$. Now, from (5) and (8) for $\phi = \varphi$ and $a = 0$ we have that $N_S(x) = \varphi^{-1}(1 - \varphi(x))$ for all $x \in [0, 1]$ and because of Lemma 2.13(i) we get $N \geqslant N_S$.

(ii) $\Longrightarrow$ (i) The proof in this direction is immediate. □

Remark 2.18. We would like to underline, that in Theorem 1.8 of [14] the authors assume that N is strict. Further, the assumption that N is continuous is crucial. As a counterexample consider the strict t-conorm $S_{\mathbf{P}}$, which is continuous and satisfies (LEM) with non-continuous negation $N_{\mathbf{D2}}$. In particular, Propositions 1 and 2 in [25] cited from [14] without any assumption on N are not correct.

Finally in this section we present some new results regarding De Morgan triples.

Definition 2.19 (*Klement et al. [21, p. 232]*). A triple (T, S, N), where T is a t-norm, S is a t-conorm and N is a strict negation, is called a De Morgan triple if

$$T(x, y) = N^{-1}(S(N(x), N(y))), \quad S(x, y) = N^{-1}(T(N(x), N(y)))$$

for all $x, y \in [0, 1]$.

Theorem 2.20 (*Klement et al. [21, p. 232]*). *For a* t-*norm* T, t-*conorm* S *and a strict fuzzy negation* N *the following statements are equivalent*:

(i) (T, S, N) *is a De Morgan triple*.
(ii) N *is a strong negation and* S *is the* N-*dual of* T, *i.e.*, $S(x, y) = N(T(N(x), N(y)))$, *for all* $x, y \in [0, 1]$.

Using the above theorem it can be shown that the following relation exists between N_T and N_S.

Theorem 2.21. *Let* T *be a left-continuous* t-*norm and* S *be a* t-*conorm. If* (T, N_T, S) *is a De Morgan triple*, *then*

(i) $N_S = N_T$ *is a strong negation*,
(ii) S *is right-continuous*.

Proof. (i) From Theorem 2.20 it follows that N_T is a strong negation. Let us assume that $N_T \neq N_S$. Then there exists $x_0 \in (0, 1)$ such that $N_T(x_0) \neq N_S(x_0)$. We consider the following two cases:

(a) If $N_T(x_0) < N_S(x_0)$, then there exists $y \in (0, 1)$ such that $N_T(x_0) < y < N_S(x_0)$. Since N_T is a bijection there exists $y_0 \in (0, 1)$ such that $N_T(y_0) = y$, i.e., $N_T(x_0) < N_T(y_0) < N_S(x_0)$. Now, by the monotonicity of S and T, S being the N_T-dual of T and the definitions of N_T, N_S we have

$$\begin{aligned} S(x_0, N_T(y_0)) \neq 1 &\Longrightarrow N_T(T(N_T(x_0), N_T \circ N_T(y_0))) \neq 1 \\ &\Longrightarrow T(N_T(x_0), y_0) \neq 0 \\ &\Longrightarrow y_0 > N_T \circ N_T(x_0) = x_0 \\ &\Longrightarrow N_T(y_0) < N_T(x_0), \end{aligned}$$

a contradiction to our assumption.

(b) On the other hand, if $N_T(x_0) > N_S(x_0)$, then there exists $y \in (0, 1)$ such that $N_T(x_0) > y > N_S(x_0)$. Because N_T is a bijection there exists $y_0 \in (0, 1)$ such that $N_T(y_0) = y$, i.e., $N_T(x_0) > N_T(y_0) > N_S(x_0)$. Similarly as above we have

$$\begin{aligned} S(x_0, N_T(y_0)) = 1 &\Longrightarrow N_T(T(N_T(x_0), N_T \circ N_T(y_0))) = 1 \\ &\Longrightarrow T(N_T(x_0), y_0) = 0 \\ &\Longrightarrow y_0 \leqslant N_T \circ N_T(x_0) = x_0 \\ &\Longrightarrow N_T(y_0) \geqslant N_T(x_0), \end{aligned}$$

a contradiction to our assumption.

(ii) If T is a left-continuous t-norm and N_T is a strong negation, then it is straightforward to see that S, as an N_T-dual of T, is right-continuous. □

3. Fuzzy implications and basic algebraic properties

In the literature, especially at the beginnings, we can find several different definitions of fuzzy implications. In this article we will use the following one, which is equivalent to the definition introduced by Fodor and Roubens [14, Definition 1.15].

Definition 3.1. A function $I: [0, 1]^2 \to [0, 1]$ is called a fuzzy implication if, for all $x, y, z \in [0, 1]$, it satisfies

$$I(x, z) \geqslant I(y, z) \quad \text{if } x \leqslant y, \tag{I1}$$

$$I(x, y) \geqslant I(x, z) \quad \text{if } y \geqslant z, \tag{I2}$$

$$I(0, 0) = 1, \quad I(1, 1) = 1, \quad I(1, 0) = 0. \tag{I3}$$

The set of all fuzzy implications is denoted by $\mathcal{FI}$.

We can easily deduce, that each fuzzy implication I is constant for $x = 0$ and for $y = 1$, i.e., I satisfies the following properties, called left and right boundary conditions, respectively:

$$I(0, y) = 1, \quad y \in [0, 1], \tag{9}$$

$$I(x, 1) = 1, \quad x \in [0, 1]. \tag{10}$$

Therefore, I satisfies also the normality condition $I(0, 1) = 1$. Consequently, every fuzzy implication restricted to the set $\{0, 1\}^2$ coincides with the classical implication, so it fulfils the binary implication truth table, i.e., Table 1.

Definition 3.2 (*cf. Trillas and Valverde [37], Dubois and Prade [10], Fodor and Roubens [14], Gottwald [15]*). A fuzzy implication I is said to satisfy

(i) the left neutrality property or is said to be left neutral, if

$$I(1, y) = y, \quad y \in [0, 1]; \tag{NP}$$

(ii) the exchange principle, if

$$I(x, I(y, z)) = I(y, I(x, z)), \quad x, y, z \in [0, 1]; \tag{EP}$$

(iii) the identity principle, if

$$I(x, x) = 1, \quad x \in [0, 1]; \tag{IP}$$

(iv) the ordering property, if

$$x \leqslant y \iff I(x, y) = 1, \quad x, y \in [0, 1]; \tag{OP}$$

(v) the law of contraposition with respect to a fuzzy negation N, $CP(N)$, if

$$I(x, y) = I(N(y), N(x)), \quad x, y \in [0, 1]. \tag{CP}$$

Just as in the case of t-norms or t-conorms, a fuzzy negation can be obtained from fuzzy implications too as follows.

Definition 3.3. If I is a fuzzy implication, then the function $N_I : [0, 1] \to [0, 1]$ defined by

$$N_I(x) = I(x, 0), \quad x \in [0, 1],$$

is called the natural negation of I.

In the following results we discuss some relationships that exist between the above properties of fuzzy implications. They will be useful in the sequel.

Lemma 3.4 (*Baczyński and Jayaram [3, Lemma 2.2], cf. Bustince et al. [6, Lemma 1]*). *Let $I : [0, 1]^2 \to [0, 1]$ be any function and N a fuzzy negation.*

(i) *If I satisfies* (I1) *and* CP (N), *then I satisfies* (I2).
(ii) *If I satisfies* (I2) *and* CP (N), *then I satisfies* (I1).
(iii) *If I satisfies* (NP) *and $CP(N)$, then I satisfies* (I3) *and $N = N_I$ is a strong negation.*

Lemma 3.5 (*Baczyński and Jayaram [3, Corollary 2.3]*). *Let I be a fuzzy implication which satisfies* (NP). *If N_I is not a strong negation, then I does not satisfy the contrapositive symmetry with any fuzzy negation.*

Lemma 3.6 (*Baczyński and Jayaram [3, Lemma 2.4], cf. Bustince et al. [6, Lemma 1]*). *Let $I : [0, 1]^2 \to [0, 1]$ be any function and N_I be a strong negation.*

(i) *If I satisfies* CP (N_I), *then I satisfies* (NP).
(ii) *If I satisfies* (EP), *then I satisfies* (I3), (NP) *and* CP (N_I).

Corollary 3.7 (*Baczyński and Jayaram [3, Corollary 2.5]*). *Let I be a fuzzy implication which satisfies* (NP) *and* (EP). *Then I satisfies* CP (N) *with some fuzzy negation N if and only if $N = N_I$ is a strong negation.*

Lemma 3.8 (*Baczyński [2, Lemma 6]*). *If a function $I : [0, 1]^2 \to [0, 1]$ satisfies* (EP) *and* (OP), *then I satisfies* (I1), (I3), (NP) *and* (IP).

