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The ⊛-Composition of Fuzzy Implications : Closures with respect to Properties, Powers and Families

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Abstract

Recently, Vemuri and Jayaram proposed a novel method of generating fuzzy implications from a given pair of fuzzy implications. Viewing this as a binary operation ⊛ on the set $\mathbb{I}$ of all fuzzy implications they obtained, for the first time, a monoid structure $(\mathbb{I}, \circledast)$ on the set $\mathbb{I}$. Some algebraic aspects of $(\mathbb{I}, \circledast)$ had already been explored and hitherto unknown representation results for the Yager's families of fuzzy implications were obtained in [Representations through a Monoid on the set of Fuzzy Implications, Fuzzy Sets and Systems, 247, 51-67]. However, the properties of fuzzy implications generated or obtained using the ⊛-composition have not been explored. In this work, the preservation of the basic properties like *neutrality, ordering and exchange principles*, the functional equations that the obtained implications satisfy, the powers w.r.t. ⊛ and their convergence, and the closures of some families of fuzzy implications w.r.t. the operation ⊛, specifically the families of (S, N)-, R-, f- and g-implications, are studied. This study shows that the ⊛-composition carries over many of the desirable properties of the original fuzzy implications to the generated fuzzy implications and further, due to the associativity of the ⊛-composition one can obtain, often, infinitely many new fuzzy implications from a single fuzzy implication through self-composition w.r.t. the ⊛-composition.

Keywords: Fuzzy implication, basic properties, functional equations, self-composition, closure, (S, N)-implications, R-implications, f-implications, g-implications.
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1. Introduction

Fuzzy implications, along with triangular norms (t-norms, in short) form the two most important fuzzy logic connectives. They are a generalisation of the classical implication and conjunction, respectively, to multi- valued logic and play an equally important role in fuzzy logic as their counterparts in classical logic.

Fuzzy implications on the unit interval $[0, 1]$ are defined as follows.

Definition 1.1 ([4], Definition 1.1.1 & [27, 16]). A function $I\colon [0,1]^2 \to [0,1]$ is called a *fuzzy implication* if it satisfies, for all $x, x_1, x_2, y, y_1, y_2 \in [0, 1]$, the following conditions:

$$\text{if } x_1 \leq x_2, \text{ then } I(x_1, y) \geq I(x_2, y)\ , \text{ i.e., } I(\,\cdot\,, y) \text{ is decreasing}\ , \tag{I1}$$

$$\text{if } y_1 \leq y_2, \text{ then } I(x, y_1) \leq I(x, y_2)\ , \text{ i.e., } I(x, \,\cdot\,) \text{ is increasing}\ , \tag{I2}$$

$$I(0,0) = 1\ , I(1,1) = 1\ , I(1,0) = 0\ . \tag{I3}$$

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The set of all fuzzy implications will be denoted by $\mathbb{I}$.

From Definition 1.1, it is clear that a fuzzy implication, when restricted to $\{0,1\}$, coincides with the classical implication. Table 1 (see also, Table 1.3 in [4]) lists some examples of basic fuzzy implications.

Name	Formula
Łukasiewicz	$I_{\mathbf{LK}}(x,y) = \min(1, 1-x+y)$
Gödel	$I_{\mathbf{GD}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{if } x > y \end{cases}$
Reichenbach	$I_{\mathbf{RC}}(x,y) = 1 - x + xy$
Kleene-Dienes	$I_{\mathbf{KD}}(x,y) = \max(1-x, y)$
Goguen	$I_{\mathbf{GG}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ \frac{y}{x}, & \text{if } x > y \end{cases}$
Rescher	$I_{\mathbf{RS}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ 0, & \text{if } x > y \end{cases}$
Yager	$I_{\mathbf{YG}}(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0 \\ y^x, & \text{if } x > 0 \text{ or } y > 0 \end{cases}$
Weber	$I_{\mathbf{WB}}(x,y) = \begin{cases} 1, & \text{if } x < 1 \\ y, & \text{if } x = 1 \end{cases}$
Fodor	$I_{\mathbf{FD}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ \max(1-x, y), & \text{if } x > y \end{cases}$
Least FI	$I_{\mathbf{0}}(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 1 \\ 0, & \text{if } x > 0 \text{ and } y < 1 \end{cases}$
Greatest FI	$I_{\mathbf{1}}(x,y) = \begin{cases} 1, & \text{if } x < 1 \text{ or } y > 0 \\ 0, & \text{if } x = 1 \text{ and } y = 0 \end{cases}$
Most Strict	$I_{\mathbf{D}}(x,y) = \begin{cases} 1, & \text{if } x = 0 \\ y, & \text{if } x > 0 \end{cases}$

Table 1: Examples of fuzzy implications (cf. Table 1.3 in [4])

Fuzzy implications play an important role in approximate reasoning, fuzzy control, decision theory, control theory, expert systems, fuzzy mathematical morphology, image processing, etc. - see for example [9, 11, 21, 22, 57, 49, 54, 56] or the recent monograph exclusively devoted to fuzzy implications [4].

The different generation methods of fuzzy implications can be broadly classified into the following three categories, viz,

(i) From binary functions on $[0,1]$, typically other fuzzy logic connectives, viz., (S,N)-, R-, QL-implications (see [4]),

(ii) From unary functions on [0,1], typically monotonic functions, for instance, Yager's f-, g-implications (see [56]), or from fuzzy negations [7, 20, 35, 46],

(iii) From fuzzy implications (see [3, 6, 13, 12, 15, 19, 39, 46]).

1.1. Motivation for this work

Obtaining fuzzy implications from given fuzzy implications, the third method listed above, can be further sub-divided into approaches that are either generative or constructive. By *generative* methods, we refer to those works which propose a closed form formula for obtaining new fuzzy implications from given ones, often

with the help of other fuzzy logic connectives, see for instance, [3, 26, 20]. By *constructive* methods, we refer to those methods that somehow depend on the underlying geometry to construct a fuzzy implication from a pair of fuzzy implications, often by specifying the values over different sub-regions of $[0,1]^2$. For instance, the threshold and vertical threshold generation methods of Massanet and Torrens [39, 37, 40, 32] fall under this category.

Recently, in [50] the authors had proposed a novel *generative* method, denoted $\circledast$, which derives fuzzy implications from a given pair of fuzzy implications.

Definition 1.2 ([50], Definition 7). For any two fuzzy implications I, J we define $I \circledast J$ as

$$(I \circledast J)(x,y) = I(x, J(x,y)), \qquad x,y \in [0,1]\,. \tag{1}$$

Note that the novelty in the proposed operation $\circledast$ arises from the following fact: *It is the first composition that does not employ any other fuzzy logic connective(s) to help in the generation and still leads to some algebraic structures on the set* $\mathbb{I}$. In fact, the operation $\circledast$ not only leads to newer implications but also to a richer algebraic structure, namely a non-idempotent monoid, on the set $\mathbb{I}$ of fuzzy implications. The algebraic aspects of the monoid $(\mathbb{I}, \circledast)$ have been already explored in [53] leading upto hitherto unknown representations of the Yager's families of fuzzy implications.

However, the properties of fuzzy implications generated or obtained using the $\circledast$-composition have not been explored. For instance, given $I, J \in \mathbb{I}$ with a certain property, the question of whether $I \circledast J$ also has this property is yet to be investigated. Thus there is a need for a comprehensive study of the preservation of the basic properties, the functional equations that the obtained implications satisfy, the powers w.r.t. $\circledast$ and their convergence, and the closures of families of fuzzy implications w.r.t. the operation $\circledast$. This forms the main motivation of this work.