Proposition 3.9 (*cf. Fodor and Roubens [14, Corollary 1.1]*). *If a function $I : [0, 1]^2 \to [0, 1]$ satisfies* (EP) *and* (OP), *then the following statements are equivalent*:

(i) *N_I is a continuous function.*
(ii) *N_I is a strong negation.*

From Proposition 3.9 and Lemmas 3.4, 3.6 and 3.8 we obtain the following very important result (see also [14, Corollary 1.2]).

Corollary 3.10. *If a function* $I\colon [0,1]^2 \to [0,1]$ *satisfies* (EP), (OP) *and* N_I *is a continuous function*, *then* $I \in \mathcal{FI}$ *and it satisfies* (NP), (IP) *and* (CP) *only with respect to* N_I, *which is a strong negation.*

The following result shows that conjugation preserves all the properties defined in this section.

Proposition 3.11 (*cf. Baczyński [2, Lemma 11]*). *Let* $\varphi \in \Phi$. *If* $I\colon [0,1]^2 \to [0,1]$ *satisfies* (NP) ((EP), (IP), (OP)), *then* I_φ *also satisfies* (NP) ((EP), (IP), (OP)). *Moreover*, *if I is continuous so is* I_φ.

In the next two sections we introduce the two main classes of fuzzy implications that have been well established in the literature, viz., (S, N)- and R-implications. After giving their definitions, we discuss some of their properties and also give their characterizations, where available. Also some representation results are obtained based on these characterizations.

4. (S, N)-implications: properties and characterizations

It is well known in the classical logic that the unary negation $\neg$ can combine with any other binary operation to generate rest of the binary operations. This distinction of the unary $\neg$ is also shared by the Boolean implication $\rightarrow$, if defined in the following usual way:

$$p \rightarrow q \equiv \neg p \vee q.$$

The definition as given above was the first to catch the attention of the researchers leading to the following class of fuzzy implications.

4.1. Definition and examples

Definition 4.1 (cf. *Trillas and Valverde [37], Dubois and Prade [10], Fodor and Roubens [14], Klir and Yuan [22] and Alsina and Trillas [1]*). A function $I\colon [0,1]^2 \to [0,1]$ is called an (S, N)-implication if there exists a t-conorm S and a fuzzy negation N such that

$$I(x, y) = S(N(x), y), \quad x, y \in [0,1]. \tag{11}$$

If N is a strong negation, then I is called a strong implication (shortly S-implication). Moreover, if an (S, N)-implication is generated from S and N, then we will often denote this by $I_{S,N}$.

It should be noted that some authors use the name S-implication, even if the negation N is not strong (see [21, Definition 11.5]). Since the name S-implication was firstly introduced in the fuzzy logic framework by Trillas and Valverde [36,37] with the restrictive assumptions (S is a continuous t-conorm and N is a strong negation), we use, in a general case, the name '(S, N)-implication' proposed by Alsina and Trillas [1].

Example 4.2. The following Table 5 lists few of the well-known (S, N)-implications along with the underlying t-conorms and fuzzy negations.

Remark 4.3 (*cf. Klir and Yuan [22, Theorem 11.1]*). It is easy to see that for a fixed fuzzy negation N, if S_1, S_2 are two comparable t-conorms such that $S_1 \leqslant S_2$, then $I_{S_1,N} \leqslant I_{S_2,N}$. Similarly, if S is a fixed t-conorm, then for two comparable fuzzy negations N_1, N_2 such that $N_1 \leqslant N_2$ we get $I_{S,N_1} \leqslant I_{S,N_2}$. Thus, from Example 2.2 and Table 5 we have that $I_{\mathbf{D}}$ and $I_{\mathbf{TD}}$ are, respectively, the least and the greatest (S, N)-implications. Further, $I_{\mathbf{KD}}$ and $I_{\mathbf{DC}}$ are, respectively, the least and the greatest S-implications obtainable from the classical negation $N_{\mathbf{C}}$. Since there does not exist any least and greatest strong negation, there does not exist any least and greatest S-implication.

4.2. Properties of (S, N)-implications

In this section we analyze (S, N)-implications with respect to the algebraic properties given in Definition 3.2. We begin with the following remark.

Table 5
Examples of basic (S, N)-implications

S	N	(S, N)-implication $I_{S,N}$	Name	Properties
$S_{\mathbf{M}}$	$N_{\mathbf{C}}$	$I_{\mathbf{KD}}(x, y) = \max(1 - x, y)$	Kleene-Dienes	(NP), (EP)
$S_{\mathbf{P}}$	$N_{\mathbf{C}}$	$I_{\mathbf{RC}}(x, y) = 1 - x + xy$	Reichenbach	(NP), (EP)
$S_{\mathbf{LK}}$	$N_{\mathbf{C}}$	$I_{\mathbf{LK}}(x, y) = \min(1 - x + y, 1)$	Łukasiewicz	(NP), (EP), (IP), (OP)
$S_{\mathbf{nM}}$	$N_{\mathbf{C}}$	$I_{\mathbf{FD}}(x, y) = \begin{cases} 1 & \text{if } x \leqslant y \\ \max(1 - x, y) & \text{if } x > y \end{cases}$	Fodor	(NP), (EP), (IP), (OP)
$S_{\mathbf{D}}$	$N_{\mathbf{C}}$	$I_{\mathbf{DC}}(x, y) = \begin{cases} y & \text{if } x = 1 \\ 1 - x & \text{if } y = 0 \\ 1 & \text{if } x < 1 \text{ and } y > 0 \end{cases}$	–	(NP), (EP), (IP)
$S_{\mathbf{M}}$	$N_{\mathbf{K}}$	$I_{\mathbf{MK}}(x, y) = \max(1 - x^2, y)$	–	(NP), (EP)
Any S	$N_{\mathbf{D1}}$	$I_{\mathbf{D}}(x, y) = \begin{cases} 1 & \text{if } x = 0 \\ y & \text{if } x > 0 \end{cases}$	Least (S, N)-implication	(NP), (EP)
Any S	$N_{\mathbf{D2}}$	$I_{\mathbf{TD}}(x, y) = \begin{cases} 1 & \text{if } x < 1 \\ y & \text{if } x = 1 \end{cases}$	Greatest (S, N)-implication	(NP), (EP), (IP)

Remark 4.4 (*see Trillas and Valverde [37] and Baczyński and Jayaram [3]*). (i) All (S, N)-implications are fuzzy implications which satisfy (NP) and (EP).

(ii) If I is an (S, N)-implication obtained from a fuzzy negation N, then $N = N_I$.

(iii) Because of Corollary 3.7 we get that an (S, N)-implication I satisfies $CP(N)$ with some fuzzy negation N if and only if $N = N_I$ is a strong negation, i.e., I is an S-implication.

Since not all (S, N)-implications satisfy the identity principle (IP) (for example $I_{\mathbf{RC}}$ and $I_{\mathbf{KD}}$), we analyze this axiom for (S, N)-implications now.

Lemma 4.5. *For a* t-*conorm S and a fuzzy negation N the following statements are equivalent*:

(i) *The* (S, N)*-implication* $I_{S,N}$ *satisfies* (IP).
(ii) *The pair* (S, N) *satisfies* (LEM).

Proof. (i) $\Longrightarrow$ (ii) If $I_{S,N}$ satisfies (IP), then $S(N(x), x) = I(x, x) = 1$, for all $x \in [0, 1]$, i.e., the pair (S, N) satisfies (LEM).

(i) $\Longrightarrow$ (ii) Conversely, if the pair (S, N) satisfies (LEM), then $I_{S,N}(x, x) = S(N(x), x) = 1$ for all $x \in [0, 1]$, i.e., $I_{S,N}$ satisfies (IP). □

Now, using the above fact and Proposition 2.17 we get

Theorem 4.6 (*cf. Trillas and Valverde [37, Theorem 3.3]*). *For a continuous* t-*conorm S and a continuous fuzzy negation N the following statements are equivalent*:

(i) *The* (S, N)*-implication* $I_{S,N}$ *satisfies* (IP).
(ii) *S is a nilpotent* t-*conorm and* $N \geqslant N_S$.

As noted earlier, not all natural generalizations of the classical implication to multi-valued logic satisfy the ordering property (OP). In the following we discuss results on (S, N)-implications with respect to their ordering property (OP).

Theorem 4.7. *For a* t-*conorm S and a fuzzy negation N the following statements are equivalent*:

(i) *The* (S, N)*-implication* $I_{S,N}$ *satisfies* (OP).
(ii) $N = N_S$ *is a strong negation and the pair* (S, N_S) *satisfies* (LEM).

Proof. (i) $\Longrightarrow$ (ii) If an (S, N)-implication $I_{S,N}$ generated from some t-conorm S and some fuzzy negation N satisfies (OP), then it satisfies (IP). By Lemma 4.5 the pair (S, N) satisfies (LEM). Therefore, from Lemma 2.13, we know that

$N(x) \geqslant N_S(x)$ and $N_S \circ N(x) \leqslant x$ for all $x \in [0,1]$. Assume, that for some $x_0 \in [0,1]$ we have $N_S \circ N(x_0) < x_0$. Then there exists $y \in [0,1]$ such that $N_S \circ N(x_0) < y < x_0$. But now, $I(x_0, y) = S(N(x_0), y) = 1$, i.e., $x_0 \leqslant y$ from (OP), which is a contradiction. Hence $x = N_S \circ N(x)$ for all $x \in [0,1]$. By virtue of Proposition 3.8 from [3] we get, that N_S is a continuous fuzzy negation and N is a strictly decreasing fuzzy negation. Further, since N_S is a negation, we get

$$N_S(N_S(x)) \geqslant N_S(N(x)) = x, \quad x \in [0,1]. \tag{12}$$

Let us fix an arbitrary $x_0 \in [0,1]$. Thus, again from the monotonicity of N_S, we have $N_S(N_S(N_S(x_0))) \leqslant N_S(x_0)$. On the other side, since (12) is true for all $x \in [0,1]$, it holds for $x = N_S(x_0)$. Therefore $N_S(N_S(N_S(x_0))) \geqslant N_S(x_0)$. Because x_0 was arbitrarily fixed, as a result we obtain

$$N_S(N_S(N_S(x))) = N_S(x), \quad x \in [0,1].$$

But N_S is a continuous negation, and so is onto, i.e., for every $y \in [0,1]$ there exists $x \in [0,1]$ such that $y = N_S(x)$. Hence

$$N_S(N_S(y)) = N_S(N_S(N_S(x))) = N_S(x) = y, \quad y \in [0,1].$$

Thus N_S is a strong negation and it is obvious now that $N = N_S$.

(ii) $\Longrightarrow$ (i) Assume that S is any t-conorm such that N_S is a strong negation and the pair (S, N_S) satisfies (LEM). We will show that the (S, N)-implication generated from S and N_S satisfies (OP). To this end fix arbitrarily $x, y \in [0,1]$ such that $x \leqslant y$. By the monotonicity of S we have

$$I_{S,N_S}(x, y) = S(N_S(x), y) \geqslant S(N_S(x), x) = 1.$$

On the other hand, if $I_{S,N_S}(x, y) = 1$ for some $x, y \in [0,1]$, then

$$1 = I_{S,N_S}(x, y) = S(N_S(x), y),$$

which implies, by Remark 2.9(iii), that $y \geqslant N_S \circ N_S(x) = x$, i.e., I_{S,N_S} satisfies (OP). □

The last result in this subsection shows some relationships between (S, N)-implications and their conjugates.

Theorem 4.8 (*Baczyński and Jayaram [3, Theorem 5.5]*). *If $I_{S,N}$ is an (S, N)-implication generated from some* t-*conorm S and some fuzzy negation N, then the Φ-conjugate of $I_{S,N}$ is also an (S, N)-implication generated from the Φ-conjugate* t-*conorm of S and the Φ-conjugate fuzzy negation of N, i.e., if $\varphi \in \Phi$, then*

$$(I_{S,N})_\varphi = I_{S_\varphi, N_\varphi}.$$

4.3. Characterizations and representations of (S, N)-implications

A first characterization of S-implications was presented by Trillas and Valverde ([37, Theorem 3.2], see also [14, Theorem 1.13]) and it can be written in the following form.

Theorem 4.9 (*Baczyński and Jayaram [3, Theorem 2.8]*). *For a function $I: [0,1]^2 \to [0,1]$ the following statements are equivalent*:

(i) *I is an S-implication.*
(ii) *I satisfies* (I1) (*or* (I2)), (NP), (EP) *and* (CP) *with respect to a strong negation N.*

The characterization of the family of all (S, N)-implications is still an open problem, but some partial results were recently obtained by the authors in [3]. In the following we list them and use them in the sequel to obtain some representation results for (S, N)-implications with certain algebraic properties.

Theorem 4.10 (*Baczyński and Jayaram [3, Theorems 2.6, 5.1 and 5.2]; Baczyński and Jayaram [4]*). *For a function* $I\colon [0,1]^2 \to [0,1]$ *the following statements are equivalent*:

(i) *I is an* (S, N)*-implication generated from some* t*-conorm S and some continuous* (*strict*, *strong*) *fuzzy negation N*.
(ii) *I satisfies* (I1) (*or* (I2)), (EP) *and the function* N_I *is a continuous* (*strict*, *strong*) *fuzzy negation*.

Moreover, *the representation of the* (S, N)*-implication* (11) *is unique in this case.*

It should be noted, that the properties in Theorem 4.10 are mutually independent (see [3,4]).
In the class of continuous function we have the following important result.

Proposition 4.11 (*Baczyński and Jayaram [3, Proposition 5.4]*). *For a function* $I\colon [0,1]^2 \to [0,1]$ *the following statements are equivalent*:

(i) *I is a continuous* (S, N)*-implication*.
(ii) *I is an* (S, N)*-implication with continuous S and N*.

As an interesting consequence of the above characterization, Theorem 4.6 and the representation of strong negations (see [35]; [14, Theorem 1.1]) we get the following corollary.

Corollary 4.12 (*cf. Bustince et al. [6, Theorem 7]*). *For a function* $I\colon [0,1]^2 \to [0,1]$ *the following statements are equivalent*:

(i) *I continuous and satisfies* (I1) (*or* (I2)), (EP), (IP) *and* N_I *is a strong negation*.
(ii) *I is an S-implication with a continuous* t*-conorm S*, *which satisfies* (IP).
(iii) *There exist* $\varphi, \psi \in \Phi$ *such that*

$$I(x, y) = \varphi^{-1}(\min(\varphi(\psi^{-1}(1 - \psi(x))) + \varphi(y), 1)), \quad x, y \in [0, 1]$$

and

$$\psi^{-1}(1 - \psi(x)) \geqslant \varphi^{-1}(1 - \varphi(x)), \quad x \in [0, 1].$$

Finally, we are able to prove one of the most important results connected with (S, N)-implications. It is usually called in the literature as Smets–Magrez Theorem, since the equivalence between points (i) and (v) was presented by Smets and Magrez in [34]. We would like to note, that the similar result has been obtained by Trillas and Valverde 2 years earlier (see [37, Theorem 3.4]). In fact, in the article [34], the authors required more conditions than it is necessary, which was shown by Fodor and Roubens (see [14, Theorem 1.15]) and Baczyński [2].

Theorem 4.13 (*cf. Trillas and Valverde [37], Smets and Magrez [34], Fodor and Roubens [14], Baczyński [2]*). *For a function* $I\colon [0,1]^2 \to [0,1]$ *the following statements are equivalent*:

(i) *I is continuous and satisfies* (EP), (OP).
(ii) *I is an* (S, N)*-implication obtained from a continuous* t*-conorm S and a continuous fuzzy negation N*, *which satisfies* (OP).
(iii) *I is a continuous* (S, N)*-implication*, *which satisfies* (OP).
(iv) *I is an* (S, N)*-implication obtained from a nilpotent* t*-conorm and its natural negation*.
(v) *I is* Φ*-conjugate with the Łukasiewicz implication* $I_{\mathbf{LK}}$, *i.e.*, *there exists* $\varphi \in \Phi$, *which is uniquely determined*, *such that*

$$I(x, y) = \varphi^{-1}(\min(1 - \varphi(x) + \varphi(y), 1)), \quad x, y \in [0, 1]. \tag{13}$$

Proof. (i) $\Longrightarrow$ (ii) Assume, that I is a continuous function which satisfies (EP) and (OP). This implies that N_I is a continuous function. By virtue of Corollary 3.10 we obtain that $I \in \mathcal{FI}$ and I satisfies (NP), (IP). Moreover N_I is a strong negation. Now, from Theorem 4.10, it follows that I is an (S, N)-implication generated from some t-conorm S and some strong negation N. Since I is continuous, Proposition 4.11 implies that S is also continuous.

(ii) $\Longleftrightarrow$ (iii) This equivalence is a consequence of Proposition 4.11.

(iv) $\Longleftrightarrow$ (v) This equivalence is obvious and follows from the representation of nilpotent t-conorms.

(ii) $\Longrightarrow$ (v) Let us assume, that I is an (S, N)-implication obtained from a continuous t-conorm S and a continuous fuzzy negation N, which satisfies (OP). From Remark 4.4 we see that I also satisfies (NP), (EP) and $N = N_I$. Again by virtue of Corollary 3.10 we get that N_I is a strong negation and I satisfies also (IP). From Corollary 4.12 there exist $\varphi, \psi \in \Phi$ such that

$$I(x, y) = \varphi^{-1}(\min(\varphi(\psi^{-1}(1 - \psi(x))) + \varphi(y), 1)), \quad x, y \in [0, 1]$$

and

$$N_I(x) = \psi^{-1}(1 - \psi(x)) \geqslant \varphi^{-1}(1 - \varphi(x)) = N_S(x), \quad x \in [0, 1].$$

Since I satisfies (OP), by virtue of Theorem 4.7 we know, that $N_I = N_S$. In particular, I has the form (13).