1.2. Main contribution of the paper

In this work, we continue our study of the recently proposed generative method [50, 53], viz., the $\circledast$-composition (Definition 1.2) on $\mathbb{I}$. While some algebraic aspects have already been explored in [53], this work can be broadly seen to discuss the closures of some subsets of $\mathbb{I}$ w.r.t. $\circledast$, or alternately, of investigating the analytical aspects of the $\circledast$-composition. Specifically, we study the preservation of various attributes of the given pair of fuzzy implications under the $\circledast$-composition, viz., (i) basic or desirable properties - for instance, *neutrality, ordering and exchange principles*, (ii) the two main functional equations involving fuzzy implications, namely, the *law of importation* and the *contraposition principle*, (iii) powers of fuzzy implications under self-composition with $\circledast$ and their convergence, and (iv) closures of some families of fuzzy implications w.r.t. the $\circledast$-composition.

This work clearly demonstrates that the $\circledast$-composition carries over many of the desirable properties of the original fuzzy implications to the generated fuzzy implications (see Section 3.2). Further, due to the associativity of the $\circledast$-composition one can obtain, often, infinitely many new fuzzy implications from a single fuzzy implication through self-composition w.r.t. the $\circledast$-composition and once again, carrying over, all the desirable properties to the newly generated fuzzy implications (see Section 5.2).

In Section 6, we study the effect of the $\circledast$-composition on fuzzy implications obtained from other generation methods. Specifically, we consider the families of (S,N)- and R-implications (Sections 6.1 and 6.2) which are representative of the first type of generation methods and the Yager's families of f- and g-implications (Sections 6.3 and 6.4) which are representative of the second type of generation methods.

In the course of this study, we have also proposed and discussed a concept of mutual exchangeability (ME) of a pair of fuzzy implications (Definition 3.9), which plays a central role in our study. The property (ME) can be seen as a generalisation of the usual exchange principle of a fuzzy implication to a pair of fuzzy implications and thus, we believe, can enable one to understand the interactions between pairs of fuzzy implications.

1.3. Outline of the paper

The organisation of this paper is as follows. In Section 2 we list what are considered to be the basic or desirable properties of fuzzy implications and review some of the major generative methods and summarise the properties preserved by them. Section 3 introduces the proposed $\circledast$-composition on the set $\mathbb{I}$ and discusses the basic properties preserved by this generation process, while Section 4 discusses the preservation of two of the main functional equations related to fuzzy implications, viz., the law of importation and contrapositive symmetry. In Section 5 we discuss the powers of fuzzy implications obtained by self-composition w.r.t. the proposed $\circledast$-composition. Following this, we discuss the closures of some families of fuzzy implications w.r.t. the proposed $\circledast$-composition in Section 6.

2. Fuzzy Implications from Fuzzy Implications: Existing Generative Methods

In this work, we restrict our focus to generative methods for obtaining new fuzzy implications from existing ones. In the literature only a few such generative methods are known. In this section, we begin by giving a brief review of the existing methods. Following this, we list out some of the main properties desirable of a fuzzy implication and tabulate the known results, vis-á-vis those properties that are preserved under the various generative methods.

2.1. Lattice of Fuzzy Implications:

The lattice operations of meet and join were the first to be employed towards generating new fuzzy implications. Bandler and Kohout [8] obtained fuzzy implications by taking the meet and join of a given fuzzy implication I and its reciprocal $I_N(x,y) = I(N(y), N(x))$, where N is a strong negation (see Definition 2.5). This method has been discussed extensively under the topic of contrapositivisation of fuzzy implications, see Fodor [15], Balasubramaniam [6].

In general, given $I, J \in \mathbb{I}$, the following 'meet' ($I \mathbin{\textcircled{\scriptsize\vee}} J$) and 'join' ($I \mathbin{\textcircled{\scriptsize\wedge}} J$) operations give rise to fuzzy implications ([4], Theorem 6.1.1):

$$(I \mathbin{\textcircled{\scriptsize\vee}} J)(x,y) = \max(I(x,y), J(x,y)), \qquad x,y \in [0,1], \qquad I, J \in \mathbb{I}\,, \tag{Latt-Max}$$

$$(I \mathbin{\textcircled{\scriptsize\wedge}} J)(x,y) = \min(I(x,y), J(x,y)), \qquad x,y \in [0,1], \qquad I, J \in \mathbb{I}\,. \tag{Latt-Min}$$

2.2. Convex Classes of Fuzzy Implications:

We know that fuzzy implications are basically functions on [0,1]. Thus one can define convex combinations of fuzzy implications in the usual manner.

Definition 2.1. Convex combination of two fuzzy implications I, J is defined as

$$K(x,y) = \lambda I(x,y) + (1-\lambda)J(x,y), \qquad x,y \in [0,1],\ \lambda \in [0,1]\,,$$

and is a fuzzy implication.

2.3. Conjugacy Classes of Fuzzy Implications:

Let Φ denote the set of all increasing bijections on $[0,1]$. Note that if $([0,1], *)$ and $([0,1], \diamond)$ are two ordered groupoids, then $\varphi(x * y) = \varphi(x) \diamond \varphi(y)$ is a groupoid homomorphism for any $\varphi \in \Phi$. Conversely, given a binary groupoid operation, one could obtain new groupoid operations from the above as follows: $x * y = \varphi^{-1}(\varphi(x) \diamond \varphi(y))$.

Viewing a fuzzy implication as a groupoid on $[0,1]$, Baczyński and Drewniak [2, 45] obtained new implications from given ones as above.

Definition 2.2 ([2], [4], Theorem 6.3.1). For any $\varphi \in \Phi$ and $I \in \mathbb{I}$, the φ-conjugate of I defined as follows is also a fuzzy implication, i.e., $I_\varphi \in \mathbb{I}$:

$$I_\varphi(x,y) = \varphi^{-1}\left(I(\varphi(x), \varphi(y))\right), \qquad x,y \in [0,1]\,.$$

Definition 2.3 (cf. [12], [4]). A fuzzy implication I is called *self-conjugate* or *invariant* if $I_\varphi = I$, for all $\varphi \in \Phi$. $\mathbb{I}_{\mathbf{inv}}$ denotes the set of all invariant fuzzy implications.

Name	Formula
minimum	$T_{\mathbf{M}}(x,y) = \min(x,y)$
algebraic product	$T_{\mathbf{P}}(x,y) = xy$
Łukasiewicz	$T_{\mathbf{LK}}(x,y) = \max(x+y-1,0)$
drastic product	$T_{\mathbf{D}}(x,y) = \begin{cases} 0, & \text{if } x,y \in [0,1) \\ \min(x,y), & \text{otherwise} \end{cases}$
nilpotent minimum	$T_{\mathbf{nM}}(x,y) = \begin{cases} 0, & \text{if } x+y \leq 1 \\ \min(x,y), & \text{otherwise} \end{cases}$

Table 2: Basic t-norms

Name	Formula
maximum	$S_{\mathbf{M}}(x,y) = \max(x,y)$
probabilistic sum	$S_{\mathbf{P}}(x,y) = x+y-xy$
Łukasiewicz	$S_{\mathbf{LK}}(x,y) = \min(x+y,1)$
drastic sum	$S_{\mathbf{D}}(x,y) = \begin{cases} 1, & \text{if } x,y \in (0,1] \\ \max(x,y), & \text{otherwise} \end{cases}$
nilpotent maximum	$S_{\mathbf{nM}}(x,y) = \begin{cases} 1, & \text{if } x+y \geq 1 \\ \max(x,y), & \text{otherwise} \end{cases}$

Table 3: Basic t-conorms

2.4. Compositions of Fuzzy Implications:

Definition 2.4 ([28], Definition 3.1). A binary operation $T(S)\colon [0,1]^2 \longrightarrow [0,1]$ is called a *t-norm(t-conorm)*, if it is increasing in both the variables, commutative, associative and has $1(0)$ as the neutral element.

In the infix notation, usually a $T(S)$ is denoted by $*(\oplus)$. Tables 2 and 3 list a few of the t-norms and t-conorms that are considered basic in the literature, which will also be useful in the sequel.

Definition 2.5 ([28], Definition 11.3). A function $N : [0,1] \longrightarrow [0,1]$ is called a *fuzzy negation* if $N(0) = 1, N(1) = 0$ and N is decreasing. A fuzzy negation N is called

(i) *strict* if, in addition, N is strictly decreasing and is continuous.
(ii) *strong* if it is an involution, i.e., $N(N(x)) = x$, for all $x \in [0,1]$.

Table 4 lists the basic fuzzy negations, which will also be useful in the sequel.

Name	Formula
Classical	$N_{\mathbf{C}}(x) = 1-x$
Least	$N_{\mathbf{D_1}}(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x > 0 \end{cases}$
Greatest	$N_{\mathbf{D_2}}(x) = \begin{cases} 1, & \text{if } x < 1 \\ 0, & \text{if } x = 1 \end{cases}$

Table 4: Basic fuzzy negations

Note that any binary function $F : [0,1]^2 \longrightarrow [0,1]$ can be treated as a binary fuzzy relation on $[0,1]$. Once again, treating a fuzzy implication I as a fuzzy relation, Baczyński and Drewniak [3] employed relational composition operators to obtain new fuzzy implications.

Definition 2.6 (cf. [3],[4], Definition 6.4.1 & Theorem 6.4.4). Let $I, J \in \mathbb{I}$ and $*$ be a t-norm. Then sup–$*$ composition of I, J is given as follows:

$$(I \overset{*}{\circ} J)(x,y) = \sup_{t\in[0,1]} (I(x,t) * J(t,y)), \qquad x,y \in [0,1]\ . \tag{COMP}$$

Further, $I \overset{*}{\circ} J \in \mathbb{I}$ if and only if $(I \overset{*}{\circ} J)(1,0) = 0$.

2.5. Basic properties of fuzzy implications and their preservation by generative methods

In the above, we had recalled some of the generative methods of obtaining fuzzy implications from fuzzy implications and the structures they impose on the set $\mathbb{I}$ of fuzzy implications. In the following we list a few of the most important properties of fuzzy implications. Note that they are also a natural generalisation of the corresponding properties of the classical implication to multi valued logic (see [4, 43, 48]). These properties play a key role in characterising different fuzzy implications as well as applications of fuzzy implications in different contexts.

Following this, we tabulate the known results w.r.t. the properties that are preserved by the above generative methods in Table 5. Note that, this section is intended to only highlight the results in brief and for the relevant results, their proofs and more details, we refer the readers to the already listed references, viz., [3, 4, 44].

Definition 2.7 (cf. [4], Definition 1.3.1).

- A fuzzy implication I is said to satisfy

(i) the left neutrality property (NP) if

$$I(1,y) = y, \qquad y \in [0,1]\ . \tag{NP}$$

(ii) the ordering property (OP), if

$$x \le y \iff I(x,y) = 1 \qquad x,y \in [0,1]\ . \tag{OP}$$

(iii) the identity principle (IP), if

$$I(x,x) = 1, \qquad x \in [0,1]\ . \tag{IP}$$

(iv) the exchange principle (EP), if

$$I(x,I(y,z)) = I(y,I(x,z)), \qquad x,y,z \in [0,1]\ . \tag{EP}$$

- A fuzzy implication I is said to be continuous if it is continuous in both the variables.

Let $\mathbb{I}_{\mathbf{NP}}$ denote the set of fuzzy implications satisfying (NP). Similarly, let the subsets $\mathbb{I}_{\mathbf{IP}}, \mathbb{I}_{\mathbf{OP}}, \mathbb{I}_{\mathbf{EP}}$ denote the set of fuzzy implications satisfying (IP), (OP) and (EP), respectively.

Property	I	J	$I \mathbin{⩒} J$	$I \mathbin{⩓} J$	Convex Combination	$I \overset{*}{\circ} J$
IP	✓	✓	✓	✓	✓	✓
OP	✓	✓	✓	✓	✓	✓
NP	✓	✓	✓	✓	✓	×
EP	✓	✓	×	×	×	×
Self conjugacy	✓	✓	✓	✓	×	×
Continuity	✓	✓	✓	✓	✓	×

Table 5: Closures of some generative methods w.r.t. different properties. For the relevant results and their proofs please see, for instance, [1, 3, 4, 14].

Remark 2.8. From Table 5 it is clear that while (NP), (IP) and (OP) are preserved by most of the generative methods, the exchange principle (EP) is not usually preserved.

Even though, the preservation of (EP) is fully characterised for some constructive methods, like vertical and horizontal threshold methods, see [39, 40], the preservation of (EP) by most of the generative methods remains as an interesting open problem to investigate, see, [41]. For more details and examples, please refer to [4], Chapter 6.

3. The $\circledast$-composition on $\mathbb{I}$

In this section, we once again recall the definition of the $\circledast$-composition proposed in [50] (also presented in [29, 42]) and show that this is indeed closed on the set $\mathbb{I}$, i.e., it does indeed generate fuzzy implications from given pair of fuzzy implications without any assumptions on the given fuzzy implications. Following this we discuss the preservation of basic properties of fuzzy implications, viz., (NP), (IP), (EP) and (OP) w.r.t. the $\circledast$-composition.

3.1. Introduction

Definition 3.1 ([50], Definition 7). Given $I, J \in \mathbb{I}$, we define $I \circledast J \colon [0,1]^2 \longrightarrow [0,1]$ as

$$(I \circledast J)(x,y) = I(x, J(x,y)), \qquad x, y \in [0,1]\,. \tag{1}$$

Theorem 3.2 ([50], Theorem 10)**.** *The function $I \circledast J$ is a fuzzy implication, i.e., $I \circledast J \in \mathbb{I}$.*

Table 6 (cf. Table 3, [53])) shows some new implications obtained from some of the basic fuzzy implications listed in Table 1 via the operation $\circledast$ defined in Definition 1.2.

I	J	$I \circledast J$
$I_{\mathbf{RC}}$	$I_{\mathbf{LK}}$	$\begin{cases} 1, & \text{if } x \le y \\ 1 - x^2 + xy, & \text{if } x > y \end{cases}$
$I_{\mathbf{GG}}$	$I_{\mathbf{RC}}$	$\begin{cases} 1, & \text{if } x \le 1 - x + xy \\ \dfrac{1 - x + xy}{x}, & \text{otherwise} \end{cases}$
$I_{\mathbf{KD}}$	$I_{\mathbf{RS}}$	$\begin{cases} 1, & \text{if } x \le y \\ 1 - x, & \text{if } x > y \end{cases}$
$I_{\mathbf{RC}}$	$I_{\mathbf{KD}}$	$\max(1 - x^2, 1 - x + xy)$
$I_{\mathbf{FD}}$	$I_{\mathbf{RC}}$	$\begin{cases} 1, & \text{if } x \le 1 - x + xy \\ 1 - x + xy & \text{otherwise} \end{cases}$
$I_{\mathbf{YG}}$	$I_{\mathbf{GD}}$	$\begin{cases} 1, & \text{if } x \le y \\ y^x, & \text{if } x > y \end{cases}$
$I_{\mathbf{GD}}$	$I_{\mathbf{LK}}$	$\begin{cases} 1, & \text{if } x \le \frac{1+y}{2} \\ 1 - x + y, & \text{otherwise} \end{cases}$

Table 6: Composition of some fuzzy implications w.r.t. $\circledast$ (see also Table 3, [53]).