(v) $\Longrightarrow$ (i) It can be easily verified that $I_{\mathbf{LK}}$ is a continuous fuzzy implication which satisfies (EP) and (OP). Because of Proposition 3.11 the function I given by (13) is also continuous and satisfies (EP), (OP). □

Remark 4.14. The continuity of the t-conorm S is important in the above theorem. Consider the Fodor implication $I_{\mathbf{FD}}$. It is a non-continuous (S, N)-implication, not conjugate with the Łukasiewicz implication and obtained from the right-continuous t-conorm $S_{\mathbf{nM}}$. But it satisfies both (EP) and (OP).

5. *R*-implications: properties and characterizations

From Section 4 we see that (S, N)-implications are the generalization of the material implication of classical two-valued logic to fuzzy logic. From the isomorphism that exists between classical two-valued logic and classical set theory one can immediately note the following set theoretic identity:

$$\overline{P} \cup Q = \overline{P \setminus Q} = \cup\{T | P \cap T \subseteq Q\},$$

where P, Q are subsets of some universal set. The above identity gives another way of defining the Boolean implication and is employed in the intuitionistic logic. Fuzzy implications obtained as the generalization of the above identity form the family of residuated implications, usually called as R-implications in the literature. In this section, we investigate properties they possess, analogous to our treatment of (S, N)-implications in Section 4.

5.1. Definition and examples

Definition 5.1 (*see Trillas and Valverde [37], Dubois and Prade [10], Fodor and Roubens [14] and Gottwald [15]*). A function $I: [0, 1]^2 \to [0, 1]$ is called an R-implication if there exists a t-norm T such that

$$I(x, y) = \sup\{t \in [0, 1] | T(x, t) \leqslant y\}, \quad x, y \in [0, 1]. \tag{14}$$

If an R-implication is generated from a t-norm T, then we will often denote this by I_T.

Firstly observe, that since for any t-norm T and all $x \in [0, 1]$ we have $T(x, 0) = 0$, the appropriate set in (14) is non-empty. It is very important to note that the name 'R-implication' is a short version of 'residual implication', and I_T is also called as 'the residuum of T'. This class of implications is related to a residuation concept from the intuitionistic logic. In fact, it has been shown that in this context this name is proper only for left-continuous t-norms.

Proposition 5.2 (*cf. Gottwald [15, Proposition 5.4.2 and Corollary 5.4.1]*). *For a* t-*norm T the following statements are equivalent*:

(i) *T is left-continuous*.
(ii) *T and I_T form an adjoint pair*, *i.e.*, *they satisfy the residuation property*

$$T(x, t) \leqslant y \iff I_T(x, y) \geqslant t, \quad x, y, t \in [0, 1]. \tag{RP}$$

Table 6
Examples of basic R-implications

t-norm T	R-implication I_T	Name	Properties
$T_{\mathbf{M}}$	$I_{\mathbf{GD}}(x,y)=\begin{cases}1 & \text{if } x\leqslant y\\ y & \text{if } x>y\end{cases}$	Gödel	(NP), (EP), (IP), (OP)
$T_{\mathbf{P}}$	$I_{\mathbf{GG}}(x,y)=\begin{cases}1 & \text{if } x\leqslant y\\ \dfrac{y}{x} & \text{if } x>y\end{cases}$	Goguen	(NP), (EP), (IP), (OP)
$T_{\mathbf{LK}}$	$I_{\mathbf{LK}}$	Łukasiewicz	(NP), (EP), (IP), (OP)
$T_{\mathbf{nM}}$	$I_{\mathbf{FD}}$	Fodor	(NP), (EP), (IP), (OP)
$T_{\mathbf{D}}$	$I_{\mathbf{TD}}$	Greatest R-implication	(NP), (EP), (IP)

(iii) *The supremum in* (14) *is the maximum*, *i.e.*,

$$I_T(x,y)=\max\{t\in[0,1]\,|\,T(x,t)\leqslant y\},$$

where the right side exists for all $x,y\in[0,1]$.

Example 5.3. Table 6 lists few of the well-known R-implications along with their t-norms from which they have been obtained. Note that $T_{\mathbf{D}}$ is not left-continuous but still $I_{\mathbf{TD}}$ is a fuzzy implication.

Remark 5.4. It is easy to observe that if T_1 and T_2 are two comparable t-norms such that $T_1\leqslant T_2$, then $I_{T_1}\geqslant I_{T_2}$. Thus, since $T_{\mathbf{D}}$ and $T_{\mathbf{M}}$ are, respectively, the least and the greatest t-norms (see [21, Remark 1.5(i)]), from Table 6 we have that $I_{\mathbf{GD}}$ and $I_{\mathbf{TD}}$ are, respectively, the least and the greatest R-implications. Further, since there does not exist any least left-continuous t-norm, there does not exist any greatest R-implication generated from a left-continuous t-norm. $I_{\mathbf{GD}}$ is again the least R-implication generated from a left-continuous t-norm.

5.2. *Properties of R-implications*

Now we examine R-implications based on the properties introduced in Section 3.

Theorem 5.5 (*cf. Fodor and Roubens [14] and Gottwald [15]*). *If T is any* t-*norm* (*not necessarily left-continuous*), *then $I_T\in\mathcal{FI}$. Moreover*, *I_T satisfies* (NP) *and* (IP).

Proof. Let T be any t-norm and let I_T be a function defined by (14). Firstly we show that I_T satisfies axioms from Definition 3.1. Let $x_1,x_2,y\in[0,1]$ be arbitrarily fixed and assume that $x_1\leqslant x_2$. We have to show that $I_T(x_1,y)\geqslant I_T(x_2,y)$, which is equivalent to the inequality

$$\sup\{t\in[0,1]\,|\,T(x_1,t)\leqslant y\}\geqslant\sup\{t\in[0,1]\,|\,T(x_2,t)\leqslant y\},$$

so it is enough to show the inclusion

$$\{t\in[0,1]\,|\,T(x_1,t)\leqslant y\}\supset\{t\in[0,1]\,|\,T(x_2,t)\leqslant y\}.$$

Take any $t\in[0,1]$ such that $T(x_2,t)\leqslant y$. Since $x_1\leqslant x_2$, from the monotonicity of a t-norm T we get $T(x_1,t)\leqslant T(x_2,t)$, thus $T(x_1,t)\leqslant y$. Therefore I_T satisfies (I1). Now assume that $x,y_1,y_2\in[0,1]$ are arbitrarily fixed and $y_1\leqslant y_2$. By similar deduction as above we get, that (I2) follows from

$$\{t\in[0,1]\,|\,T(x,t)\leqslant y_1\}\subset\{t\in[0,1]\,|\,T(x,t)\leqslant y_2\},$$

which is the consequence of the assumption, that $y_1\leqslant y_2$. Moreover,

$$I_T(0,0)=\sup\{t\in[0,1]\,|\,T(0,t)\leqslant 0\}=\sup\{t\in[0,1]\,|\,0\leqslant 0\}=1,$$

$$I_T(1,1)=\sup\{t\in[0,1]\,|\,T(1,t)\leqslant 1\}=\sup\{t\in[0,1]\,|\,t\leqslant 1\}=1,$$

$$I_T(1,0)=\sup\{t\in[0,1]\,|\,T(1,t)\leqslant 0\}=\sup\{t\in[0,1]\,|\,t\leqslant 0\}=0,$$

which shows, that $I_T \in \mathcal{FI}$. Further, for any $y \in [0, 1]$ we get

$$I_T(1, y) = \sup\{t \in [0, 1] | T(1, t) \leqslant y\} = \sup\{t \in [0, 1] | t \leqslant y\} = y,$$

so I_T satisfies (NP). Finally, since $T(x, 1) = x$ for all $x \in [0, 1]$, we have

$$I_T(x, x) = \sup\{t \in [0, 1] | T(x, t) \leqslant x\} = 1. \qquad \square$$

Without additional assumptions on a t-norm T, the residual implication I_T may not satisfy other basic properties.

Example 5.6. (i) The R-implication $I_{\mathbf{TD}}$ from Table 6 satisfies (EP), but does not satisfy (OP). Moreover, since the natural negation of $I_{\mathbf{TD}}$ is discontinuous, from Lemma 3.5 we see that $I_{\mathbf{TD}}$ does not satisfy (CP) with any fuzzy negation.

(ii) Consider the non-left-continuous t-norm given in [21], Example 1.24(i) as follows:

$$T_{\mathbf{B}^*}(x, y) = \begin{cases} 0 & \text{if } (x, y) \in (0, 0.5)^2, \\ \min(x, y) & \text{otherwise,} \end{cases} \qquad x, y \in [0, 1].$$

Then the R-implication generated from $T_{\mathbf{B}^*}$ is given by:

$$I_{\mathbf{TB}^*}(x, y) = \begin{cases} 1 & \text{if } x \leqslant y, \\ 0.5 & \text{if } x > y \text{ and } x \in [0, 0.5), \\ y & \text{otherwise,} \end{cases} \qquad x, y \in [0, 1].$$

Obviously, $I_{\mathbf{TB}^*}$ satisfies (OP) but not (EP). To see this, let $x = 0.4$, $y = 0.5$ and $z = 0.3$. In this situation we have

$$I_{\mathbf{TB}^*}(x, I_{\mathbf{TB}^*}(y, z)) = I_{\mathbf{TB}^*}(0.4, I_{\mathbf{TB}^*}(0.5, 0.3)) = I_{\mathbf{TB}^*}(0.4, 0.3) = 0.5,$$

while

$$I_{\mathbf{TB}^*}(y, I_{\mathbf{TB}^*}(x, z)) = I_{\mathbf{TB}^*}(0.5, I_{\mathbf{TB}^*}(0.4, 0.3)) = I_{\mathbf{TB}^*}(0.5, 0.5) = 1.$$

(iii) Consider now the non-left-continuous t-norm T given in [21], Example 1.24(ii) as follows:

$$T_{\mathbf{B}}(x, y) = \begin{cases} 0 & \text{if } (x, y) \in (0, 1)^2 \setminus [0.5, 1)^2, \\ \min(x, y) & \text{otherwise,} \end{cases} \qquad x, y \in [0, 1].$$

Then the R-implication generated from $T_{\mathbf{B}}$ is given by:

$$I_{\mathbf{TB}}(x, y) = \begin{cases} 1 & \text{if } x \leqslant y \text{ or } x, y \in [0, 0.5), \\ 0.5 & \text{if } x \in [0.5, 1) \text{ and } y \in [0, 0.5), \\ y & \text{otherwise,} \end{cases} \qquad x, y \in [0, 1].$$

It is obvious that $I_{\mathbf{TB}}$ does not satisfy (OP). Now, putting $x = 0.8$, $y = 0.5$ and $z = 0.3$, we have

$$I_{\mathbf{TB}}(x, I_{\mathbf{TB}}(y, z)) = I_{\mathbf{TB}}(0.8, I_{\mathbf{TB}}(0.5, 0.3)) = I_{\mathbf{TB}}(0.8, 0.5) = 0.5,$$

while

$$I_{\mathbf{TB}}(y, I_{\mathbf{TB}}(x, z)) = I_{\mathbf{TB}}(0.5, I_{\mathbf{TB}}(0.8, 0.3)) = I_{\mathbf{TB}}(0.5, 0.5) = 1.$$

Hence $I_{\mathbf{TB}}$ does not satisfy (EP), too.

(iv) Interestingly, if we consider the following non-left-continuous nilpotent minimum t-norm (see [26, p. 851])

$$T_{\mathbf{nM}^*}(x, y) = \begin{cases} 0 & \text{if } x + y < 1, \\ \min(x, y) & \text{otherwise,} \end{cases} \qquad x, y \in [0, 1],$$

then the R-implication generated from $T_{\mathbf{nM}^*}$ is the Fodor implication $I_{\mathbf{FD}}$, which satisfies both (EP) and (OP).

Remark 5.7. Though both the t-norms $T_{\mathbf{B}^*}$ and $T_{\mathbf{B}}$ are not left-continuous, $T_{\mathbf{B}^*}$ is a border continuous t-norm which, as we show below, is an equivalent condition for the corresponding I_T to have (OP).

Proposition 5.8. *For a* t-*norm T the following statements are equivalent*:

(i) *The R-implication I_T satisfies* (OP).
(ii) *T is border continuous.*

Proof. Let T be any t-norm and for any fixed $x \in [0, 1]$, consider the vertical segment $T_x(.) = T(x, .)$. Obviously, T_x is a one-variable function from $[0, 1]$ to $[0, x]$. Now notice that if T is a border continuous t-norm, then for every $x \in (0, 1)$ there exists a neighborhood $U_x = (a_x, 1]$, where $a_x \in (0, 1)$ is dependent on the chosen x, such that T_x is continuous on U_x.

(i) $\Longrightarrow$ (ii) Let I_T satisfy (OP). On the contrary, if T is not border continuous, then there exists an $x_0 \in (0, 1)$ such that $\lim_{y\to 1^-} T(x_0, y) = z < x_0$. Now, by definition $I_T(x_0, z) = \sup\{t \in [0, 1] | T(x_0, t) \leqslant z\} = 1$, a contradiction to the fact that I_T satisfies (OP).

(ii) $\Longrightarrow$ (i) Let T be a border continuous t-norm. On the contrary, let I_T not satisfy the ordering property (OP). Since for any T we have that $x \leqslant y \Longrightarrow I_T(x, y) = 1$, there exists $x_0, y_0 \in (0, 1)$ such that $x_0 > y_0$ and $I_T(x_0, y_0) = 1$. Let $x' = T_{x_0}(a_{x_0}) = T(x_0, a_{x_0}) \leqslant x_0$. Now, we have two cases. If $y_0 > x'$ then there exists $t \in U_{x_0}$ and $t \neq 1$ such that $T_{x_0}(t) = y_0$ contradicting our assumption that $I_T(x_0, y_0) = 1$. On the other hand, if $y_0 \leqslant x'$ then by definition $I_T(x_0, y_0) \leqslant a_x < 1$. Hence I_T satisfies (OP). □

From Theorem 5.5 and straightforward calculations we get

Theorem 5.9 (*see Fodor and Roubens [14, Theorem 1.14]*). *If I_T is an R-implication based on a left-continuous* t-*norm T, then $I_T \in \mathcal{FI}$ and I_T satisfies* (NP), (EP), (IP), (OP) *and the following inequality*:

$$I_T(x, T(x, y)) \geqslant y, \quad x, y \in [0, 1]. \tag{15}$$

Moreover, I_T is left-continuous with respect to the first variable and right-continuous with respect to the second variable.

It should be noted that (15) is important in many applications of R-implications (see [32,22]).

Once again, as in the case of (S, N)-implications, it can be easily shown that the conjugate of an R-implication is also an R-implication.

Proposition 5.10 (*cf. Baczyński [2, Proposition 12]*). *If I_T is an R-implication based on some* t-*norm T, then the Φ-conjugate of I_T is also an R-implication generated from the Φ-conjugate* t-*norm of T, i.e., if $\varphi \in \Phi$, then*

$$(I_T)_\varphi = I_{T_\varphi}.$$

From Proposition 2.31 of [21] and the above Proposition 5.10 we have (see also [8]):

Proposition 5.11. *For a* t-*norm T the following statements are equivalent*:

(i) $(I_T)_\varphi = I_T$, *for all* $\varphi \in \Phi$.
(ii) $I_T = I_{\mathbf{TD}}$ *or* $I_T = I_{\mathbf{GD}}$.

5.3. Characterizations and representations of R-implications

Our main goal in this subsection is to present the characterization of R-implications. In fact, presently such a characterization is available only for R-implications obtained from left-continuous t-norms. We also discuss the representations of R-implications for some special classes of left-continuous t-norms. To do this we consider the dual situation now, i.e., the method of obtaining t-norms from fuzzy implications by a residuation principle. Since for every fuzzy implication I we have (10), the following function $T_I\colon [0, 1]^2 \to [0, 1]$ defined by

$$T_I(x, y) = \inf\{t \in [0, 1] | I(x, t) \geqslant y\}, \quad x, y \in [0, 1] \tag{16}$$

is a well-defined function of two variables and similarly to Proposition 5.2 we have

Proposition 5.12 (*Baczyński [2, Corollary 10]*). *For a fuzzy implication I the following statements are equivalent*:

(i) *I is right-continuous with respect to the second variable*.
(ii) *I and T_I form an adjoint pair, i.e., they satisfy* (RP).
(iii) *The infimum in* (16) *is the minimum, i.e.,*

$$T_I(x, y) = \min\{t \in [0, 1] | I(x, t) \geqslant y\}, \tag{17}$$

where the right side exists for all $x, y \in [0, 1]$.

Remark 5.13. It is interesting to note that formula (16) does not always generate a t-norm. For example, if I is the Reichenbach implication $\boldsymbol{I_{RC}}$, then for $x > 0$ we obtain $T_{I_{\mathbf{RC}}}(x, 1) = 1$, so $T_{I_{\mathbf{RC}}}$ is not a t-norm.

Using similar techniques as in the proof of Theorem 5.9 one can prove the following result.