3.2. Basic properties preserved by $\circledast$-generated implications

Given that $I, J \in \mathbb{I}$ satisfy a certain property $\mathbf{P}$, we now investigate whether $I \circledast J$ satisfies the same property or not. If not, then we attempt to characterise those implications I, J satisfying the property $\mathbf{P}$ such that $I \circledast J$ also satisfies the same property.

Lemma 3.3 ([53], Lemma 2.8)**.** *If $I, J \in \mathbb{I}$ satisfy* (NP)*/*(IP)*, self-conjugacy, continuity) then $I \circledast J$ satisfies the same property.*

3.2.1. The Ordering Property and $\circledast$-generated implications

While the operation $\circledast$ preserves (NP) and (IP), this is not true with either (OP) or (EP), as is made clear from the following remark.

Remark 3.4. (i) From Table 1, it is clear that both $I = I_{\mathbf{GD}}, J = I_{\mathbf{LK}}$ satisfy (OP). However, $I \circledast J$ does not satisfy (OP) because $(I \circledast J)(0.4, 0.2) = 1$ but $0.4 > 0.2$ (see Table 6 for its explicit formulae or Example 2.9 (i) in [53]).

(ii) However, note that in the above example, $J \circledast I$ satisfies (OP), since $J \circledast I = J = I_{\mathbf{LK}}$, satisfies (OP). In fact, it is easy to check that $I \circledast I_{\mathbf{GD}} = I$ for all $I \in \mathbb{I}_{\mathbf{OP}}$.

(iii) Yet another example where $I \circledast J$ is neither I nor J, can be obtained by taking $I = I_{\mathbf{GG}}$, the Goguen implication, and

$$J(x, y) = \begin{cases} 1, & \text{if } x \leq y, \\ y^2, & \text{if } x > y \end{cases} .$$

Now, both I, J satisfy (OP) and so also their $\circledast$ composition given by

$$(I \circledast J)(x, y) = \begin{cases} 1, & \text{if } x \leq y, \\ \dfrac{y^2}{x}, & \text{if } x > y \end{cases} .$$

(iv) It is also interesting to note that even when not both of I, J satisfy (OP), one can have that $I \circledast J$ satisfies (OP). To see this, let $I = I_{\mathbf{GG}}$, the Goguen implication which satisfies (OP), and $I = I_{\mathbf{RC}}$, the Reichenbach implication which does not satisfy (OP). Now, $I \circledast J = I_{\mathbf{LK}}$ which does satisfy (OP).

The following result characterises all fuzzy implications $I, J \in \mathbb{I}_{\mathbf{OP}}$ such that $I \circledast J \in \mathbb{I}_{\mathbf{OP}}$.

Theorem 3.5. *Let $I, J \in \mathbb{I}$ satisfy* (OP)*. Then the following statements are equivalent.*

(i) *$I \circledast J$ satisfies* (OP)*.*

(ii) *J satisfies the following:*

$$x > J(x, y) \text{ for all } x > y \ . \tag{2}$$

(iii) *$J(x, y) \leq y$ for all $x > y$.*

Proof. (i) $\implies$ (ii): Let $I \circledast J$ satisfy (OP). Then $(I \circledast J)(x, y) = 1 \iff x \leq y$,

$$i.e., I(x, J(x, y)) = 1 \iff x \leq y \ ,$$

$$i.e., x \leq J(x, y) \iff x \leq y \ ,$$

which implies $x > J(x, y)$ for all $x > y$.

(ii) $\implies$ (iii): Let J satisfy (2). If $x > y$, then there exists $\varepsilon > 0$, arbitrarily small, such that $x > y + \varepsilon > y$. Now, from the antitonicity of J in the first variable and (2), we have $J(x, y) \leq J(y + \varepsilon, y) < y + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we see that $J(x, y) \leq y$ for all $x > y$.

(iii) $\implies$ (i): Let J satisfy $J(x, y) \leq y$ for all $x > y$.

- Let $x \leq y$. Then $J(x, y) = 1$ and consequently, $(I \circledast J)(x, y) = I(x, J(x, y)) = 1$.
- Let $x > y$. Then we have $J(x, y) \leq y < x$. From (OP) of I, it follows that $I(x, J(x, y)) < 1$.

In other words, we have $x > y \iff (I \circledast J)(x, y) < 1$ and hence $I \circledast J$ satisfies (OP). □

Example 3.6. (i) Let us denote by $\mathbb{I}^{\circ}_{\Psi} \subset \mathbb{I}_{\mathbf{OP}}$ such that every $I \in \mathbb{I}^{\circ}_{\Psi}$ is of the following form

$$I(x, y) = \begin{cases} 1, & \text{if } x \leq y, \\ \psi(y), & \text{if } x > y, \end{cases}$$

where $\psi \in \Psi$, the set of all increasing, in the sense of non-decreasing, functions defined on $[0, 1]$ such that $\psi(0) = 0$ and $\psi(x) \leq x$ for all $x \in [0, 1]$. Note that when $\psi(x) = x$ for all $x \in [0, 1]$, we get $I = I_{\mathbf{GD}}$ and when $\psi(x) = 0$, for all $x \in [0, 1]$ we obtain $I_{\mathbf{RS}}$. Clearly, every $I \in \mathbb{I}^{\circ}_{\Psi}$ satisfies (2) and hence, if $I, J \in \mathbb{I}^{\circ}_{\Psi}$ then $I \circledast J$ satisfies (OP). In fact, $I \circledast J \in \mathbb{I}^{\circ}_{\Psi}$.

(ii) However, $\mathbb{I}^{\circ}_{\Psi}$ does not contain all fuzzy implications satisfying (2). To see this consider the following fuzzy implication which does satisfy (2) but does not belong to $\mathbb{I}^{\circ}_{\Psi}$:

$$I(x,y) = \begin{cases} 1, & \text{if } x \leq y, \\ \min(y, 1-\frac{x}{2}), & \text{if } x > y\ . \end{cases}$$

3.2.2. The Exchange Principle (EP) *and* $\circledast$*-generated implications*

Among the basic properties of fuzzy implications, the exchange principle (EP) is the most important. Along with the ordering property (OP), (EP) implies many other properties. For instance, the following result from [4] shows that (OP) and (EP) are almost sufficient to make an arbitrary binary function on $[0,1]$ into a fuzzy implication with all the desirable properties.

Lemma 3.7 ([4], Lemma 1.3.4)**.** *If a function* $I\colon [0,1]^2 \longrightarrow [0,1]$ *satisfies* (EP) *and* (OP)*, then* I *satisfies* (I1)*,* (I3)*,* (NP) *and* (IP)*.*

Once again, as in the case of (OP), the following remark shows that the $\circledast$-composition does not always preserve (EP).

Remark 3.8. (i) From Table 1.4 in [4], one notes that both the fuzzy implications $I = I_{\mathbf{RC}}, J = I_{\mathbf{KD}}$ satisfy (EP). Table 6 gives the formula for $I_{\mathbf{RC}} \circledast I_{\mathbf{KD}}$. However, $(I_{\mathbf{RC}} \circledast I_{\mathbf{KD}})(0.3, (I_{\mathbf{RC}} \circledast I_{\mathbf{KD}})(0.8, 0.5)) = 0.91$ where as $(I_{\mathbf{RC}} \circledast I_{\mathbf{KD}})(0.8, (I_{\mathbf{RC}} \circledast I_{\mathbf{KD}})(0.3, 0.5)) = 0.928$. Thus showing that $I_{\mathbf{RC}} \circledast I_{\mathbf{KD}}$ does not satisfy (EP) even if I, J satisfy (EP) (see also Example 2.9 (ii) in [53]).

(ii) Once again, as in the case of (OP), observe that for the same I, J their $\circledast$-composition $J \circledast I$ satisfies (EP), since $I_{\mathbf{KD}} \circledast I_{\mathbf{RC}} = I_{\mathbf{RC}}$.