Theorem 5.14 (*cf. Fodor and Roubens [14, Theorem 1.14] or Gottwald [15, Theorem 5.4.1]*). *If $I \in \mathcal{FI}$ satisfies* (EP), (OP) *and is right-continuous with respect to the second variable, then T_I defined by* (17) *is a left-continuous* t-*norm. Moreover $I = I_{T_I}$, i.e.,*

$$I(x, y) = \max\{t \in [0, 1] | T_I(x, t) \leqslant y\}, \quad x, y \in [0, 1].$$

From Theorems 5.9 and 5.14 we get the following well-known characterization of *R*-implications generated from left-continuous t-norms.

Corollary 5.15 (*cf. Miyakoshi and Shimbo [30], see also Fodor and Roubens [14, Theorem 1.14]*). *For a function $I: [0, 1]^2 \to [0, 1]$ the following statements are equivalent*:

(i) *I is an R-implication generated from a left-continuous* t-*norm*.
(ii) *I satisfies* (I2), (EP), (OP) *and is right-continuous with respect to the second variable*.

It should be noted that in contrast to the characterization of (S, N)-implications the problem of mutual-independence of the above properties is still an open problem.

We also have the following connection between a left-continuous t-norm *T* and the *R*-implication generated from *T*.

Lemma 5.16. *If T is a left-continuous* t-*norm, then $T = T_{I_T}$.*

Once again, it should be noted that in the case when T is a non-left-continuous t-norm, the T_{I_T} obtained can be different from T. For example, consider the t-norm $\boldsymbol{T_{nM^*}}$ from Example 5.6(iv). Now, the *R*-implication is equal to the Fodor implication $I_{\boldsymbol{T_{nM^*}}} = \boldsymbol{I_{FD}}$, while $T_{\boldsymbol{I_{FD}}} = \boldsymbol{T_{nM}} \neq \boldsymbol{T_{nM^*}}$.

From the above results we obtain the following characterization of left-continuous t-norms.

Corollary 5.17 (*see Baczyński [2, Corollary 10]*). *For a function $T: [0, 1]^2 \to [0, 1]$ the following statements are equivalent*:

(i) *T is a left-continuous* t-*norm*.
(ii) *There exists a fuzzy implication I, which satisfies* (EP), (OP) *and is right-continuous with respect to the second variable, such that T is given by* (17).

The following interesting result has also been proven in [2].

Theorem 5.18 (*Baczyński [2, Theorem 15]*). *If a function $I: [0, 1]^2 \to [0, 1]$ satisfies* (OP), (EP) *and N_I is strong, then a function $T: [0, 1]^2 \to [0, 1]$ defined as*

$$T(x, y) = N_I(I(x, N_I(y))), \quad x, y \in [0, 1],$$

is a t-*norm. Additionally, T and I satisfy* (RP).

For some classes of t-norms we have the following representations of *R*-implications.

Theorem 5.19 (*Fodor and Roubens [14, Theorem 1.16]*). *If T is a continuous Archimedean* t-*norm with the additive generator t*, *i.e.*, *a continuous*, *strictly decreasing function* $t:[0,1]\to[0,1]$ *with* $t(1)=0$, *then*

$$I_T(x,y)=t^{-1}(\max(t(y)-t(x),0)),\quad x,y\in[0,1]. \tag{18}$$

Theorem 5.20 (*De Baets and Mesiar [7], cf. Gottwald [15, Proposition 5.4.3]*). *If T is a continuous* t-*norm with the ordinal sum structure* (3) *as given in Theorem* 2.7, *then*

$$I_T(x,y)=\begin{cases}\psi^{-1}(I_{T_\alpha}(\psi_\alpha(x),\psi_\alpha(y))) & \text{if } x>y \text{ and } x,y\in[a_\alpha,e_\alpha],\\ I_{\mathbf{GD}}(x,y) & \text{otherwise},\end{cases}$$

for all $x,y\in[0,1]$, *where* ψ_α *is the affine transformation defined by* $\psi_\alpha(x)=\frac{x-a_\alpha}{e_\alpha-a_\alpha}$.

In the case, when a left-continuous t-norm T is represented as an ordinal sum of left-continuous t-subnorms a similar representation for I_T was obtained (see [29, Theorem 5]).

It is important to note that for continuous R-implications we have the following result, which is an other version of Theorem 4.13.

Theorem 5.21 (*cf. Miyakoshi and Shimbo [30]; Fodor [13, Corollary 2]*). *For a function* $I:[0,1]^2\to[0,1]$ *the following statements are equivalent*:

(i) *I is continuous and satisfies* (EP), (OP).
(ii) *I is a continuous R-implication based on some left-continuous* t-*norm*.
(iii) *I is an R-implication based on some continuous* t-*norm*, *with a strong natural negation* N_I.
(*iv*) *I is an R-implication based on some nilpotent* t-*norm*.
(*v*) *I is Φ-conjugate with the Łukasiewicz implication* $I_{\mathbf{LK}}$.

6. Intersections among families of (S, N)- and R-implications

In this section, we discuss the different overlaps that exist between the above families. Let us denote the different families of fuzzy implications as follows:

- $\mathbb{I}_{\mathbb{S},\mathbb{N}}$—the family of all (S, N)-implications;
- $^{\mathbb{C}}\mathbb{I}_{\mathbb{S},\mathbb{N}}$—the family of all continuous (S, N)-implications;
- $\mathbb{I}_{\mathbb{S}_\mathbb{C},\mathbb{N}_\mathbb{C}}$—the family of all (S, N)-implications obtained from continuous t-conorms and continuous negations;
- $\mathbb{I}_{\mathbb{S}}$—the family of all S-implications;
- $\mathbb{I}_{\mathbb{S},\mathbb{N}_\mathbb{S}}$—the family of all (S, N)-implications obtained from t-conorms and their natural negations;
- $\mathbb{I}_{\mathbb{S}^*,\mathbb{N}_\mathbb{S}^*}$—the family of all (S, N)-implications obtained from right-continuous t-conorms and their natural negations which are strong;
- $\mathbb{I}_{\mathbb{T}}$—the family of all R-implications;
- $^{\mathbb{C}}\mathbb{I}_{\mathbb{T}}$—the family of all continuous R-implications;
- $\mathbb{I}_{\mathbb{T}_{\mathbb{LC}}}$—the family of all R-implications obtained from left-continuous t-norms;
- $^{\mathbb{C}}\mathbb{I}_{\mathbb{T}_{\mathbb{LC}}}$—the family of all continuous R-implications obtained from left-continuous t-norms;
- $\mathbb{I}_{\mathbb{T}_\mathbb{C}}$—the family of all R-implications obtained from continuous t-norms;
- $^{\mathbb{C}}\mathbb{I}_{\mathbb{T}_\mathbb{C}}$—the family of all continuous R-implications obtained from continuous t-norms;
- $\mathbb{I}_{\mathbb{LK}}$—the family of all implications Φ-conjugate with the Łukasiewicz implication $I_{\mathbf{LK}}$.

In the following two subsections, we summarize the known intersections between the above subfamilies of (S, N)- and R-implications based on the results cited and obtained earlier, which is also diagrammatically represented in Figs. 1 and 2. In the next two subsections we discuss the relationships between families of (S, N)- and R-implications. The final result will also be diagrammatically represented.

6.1. Intersections between subfamilies of (S, N)-implications

Because of Proposition 4.11 we get

$$^{\mathbb{C}}\mathbb{I}_{\mathbb{S},\mathbb{N}}=\mathbb{I}_{\mathbb{S}_\mathbb{C},\mathbb{N}_\mathbb{C}}.$$

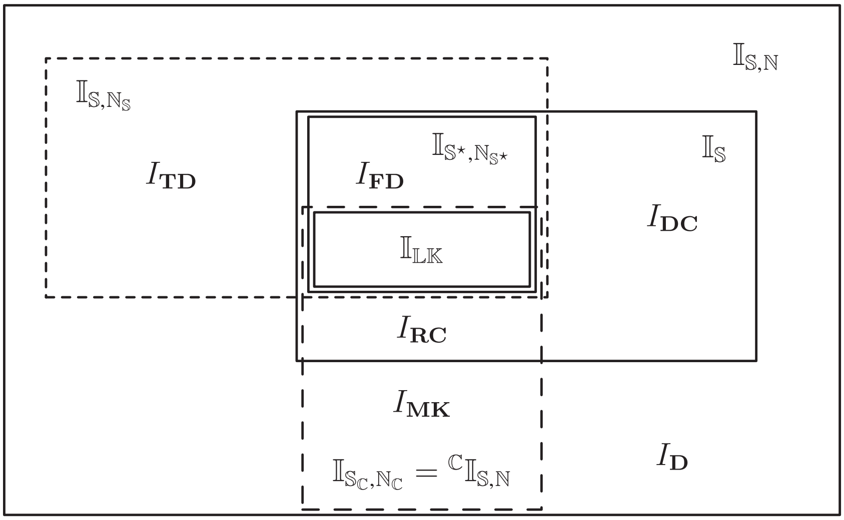

Fig. 1. Intersections among the subfamilies of (S, N)-implications.