From the above, we see that the $\circledast$-composition does not always preserve (EP). In the following we define a property of a pair of fuzzy implications I, J which turns out to be a sufficient condition for the preservation of (EP) by the $\circledast$-composition. In fact, as we will see later, this property plays an important role in the sequel.

Definition 3.9. A pair (I, J) of fuzzy implications is said to be *mutually exchangeable* if

$$I(x, J(y,z)) = J(y, I(x,z)), \qquad x, y, z \in [0,1]. \tag{ME}$$

Remark 3.10. (i) Note that Definition 3.9 is different from the generalised exchange property (GEP) discussed in Proposition 5.5 of [44]. However, when $I = J \in \mathbb{I}$ both (ME) and the (GEP) of [44] reduce to the usual (EP) of I.

(ii) If I, J are mutually exchangeable, then $I \circledast J = J \circledast I$, i.e., I, J are commuting elements of $\circledast$. To see this, let $x = y$ in (ME), which then becomes $I(x, J(x,z)) = J(x, I(x,z))$. i.e., $(I \circledast J)(x,z) = (J \circledast I)(x,z)$, for all $x, z \in [0,1]$.

(iii) The following example illustrates that there exist distinct $I, J \in \mathbb{I}$ that satisfy (ME) such that $I \circledast J$ also satisfies (ME). Let $0 \leq \epsilon \leq \delta < 1$. Now observe that the implications defined by

$$I(x,y) = \begin{cases} 1, & \text{if } x \leq \epsilon, \\ y^2, & \text{if } x > \epsilon \end{cases} \text{ and } J(x,y) = \begin{cases} 1, & \text{if } x \leq \delta, \\ y^3, & \text{if } x > \delta \end{cases}$$

are such that the pair (I, J) satisfies (ME), but $I \circledast J$ given by

$$(I \circledast J)(x,y) = \begin{cases} 1, & \text{if } x \leq \delta\ , \\ y^6, & \text{if } x > \delta\ , \end{cases}$$

is neither equal to I nor J.

(iv) In fact, one could define the following class of implications, for every $\varepsilon \in [0,1)$,

$$I_\varepsilon(x,y) = \begin{cases} 1, & \text{if } x \leq \varepsilon \ , \\ \varphi_\varepsilon(y), & \text{if } x > \varepsilon \ , \end{cases}$$

where $\varphi_\varepsilon \in \Phi$. Then it is clear that for any $\varepsilon_1 \neq \varepsilon_2 \in [0,1)$, we have that the pair $(I_{\varepsilon_1}, I_{\varepsilon_2})$ satisfies (ME) and that their $\circledast$-composition is neither of them as long as $\varphi_{\varepsilon_1}, \varphi_{\varepsilon_2}$ are different from the identity function. In fact,

$$(I_{\varepsilon_1} \circledast I_{\varepsilon_2})(x,y) = \begin{cases} 1, & \text{if } x \leq \max(\varepsilon_1, \varepsilon_2) \ , \\ (\varphi_{\varepsilon_1} \circ \varphi_{\varepsilon_2})(y), & \text{if } x > \max(\varepsilon_1, \varepsilon_2) \ . \end{cases} \tag{3}$$

Theorem 3.11. *Let $I, J \in \mathbb{I}$ satisfy* (EP) *and be mutually exchangeable, i.e., satisfy* (ME). *Then $I \circledast J$ satisfies* (EP).

Proof. Let $I, J \in \mathbb{I}_{\mathbf{EP}}$ satisfy (ME) and $x, y, z \in [0,1]$.

$$\begin{aligned} (I \circledast J)(x, (I \circledast J)(y,z)) &= I(x, J(x, (I \circledast J)(y,z))) \\ &= I(x, J(x, I(y, J(y,z)))) \\ &= I(x, I(y, J(x, J(y,z)))) \\ &= I(y, I(x, J(y, J(x,z)))) \\ &= I(y, J(y, I(x, J(x,z)))) \\ &= (I \circledast J)(y, (I \circledast J)(x,z)) \ . \end{aligned}$$

Thus $I \circledast J$ satisfies (EP). □

Remark 3.12. The condition that $I, J \in \mathbb{I}_{\mathbf{EP}}$ satisfy (ME) for $I \circledast J$ to satisfy (EP) is only sufficient but not necessary. To see this, let

$$I(x,y) = \begin{cases} 1, & \text{if } x \leq 0.3 \ , \\ y^2, & \text{if } x > 0.3 \ , \end{cases} \text{ and } J(x,y) = \begin{cases} 1, & \text{if } x \leq 0.5 \ , \\ \sin(\frac{\pi y}{2}), & \text{if } x > 0.5 \ . \end{cases}$$

Now, $I \circledast J$ is given by (cf. formula (3) above)

$$(I \circledast J)(x,y) = \begin{cases} 1, & \text{if } x \leq 0.5 \ , \\ \sin^2(\frac{\pi y}{2}), & \text{if } x > 0.5 \ . \end{cases}$$

It is easy to check that $I, J, I \circledast J \in \mathbb{I}_{\mathbf{EP}}$. However, if $x = 0.6, y = 0.7, z = 0.8$, then $I(x, J(y,z)) = 0.9045$ and $J(y, I(x,z)) = 0.8443$. Thus I, J fail to satisfy (ME).

So far, we have studied the basic properties of fuzzy implications w.r.t. the $\circledast$-composition. The summary of the properties of fuzzy implications w.r.t. the $\circledast$-composition is shown in Table 7.

4. Functional Equations and the $\circledast$-generated implications

The study of functional equations involving fuzzy implications has attracted much attention not only due to their theoretical aesthetics but also due to their applicational value. In this section, we present two of the most important functional equations involving fuzzy implications, viz., the law of importation (LI) and contraposition w.r.t. a strong negation N, CP(N), and study the following question: *If a given pair of fuzzy implications (I, J) satisfies a functional equation, does $I \circledast J$ also satisfy the same functional equation?*

Property	I	J	$I \circledast J$	Remark
IP	✓	✓	✓	
OP	✓	✓	×	Theorem 3.5
NP	✓	✓	✓	
EP	✓	✓	×	Theorem 3.11
Self conjugacy	✓	✓	✓	
Continuity	✓	✓	✓	

Table 7: Closure of $\circledast$ w.r.t. different properties.

4.1. The Law of Importation and $\circledast$-generated implications

The law of importation (LI) has been shown to play a major role in the computational efficiency of fuzzy relational inference mechanisms that employ fuzzy implications to relate antecedents and consequents, see for instance, [49, 31].

Definition 4.1 ([4], Definition 1.5.1). An implication I is said to satisfy the law of importation (LI) w.r.t. a t-norm T, if

$$I(x, I(y, z)) = I(T(x, y), z), \qquad x, y, z \in [0, 1]. \tag{LI}$$

In the literature, one finds many weaker versions of the law of importation (LI) where the t-norm T is generalised to a commutative conjunctor, for instance, see the version presented in Massanet and Torrens [34]. However, here in this work we deal only with the classical version of the law of importation (LI), i.e., where the conjunctor is a t-norm T. Note that any $I \in \mathbb{I}$ that satisfies the law of importation (LI) automatically satisfies (EP) too, while the converse is not true, see for instance, [4, 34].

Remark 4.2. Note that even if $I, J \in \mathbb{I}$ satisfy (LI) w.r.t. the same t-norm T, $I \circledast J$ may satisfy (LI) w.r.t. no t-norm, the same t-norm T or even a different t-norm T'.

(i) Let $I = I_{\mathbf{RC}}, J = I_{\mathbf{YG}}$. It follows from Table 7.1 in [4], that both I, J satisfy (LI) w.r.t. the product t-norm $T_{\mathbf{P}}(x, y) = xy$. However, $I \circledast J$ given by

$$(I \circledast J)(x, y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0, \\ 1 - x + xy^x, & \text{if } x > 0 \text{ or } y > 0 \end{cases}$$

does not satisfy (EP) since

$$(I \circledast J)(0.2, (I \circledast J)(0.3, 0.4)) = 0.9487 \neq 0.8752 = (I \circledast J)(0.3, (I \circledast J)(0.2, 0.4)) \ .$$

It follows from Remark 7.3.1 in [4], that $I \circledast J$ does not satisfy (LI) w.r.t. any t-norm T.

(ii) Let $I = I_{\mathbf{RC}}, J = I_{\mathbf{GG}}$. It follows from Table 7.1 in [4] that I, J satisfy (LI) w.r.t. the t-norm $T = T_{\mathbf{P}}$. Now, $I_{\mathbf{RC}} \circledast I_{\mathbf{GG}} = I_{\mathbf{LK}}$. From Theorem 7.3.5 in [4], it follows that $I_{\mathbf{LK}}$ satisfies (LI) w.r.t. only the Lukasiewicz t-norm $T_{\mathbf{LK}}(x, y) = \max(0, x + y - 1)$, which means that $I_{\mathbf{RC}} \circledast I_{\mathbf{GG}}$ does not satisfy (LI) w.r.t. product t-norm $T_{\mathbf{P}}$ but with a different t-norm $T_{\mathbf{LK}}$.

(iii) Consider the fuzzy implication $I_{(n)}(x, y) = 1 - x^n + x^n y$, for some arbitrary but fixed $n \in \mathbb{N}$. Then

$$\begin{aligned} I_{(n)}(T_{\mathbf{P}}(x, y), z) &= I_{(n)}(xy, z) = 1 - x^n y^n + x^n y^n z \ , \text{ and} \\ I_{(n)}(x, I_{(n)}(y, z)) &= I_{(n)}(x, 1 - y^n + y^n z) = 1 - x^n + x^n(1 - y^n + y^n z) = 1 - x^n y^n + x^n y^n z \ . \end{aligned}$$

Thus $I_{(n)}$ satisfies (LI) w.r.t. $T_{\mathbf{P}}$.
Now, let $I(x, y) = I_{(1)}(x, y) = I_{\mathbf{RC}}(x, y) = 1 - x + xy$, and $J(x, y) = I_{(2)}(x, y) = 1 - x^2 + x^2 y$. From above, it follows that I, J both satisfy (LI) w.r.t. $T_{\mathbf{P}}$. Now $(I \circledast J)(x, y) = 1 - x^3 + x^3 y = I_{(3)}(x, y)$, which also satisfies (LI) w.r.t. $T_{\mathbf{P}}$.

(iv) Finally, let us consider $I, J \in \mathbb{I}$ be defined as

$$I(x,y) = \begin{cases} 1, & \text{if } x < 1 \ , \\ \sin(\frac{\pi y}{2}), & \text{if } x = 1 \ , \end{cases} \text{ and } J(x,y) = \begin{cases} 1, & \text{if } x < 1 \ , \\ y^3, & \text{if } x = 1 \ . \end{cases}$$

Then it is easy to check that all of I, J and $I \circledast J$ (as given below) satisfy (LI) w.r.t. any t-norm T:

$$(I \circledast J)(x,y) = \begin{cases} 1, & \text{if } x < 1 \ , \\ \sin(\frac{\pi y^3}{2}), & \text{if } x = 1 \ . \end{cases}$$

The following result contains a sufficient condition on the implications I, J satisfying (LI) w.r.t. the same t-norm T under which their composition $I \circledast J$ also satisfies (LI) w.r.t. the same T. Once again, we see that (ME) plays an important role.

Lemma 4.3. *Let $I, J \in \mathbb{I}$ satisfy* (LI) *w.r.t. a t-norm T. If I, J satisfy* (ME) *then $I \circledast J$ satisfies* (LI) *w.r.t. the same t-norm T.*

Proof. Let $I, J \in \mathbb{I}$ satisfy (LI) w.r.t. a t-norm T and satisfy (ME).

$$\begin{aligned} (I \circledast J)(T(x,y),z) &= I(T(x,y), J(T(x,y),z)) = I(x, I(y, J(T(x,y),z))) \\ &= I(x, I(y, J(x, J(y,z)))) = I(x, J(x, I(y, J(y,z)))) \qquad \lceil \text{ using (ME)} \\ &= I(x, J(x, (I \circledast J)(y,z))) \\ &= (I \circledast J)(x, (I \circledast J)(y,z)). \end{aligned}$$

This completes the proof. □

Remark 4.4. Note that, in Lemma 4.3, (ME) is only sufficient and not necessary. To see this, let $I, J \in \mathbb{I}$ be as given in Remark 4.2(iii). Then it follows that

$$I(x, J(y,z)) = \begin{cases} 1, & \text{if } x < 1 \text{ or } y < 1 \ , \\ \sin(\frac{\pi z^3}{2}), & \text{if } x = 1 \text{ and } \ y = 1 \ , \end{cases}$$

and

$$J(y, I(x,z)) = \begin{cases} 1, & \text{if } x < 1 \text{ or } y < 1 \ , \\ \sin^3(\frac{\pi z}{2}), & \text{if } x = 1 \text{ and } \ y = 1 \ , \end{cases}$$

are not identically the same for all $x, y, z \in [0,1]$. To see this, let $z = \frac{1}{2}$, for instance.

The following result gives a condition on I, J such that (ME) also becomes necessary for (LI).

Theorem 4.5. *Let $I, J \in \mathbb{I}$ satisfy* (LI) *w.r.t. a t-norm T. Further, let $I(x,\cdot)$ be one-one and $J(x,\cdot)$ be both one-one and onto, i.e., $J(x,\cdot)$ is an increasing bijection on $[0,1]$ for all $x \in (0,1]$. Then the following statements are equivalent.*

(i) *I, J satisfy* (ME).
(ii) *$I \circledast J$ satisfies* (LI) *w.r.t. same T.*

Proof. (i) $\implies$ (ii): Follows from Lemma 4.3.
(ii) $\implies$ (i): Let $I \circledast J$ satisfy (LI) w.r.t. same T. Then, for any $x, y \in (0,1)$,

$$\begin{aligned} (I \circledast J)(T(x,y),z) &= (I \circledast J)(x, (I \circledast J)(y,z)) \\ &\implies I(x, I(y, J(x, J(y,z)))) = I(x, J(x, I(y, J(y,z)))) \\ &\implies I(y, J(x, J(y,z)) = J(x, I(y, J(y,z))) \ . \qquad \lceil \cdot\cdot \ I(x,\cdot) \text{ is one-one} \end{aligned}$$

Since $J(x,\cdot)$ is a bijection on $[0,1]$, for every $t \in (0,1)$ and any $y \in (0,1)$ there exists a $z \in (0,1)$ such that $t = J(y,z)$. Hence, we have $I(y, J(x,t)) = J(x, I(y,t))$ for all $x, y, t \in [0,1]$, i.e., I, J are mutually exchangeable. □

4.2. Contrapositive Symmetry and $\circledast$-generated implications

Contrapositive symmetry of implications is a tautology in classical logic. Contrapositive symmetry of fuzzy implications w.r.t. an involutive or a strong negation plays an equally important role in fuzzy logic as its classical counterpart - especially in t-norm based multi-valued logics, see for instance, [17, 18, 23, 24, 25, 30].

Once again, many generalisations and weaker versions of the law of contraposition are considered in the literature, see for instance, [4], Section 1.5. However, here we consider only the classical law of contraposition where the involved negation is strong.