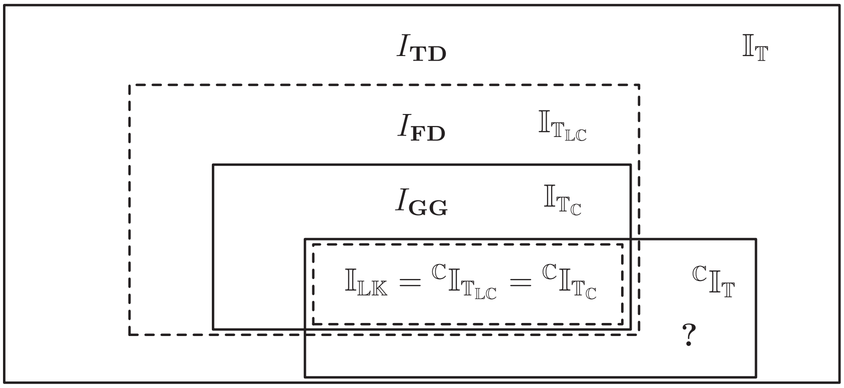

Fig. 2. Intersections among the subfamilies of R-implications.

From Tables 4 and 5, Remark 4.4(ii) and Proposition 3.11 we have that

$$\begin{aligned} \mathbb{I}_{\mathbb{LK}} &= \mathbb{I}_{\mathbb{S},\mathbb{N}_\mathbb{S}} \cap \mathbb{I}_{\mathbb{S}_\mathbb{C},\mathbb{N}_\mathbb{C}} \\ &= \mathbb{I}_{\mathbb{S}_\mathbb{C},\mathbb{N}_\mathbb{C}} \cap \mathbb{I}_{\mathbb{S}^*,\mathbb{N}^*_\mathbb{S}} \\ &\subsetneq \mathbb{I}_{\mathbb{S}^*,\mathbb{N}^*_\mathbb{S}} = \mathbb{I}_{\mathbb{S},\mathbb{N}_\mathbb{S}} \cap \mathbb{I}_\mathbb{S}. \end{aligned}$$

6.2. *Intersections between subfamilies of R-implications*

By Theorem 5.21 we have

$$^{\mathbb{C}}\mathbb{I}_{\mathbb{T}_{\mathbb{LC}}} = \mathbb{I}_{\mathbb{LK}}.$$

Quite obviously, we have the following containments:

$$^{\mathbb{C}}\mathbb{I}_{\mathbb{T}_{\mathbb{LC}}} = \mathbb{I}_{\mathbb{LK}} \subsetneq \mathbb{I}_{\mathbb{T}_\mathbb{C}} \subsetneq \mathbb{I}_{\mathbb{T}_{\mathbb{LC}}} \subsetneq \mathbb{I}_\mathbb{T}.$$

Similarly we get

$$^{\mathbb{C}}\mathbb{I}_\mathbb{T} \cap \mathbb{I}_{\mathbb{T}_\mathbb{C}} = {}^{\mathbb{C}}\mathbb{I}_\mathbb{T} \cap \mathbb{I}_{\mathbb{T}_{\mathbb{LC}}} = {}^{\mathbb{C}}\mathbb{I}_{\mathbb{T}_\mathbb{C}} = {}^{\mathbb{C}}\mathbb{I}_{\mathbb{T}_{\mathbb{LC}}} = \mathbb{I}_{\mathbb{LK}}.$$

It is still an open problem to find, if there exists a continuous R-implication generated from a non-continuous t-norm.

6.3. Intersections between families of (S, N)- and R-implications: known results

One of the first works on the intersection of S- and R-implications was done by Dubois and Prade [9], wherein they have shown that S-implications and R-implications could be merged into a single family, provided that the class of triangular norms is enlarged to non-commutative conjunction operations. See also the follow-up works of Fodor [11,12].

Firstly, note that since $I_{\mathbf{TD}}$ is both an (S, N)- and R-implications, we have

$$I_{\mathbf{TD}} \in \mathbb{I}_{\mathbb{S},\mathbb{N}} \cap \mathbb{I}_{\mathbb{T}} \neq \emptyset,$$

i.e., the intersection of (S, N)- and R-implications is non-empty. Because of Theorems 4.9, 5.9 and 5.21 we get

$$\mathbb{I}_{\mathbb{S}} \cap \mathbb{I}_{\mathbb{T}_{\mathbb{C}}} = \mathbb{I}_{\mathbb{LK}}.$$

On the other hand, by Theorem 4.13 and Corollary 5.15 we have

$$^{\mathbb{C}}\mathbb{I}_{\mathbb{S},\mathbb{N}} \cap \mathbb{I}_{\mathbb{T}_{\mathbb{LC}}} = \mathbb{I}_{\mathbb{LK}}.$$

The weaker version of the above two results is well known in the scientific literature and, in general, we say that the only continuous S-implication and R-implication (generated from a left-continuous t-norm) is the Łukasiewicz implication (up to a conjugation). But there are still some open problems connected with the intersection of (S, N)-implications (S-implications) and R-implications. It follows from the fact that there are many R-implications obtained from t-norms that are left-continuous but non-continuous and are still S-implications.

Example 6.1. The Fodor implication $I_{\mathbf{FD}}$ is an example of non-continuous fuzzy implication, which is both an R-implication and an S-implication.

A left-continuous t-norm T is said to have a strong induced negation, if the natural negation N_T is strong. Let us denote the following family of fuzzy implications as follows:

- $\mathbb{I}_{\mathbb{T}^*}$—the family of all R-implications obtained from left-continuous t-norms having strong induced negations.

It is clear now from our discussion, that

$$\mathbb{I}_{\mathbb{S}} \cap \mathbb{I}_{\mathbb{T}_{\mathbb{LC}}} = \mathbb{I}_{\mathbb{T}^*} \supsetneq \mathbb{I}_{\mathbb{LK}}.$$

Many families of left-continuous t-norms (up to a conjugation) with strong induced negations are known in the literature (cf. [18, p. 36], see also [20, Theorem 1], and [25]). The first of such families is the nilpotent class of t-norms, i.e., they are Φ-conjugate with the Łukasiewicz t-norm $T_{\mathbf{LK}}$. Another family consists of t-norms that are Φ-conjugate with the nilpotent minimum t-norm $T_{\mathbf{nM}}$. Yet another family is the class of t-norms that are Φ-conjugate with the Jenei t-norm family $(T_{\mathbf{J}})_\lambda$, for $\lambda \in [0, 0.5]$, where

$$(T_{\mathbf{J}})_\lambda(x, y) = \begin{cases} 0 & \text{if } x + y \leqslant 1, \\ x + y - 1 + \lambda & \text{if } x + y > 1 \text{ and } x, y \in (\lambda, 1 - \lambda], \\ \min(x, y) & \text{otherwise,} \end{cases} \qquad x, y \in [0, 1].$$

Note that $(T_{\mathbf{J}})_0 = T_{\mathbf{LK}}$ and $(T_{\mathbf{J}})_{0.5} = T_{\mathbf{nM}}$. Recently, Maes and De Baets [25] while studying fuzzified normal forms obtained the following family of t-norms $(T_{\mathbf{MD}})_\lambda$, for $\lambda \in [0, 0.5)$, where

$$(T_{\mathbf{MD}})_\lambda(x, y) = \begin{cases} 0 & \text{if } x + y \leqslant 1, \\ \min(x, y) & \text{if } x + y > 1 \text{ and } \min(x, y) \in (\lambda, 1 - \lambda], \\ x + y - 1 & \text{if } x + y > 1 \text{ and } (x + y \geqslant 2 - \lambda \text{ or } \min(x, y) \in [0, \lambda]), \\ 1 - \lambda & \text{otherwise,} \end{cases}$$

for $x, y \in [0, 1]$. Note also that $(T_{\mathbf{MD}})_0 = T_{\mathbf{nM}}$.

For the methods of obtaining these families see the works of Jenei [16,17,19,20] connected with the rotation and the rotation-annihilation and the works of Maes and De Baets [25,26,28] connected with the triple rotation. In fact, it can be shown that every t-norm $(T_{\mathbf{MD}})_\lambda$ can be obtained as a rotation-annihilation of a particular ordinal sum of the Łukasiewicz t-norm $T_{\mathbf{LK}}$ (see [25, p. 384]).