Definition 4.6 ([4], Definition 7.3). A fuzzy implication I is said to satisfy the contrapositive symmetry w.r.t. a fuzzy negation N if

$$I(x,y) = I(N(y),N(x)), \qquad x,y \in [0,1]\ . \tag{CP}$$

In such a case, we often write that I satisfies CP(N).

Just as (EP) and (LI) are closely related, so are (CP) and (NP). In fact, as the following result shows if a neutral fuzzy implication satisfies (CP) w.r.t. some fuzzy negation N, then it does so only with its natural negation which should be necessarily strong.

Lemma 4.7 ([4], Lemma 1.5.4)**.** *Let $I \in \mathbb{I}_{\mathbf{NP}}$ satisfy* (CP) *w.r.t. a fuzzy negation N. Then $N_I = N$ and N_I is strong.*

However, it should be emphasised, that even if a fuzzy implication I does not satisfy (NP), it still can satisfy (CP) with some fuzzy, even strong, negation, see [4], Example 1.5.10.

Remark 4.8. Once again, note that even if $I, J \in \mathbb{I}$ satisfy (CP) w.r.t. a fuzzy negation N, $I \circledast J$ may satisfy (CP) w.r.t. no fuzzy negation N or the same fuzzy negation N.

(i) Let $I = I_{\mathbf{RC}}$ and $J = I_{\mathbf{KD}}$. It is easy to check that $I_{\mathbf{RC}}$ and $I_{\mathbf{KD}}$ both satisfy CP($N_{\mathbf{C}}$), i.e., (CP) w.r.t. the classical strong negation $N_{\mathbf{C}}(x) = 1 - x$. Now, from the definition of $\circledast$, it follows that, $(I_{\mathbf{RC}} \circledast I_{\mathbf{KD}})(x,y) = \max(1-x^2, 1-x+xy)$ which has (NP). From Lemma 4.7 above, we see that since $I_{\mathbf{RC}} \circledast I_{\mathbf{KD}}$ has (NP), if it satisfies (CP) w.r.t. some fuzzy negation N, then its natural negation should be strong. However, we see that $N_{I_{\mathbf{RC}} \circledast I_{\mathbf{KD}}}(x) = (I_{\mathbf{RC}} \circledast I_{\mathbf{KD}})(x,0) = \max(1-x^2, 1-x) = 1-x^2$, which is only strict but not strong. Hence, $I_{\mathbf{RC}} \circledast I_{\mathbf{KD}}$ does not satisfy (CP) w.r.t. *any* fuzzy negation N.

(ii) Interestingly, if $I = I_{\mathbf{KD}}$ and $J = I_{\mathbf{RC}}$ then $I \circledast J = I_{\mathbf{KD}} \circledast I_{\mathbf{RC}} = I_{\mathbf{RC}}$, which satisfies (CP) w.r.t. the same negation N.

In the rest of this section, we consider only neutral fuzzy implications, i.e., $I, J \in \mathbb{I}_{\mathbf{NP}}$. If such a pair also satisfies (CP) w.r.t. the same fuzzy negation N, then the following result gives a necessary condition for $I \circledast J$ to satisfy (CP).

Theorem 4.9. *Let $I, J \in \mathbb{I}_{\mathbf{NP}}$ satisfy* (CP) *w.r.t. a fuzzy negation N. If $I \circledast J$ also satisfies CP(N) then $I(x, N(x)) = N(x)$ for all $x \in [0,1]$.*

Proof. Firstly, from Lemma 3.3, we see that $I \circledast J \in \mathbb{I}_{\mathbf{NP}}$ and from Lemma 4.7 that $N_J = N$. Further, since $I \circledast J$ satisfies (CP) w.r.t. N, once again from Lemma 4.7 we have that $N_{I \circledast J}(x) = I(x, J(x,0)) = N(x)$ or equivalently, $I(x, N(x)) = N(x)$. □

In the following we show that if the considered pair $I, J \in \mathbb{I}_{\mathbf{NP}}$ also posseses other desirable properties like (EP) or (OP), then one obtains much stronger results. The following result is helpful in the characterisation results given below. The family of (S,N)-implications will be dealt with presently in Section 6.1 below.

Theorem 4.10 (cf. [10], Theorem 5)**.** *Let I be an (S,N)-implication, i.e., $I(x,y) = S(N(x),y)$, where S is a t-conorm and N is a negation. If N is also strong, then $I(x, N(x)) = N(x)$ if and only if $S = \max$.*

Theorem 4.11. *Let $I, J \in \mathbb{I}_{\mathbf{NP}} \bigcap \mathbb{I}_{\mathbf{EP}}$ satisfy* (CP) *w.r.t. a fuzzy negation N. Then the following statements are equivalent:*

(i) *$I \circledast J$ satisfies CP(N).*
(ii) *$I(x, y) = \max(N(x), y)$.*
(iii) *$I \circledast J = J$.*

Proof. (i) $\Longrightarrow$ (ii): Since I has (NP) and CP(N), we know from Lemma 4.7 that $N = N_I$ is strong. Further, since I satisfies (EP), by the characterisation result for (S, N)-implications, viz., Theorem 6.3, we see that I is an (S, N)-implication. Now, since $I \circledast J$ satisfies CP(N) with a strong negation N, from Theorem 4.9 we have that $I(x, N(x)) = N(x)$ for all $x \in [0, 1]$ and from Theorem 4.10 above we have that $I(x, y) = \max(N(x), y)$.

(ii) $\Longrightarrow$ (iii): Let $I(x, y) = \max(N(x), y)$ for all $x, y \in [0, 1]$. From Lemma 4.7, it follows that $N_J = N$ is strong. Since J satisfies (EP), by the characterisation result for (S, N)-implications, viz., Theorem 6.3, it follows that J is an (S, N)-implication with the strong negation N, say, $J(x, y) = S(N(x), y)$ for some t-conorm S. Now, $(I \circledast J)(x, y) = I(x, J(x, y)) = \max(N(x), S(N(x), y)) = S(N(x), y) = J(x, y)$ for all $x, y \in [0, 1]$.

(iii) $\Longrightarrow$ (i): It is straight-forward now.

□

Theorem 4.12. *Let $I, J \in \mathbb{I}_{\mathbf{NP}} \bigcap \mathbb{I}_{\mathbf{OP}}$ satisfy* (CP) *w.r.t. the same fuzzy negation N. Then $I \circledast J$ does not satisfy* (CP) *w.r.t. any negation N.*

Proof. On the one hand, if $I \circledast J$ satisfies (OP), then from Theorem 4.9 we see that $I(x, N(x)) = N(x)$ for all $x \in [0, 1]$. In particular, if we take $x = e \in (0, 1)$, the equilibrium point of N, i.e., $N(e) = e$, we have that $I(e, N(e)) = I(e, e) = N(e) = e < 1$, a contradiction to the fact that I satisfies (OP).

On the other hand, if $I \circledast J$ does not satisfy (OP), then from Theorem 3.5, we know $J(x_0, 0) > x_0$ for some $x_0 \in (0, 1)$. Now,

$$(I \circledast J)(x_0, 0) = I(x_0, J(x_0, 0)) \geq I(x_0, x_0) = 1 \qquad \lceil \text{ by (OP) of } I$$
$$\text{while, } (I \circledast J)(1, N(x_0)) = I(1, J(1, N(x_0))) = N(x_0) \ ,$$

a contradiction to the fact that $0 < x_0 < 1$. □

5. Powers of elements of $\mathbb{I}$ and Convergence.

From Theorem 3.2, it follows that if $I, J \in \mathbb{I}$ then $I \circledast J \in \mathbb{I}$. Further, if $J = I$ then $I \circledast I$ is also an implication on $[0, 1]$. Since the binary operation $\circledast$ is associative in $\mathbb{I}$, see [53], one can define the powers of a fuzzy implication I w.r.t. $\circledast$ in the following natural way.