6.4. Intersections between families of (S, N)- and R-implications: new results

A characterization of the fuzzy implications that fall under the following intersections

$$\mathbb{I}_{\mathbb{S}} \cap \mathbb{I}_{\mathbb{T}_{\mathbb{LC}}} \quad \text{and} \quad \mathbb{I}_{\mathbb{S},\mathbb{N}} \cap \mathbb{I}_{\mathbb{T}_{\mathbb{LC}}}$$

has not been known so far. In this section we investigate the above intersections and show that

$$\mathbb{I}_{\mathbb{S}} \cap \mathbb{I}_{\mathbb{T}_{\mathbb{LC}}} = \mathbb{I}_{\mathbb{S},\mathbb{N}} \cap \mathbb{I}_{\mathbb{T}_{\mathbb{LC}}} = \mathbb{I}_{\mathbb{N}_{\mathbb{T}}(\mathbb{T}),\mathbb{N}_{\mathbb{T}}} = \mathbb{I}_{\mathbb{T}^*},$$

where $\mathbb{I}_{\mathbb{N}_{\mathbb{T}}(\mathbb{T}),\mathbb{N}_{\mathbb{T}}}$ is the subfamily of fuzzy implications defined as follows:

- $\mathbb{I}_{\mathbb{N}_{\mathbb{T}}(\mathbb{T}),\mathbb{N}_{\mathbb{T}}}$—the family of all (S, N)-implications obtained from the N_T-dual t-conorms of the left-continuous t-norm T whose natural negation N_T is strong.

Theorem 6.2. *For a left-continuous* t-*norm T, a* t-*conorm S and a fuzzy negation N the following statements are equivalent*:

(i) *The R-implication I_T is also an (S, N)-implication $I_{S,N}$, i.e., $I_T = I_{S,N}$.*
(ii) *$N = N_T$ is a strong negation and (T, N, S) form a De Morgan triple.*

Proof. (i) $\Longrightarrow$ (ii) Assume that there exist a left-continuous t-norm T, a fuzzy negation N and a t-conorm S such that $I_T = I_{S,N}$. For better readability we denote this function by I. Since I is an R-implication, from Corollary 5.15 it satisfies (OP). Since I is also an (S, N)-implication, by Theorem 4.7, we have that $N = N_S$ is a strong negation. On the other side by virtue of Remark 4.4(ii) we get

$$N(x) = N_{I_{S,N}}(x) = I_{S,N}(x, 0) = I_T(x, 0) = N_T(x), \quad x \in [0, 1].$$

Further, since I satisfies (OP), (EP) and N_I is strong, from Theorem 5.18 we have

$$T(x, y) = N_I(I(x, N_I(y))) = N_I(S(N_I(x), N_I(y))) = N_T(S(N_T(x), N_T(y))) \\ = N(S(N(x), N(y))),$$

i.e., (T, N_T, S) form a De Morgan triple.

(ii) $\Longrightarrow$ (i) Firstly see, that the R-implication I_T is an (S, N)-implication. Indeed, since I_T satisfies (I1), (EP) and its natural negation $N_{I_T} = N_T$ is a strong negation, by Theorem 4.10 we get that I_T is an (S, N)-implication, i.e., $I_T = I_{S',N'}$ for an appropriate t-conorm S' and a strong negation N'. Observe now that $I_T = I_{S,N_S}$. Indeed, if (T, N, S) form a De Morgan triple, then from our assumptions and Theorem 2.21 it follows that S is a right-continuous t-conorm such that $N = N_S$ is a strong negation. Therefore

$$N_S(x) = N(x) = N_T(x) = N_{I_T}(x) = I_T(x, 0) = I_{S',N'}(x, 0) = N_{I_{S',N'}}(x) = N'(x),$$

for all $x \in [0, 1]$. Hence $I_T = I_{S',N_S}$. Finally, from the proof of (i) $\Longrightarrow$ (ii) above we know that T is the N_S dual of S' and by our assumption T is the N_S dual of S. Hence $S = S'$, i.e., $I_T = I_{S,N_S}$. □

In fact, using Theorems 2.20 and 2.21 the above result can be restated as follows.

Theorem 6.3. *For a left-continuous* t-*norm T and a* t-*conorm S the following statements are equivalent*:

(i) *The R-implication I_T is also an (S, N)-implication $I_{S,N}$.*
(ii) *The R-implication I_T is also an S-implication $I_{S,N}$ with the strong negation N_T.*
(iii) *(T, N_T, S) form a De Morgan triple.*

Remark 6.4. (i) The left-continuity of T is very important in the above theorems. For example, consider any t-conorm S whose natural negation $N_S \neq N_{\mathbf{D2}}$. But, $I_{S,N_{\mathbf{D2}}} = I_{\mathbf{TD}}$ which is also an R-implication obtained from the non-left-continuous t-norm $T_{\mathbf{D}}$. It is obvious that the triple $(T_{\mathbf{D}}, N_{\mathbf{D2}}, S)$ do not form a De Morgan triple.

(ii) It should also be noted that even if T is a non-left-continuous t-norm, the N_T can still be strong. Once again, consider the t-norm $T_{\mathbf{nM}^*}$ from Example 5.6 (iv). Now, $N_{T_{\mathbf{nM}^*}} = N_{\mathbf{C}}$ and $I_{T_{\mathbf{nM}^*}} = I_{\mathbf{FD}}$, which is an (S, N)-implication.

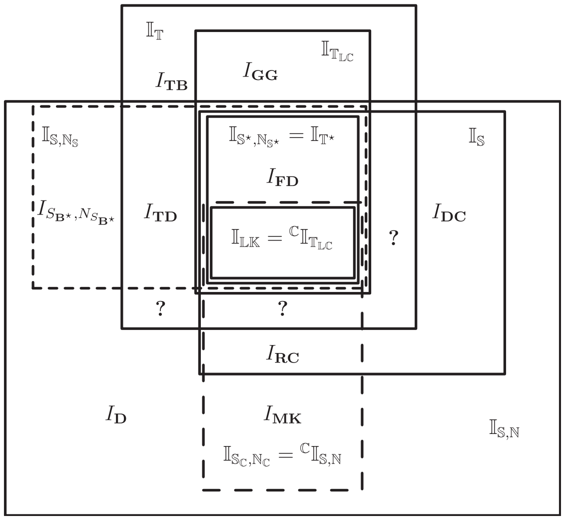

Fig. 3. Intersections between families of (S, N)- and R-implications.

Theorem 6.5. *For a right-continuous* t-*conorm* S *and a* t-*norm* T *the following statements are equivalent*:

(i) *The* (S, N)-*implication* I_{S,N_S} *is also the* R-*implication* I_T.
(ii) (S, N_S, T) *form a De Morgan triple*.

The results presented in this section are also diagrammatically represented in Fig. 3, for which the following example will be useful.

Example 6.6. Consider the following non-right-continuous t-conorm which is $N_{\mathbf{C}}$-dual of the t-norm $T_{\mathbf{B}^*}$:

$$S_{\mathbf{B}^*}(x, y) = \begin{cases} 1 & \text{if } (x, y) \in (0.5, 1)^2, \\ \max(x, y) & \text{otherwise}, \end{cases} \qquad x, y \in [0, 1].$$

Then its natural negation is the discontinuous function given by

$$N_{S_{\mathbf{B}^*}}(x) = \begin{cases} 1 & \text{if } x \in (0, 0.5), \\ 0.5 & \text{if } x \in (0.5, 1), \\ 0 & \text{if } x = 1, \end{cases} \qquad x \in [0, 1].$$

The (S, N)-implication obtained from $S_{\mathbf{B}^*}$ and $N_{S_{\mathbf{B}^*}}$ is given by

$$I_{S_{\mathbf{B}^*},N_{S_{\mathbf{B}^*}}}(x, y) = \begin{cases} 1 & \text{if } x \in [0, 0.5], \\ 0.5 & \text{if } x \in (0.5, 1) \text{ and } y \in [0, 0.5], \\ y & \text{otherwise}, \end{cases} \qquad x, y \in [0, 1].$$

Obviously, $I_{S_{\mathbf{B}^*},N_{S_{\mathbf{B}^*}}}$ does not satisfy (IP), and therefore is not an R-implication.

7. Concluding remarks

In this work, we have given what should be seen as the 'state-of-the-art' review of two of the well-studied families of fuzzy implications, viz., (S, N)- and R-implications, detailing their definitions, properties, characterizations and some representation results, illustrated ably with examples. Also, we have obtained the exact intersections between some subfamilies of the above classes of fuzzy implications.

The authors in [3] note that the method employed by them to obtain the above characterizations, viz., Theorem 4.10, cannot be adopted for characterizing (S, N)-implications from non-continuous negations (see [3, Remark 4.4]). Moreover, the representation of (S, N)-implications in this case may not be unique. Also, characterization of R-implications generated from non-left-continuous t-norms is still unknown. All this discussion leaves us with the following open problems.

Problem 7.1. *Characterize the following*:

(i) (S, N)-*implications generated from non-continuous negations*;
(ii) R-*implications generated from non-left-continuous* t-*norms*;
(iii) *continuous* R-*implications generated from non-left-continuous* t-*norms*;
(iv) *the non-empty intersection* $\mathbb{I}_{\mathbb{S},\mathbb{N}} \cap \mathbb{I}_{\mathbb{T}}$.

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