Definition 5.1. Let $I \in \mathbb{I}$. For any $n \in \mathbb{N}$, we define the n-th power of I w.r.t. the binary operation $\circledast$ as follows:

$$I_{\circledast}^{[n]}(x, y) = I\left(x, I_{\circledast}^{[n-1]}(x, y)\right) = I_{\circledast}^{[n-1]}\left(x, I(x, y)\right) \ , \qquad x, y \in [0, 1] \ . \tag{4}$$

In this section, we firstly explore the limiting case behaviour of $I_{\circledast}^{[n]}$ for an $I \in \mathbb{I}$, in general, and also when I satisfies some of the other desirable properties. Further, we also investigate the following question: If an $I \in \mathbb{I}$ satisfies a particular property $\mathbf{P}$, say (EP), then whether $I_{\circledast}^{[n]}$ satisfies the same property for all $n \in \mathbb{N}$.

It is immediately clear that it is possible for some $I \in \mathbb{I}$ to be such that $I \circledast I = I$ (see, for instance, [47]). Towards this end, we define the following characteristic of an $I \in \mathbb{I}$ w.r.t. the $\circledast$- composition.

Implication(I)	Order $\mathcal{O}(I)$	$\lim_{n\to\infty} I_{\circledast}^{[n]}$
$I_{\mathbf{RC}}$	∞	$I_{\mathbf{WB}}$
$I_{\mathbf{KD}}$	1	$I_{\mathbf{KD}}$
$I_{\mathbf{FD}}$	2	$I_{\mathbf{FD}}^2$
$I_{\mathbf{GD}}$	1	$I_{\mathbf{GD}}$
$I_{\mathbf{GG}}$	∞	$I_{\mathbf{WB}}$
$I_{\mathbf{RS}}$	1	$I_{\mathbf{RS}}$
$I_{\mathbf{LK}}$	∞	$I_{\mathbf{WB}}$
$I_{\mathbf{WB}}$	∞	$I_{\mathbf{WB}}$
$I_{\mathbf{YG}}$	∞	$I_{\mathbf{WB}}$
$I_{\mathbf{1}}$	1	$I_{\mathbf{1}}$
$I_{\mathbf{0}}$	1	$I_{\mathbf{0}}$

Table 8: Powers of the basic fuzzy implications w.r.t. $\circledast$ and their orders.

5.1. Order of a fuzzy implication I w.r.t. $\circledast$

Definition 5.2. An I is said to be of *fixed point order* n w.r.t. $\circledast$ if there exists an $n \in \mathbb{N}$ such that n is the smallest integer for which $I_{\circledast}^{[n]} = I_{\circledast}^{[n+1]}$. We denote it by $\mathcal{O}(I)$ and refer to it just as the *order* of an I. Note that if, for a given I, no such n exists then we write $\mathcal{O}(I) = \infty$.

Table 8 tabulates the orders and the limiting case behaviour of the basic fuzzy implications listed in Table 1. Note that $I_{\mathbf{FD}}^2$ in Table 8 is obtained as

$$(I_{\mathbf{FD}} \circledast I_{\mathbf{FD}})(x,y) = I_{\mathbf{FD}}^2(x,y) = \begin{cases} 1, & \text{if } \ x \le y \text{ or } x \in [0,0.5] \ , \\ \max(1-x,y), & \text{otherwise} \ . \end{cases}$$

If $\mathcal{O}(I) = 1$ then $I_{\circledast}^{[2]} = I$ and we do not obtain any new implications from I. In algebraic terms, such $I \in \mathbb{I}$ form the set of idempotents in the monoid $(\mathbb{I}, \circledast)$. Note that the characterisation of all such idempotent elements is a non-trivial task and will be presented in a future work. Some partial results are already available in [51, 52]. In the case when I comes from a specific family of fuzzy implications this functional equation has been dealt with by many authors, see, [33, 36, 46, 55]. Since our motivation in this work is to obtain new fuzzy implications from given ones, in the sequel we study only the case when $\mathcal{O}(I) > 1$.

It can be observed from Table 8, that for most of the basic fuzzy implications I listed in Table 1, $I_{\circledast}^{[n]}$ do converge to $I_{\mathbf{WB}}$ in the limiting case. The following result explores the context in which this is true for any general fuzzy implication.

Theorem 5.3. *Let $I \in \mathbb{I}$ satisfy* (LI) *w.r.t. a t-norm T.*

(i) *Then $I_{\circledast}^{[n]}(x,y) = I\left(x_T^{[n]}, y\right)$, where $x_T^{[n]} = T\left(x, x_T^{[n-1]}\right)$ and $x_T^{[1]} = x$ for any $x \in [0,1]$.*

(ii) *Further, let T be Archimedean, i.e., for any $x, y \in (0,1)$ there exists an $n \in \mathbb{N}$ such that $x_T^{[n]} < y$. Then $\lim_{n\to\infty} I_{\circledast}^{[n]}(x,y) = \begin{cases} 1, & \text{if } \ x < 1 \ , \\ I(1,y), & \text{if } \ x = 1 \ . \end{cases}$*

(iii) *Moreover, if $I \in \mathbb{I}_{\mathbf{NP}}$ then $\lim_{n\to\infty} I_{\circledast}^{[n]} = I_{\mathbf{WB}}$.*

Proof. (i) Let $x, y \in [0,1]$. Then $I_{\circledast}^{[2]}(x,y) = I(x, I(x,y)) = I(T(x,x), y)$, since I satisfies (LI). Thus $I_{\circledast}^{[2]}(x,y) = I(x_T^{[2]}, y)$. By induction, we obtain $I_{\circledast}^{[n]}(x,y) = I(x_T^{[n]}, y)$.

(ii) Let $\epsilon > 0$ and $x < 1$. Since T is Archimedean, for any $\epsilon > 0$, there exists an $m \in \mathbb{N}$ such that $x_T^{[m]} < \epsilon$. Thus, for any $y \in [0,1]$, $I_{\circledast}^{[n]}(x,y) = I(x_T^{[n]}, y) \longrightarrow I(0,y)$, as $n \longrightarrow \infty$ and $\lim_{n\to\infty} I_{\circledast}^{[n]}(x,y) = 1$.
If $x = 1$, then $I_{\circledast}^{[2]}(1,y) = I(1, I(1,y)) = I(T(1,1), y) = I(1,y)$ and in general, $I_{\circledast}^{[n]}(1,y) = I(1,y)$.

(iii) Follows from (ii) and the fact that $I(1,y) = y$.

□

5.2. Properties preserved by $I_{\circledast}^{[n]}$.

In this subsection, given an $I \in \mathbb{I}$ satisfying a particular property **P**, we investigate whether all the powers $I_{\circledast}^{[n]}$ of I satisfy the same property or not.

Lemma 5.4. *If I satisfies* (EP) *then the pair $(I, I_{\circledast}^{[n]})$ satisfies* (ME)*.i.e.,*

$$I(x, I_{\circledast}^{[n]}(y,z)) = I_{\circledast}^{[n]}(y, I(x,z)), \qquad \text{for all } x,y,z \in [0,1], n \in \mathbb{N}. \tag{5}$$

Proof. We prove this by using mathematical induction on n. For, $n = 1$, I satisfies (5), from the (EP) of I. Assume that I satisfies (5) for $n = k-1$. Now,

$$\begin{aligned} I(x, I_{\circledast}^{[k]}(y,z)) &= I(x, I(y, I_{\circledast}^{[k-1]}(y,z))) \\ &= I(x, I_{\circledast}^{[k-1]}(y, I(y,z))) \\ &= I_{\circledast}^{[k-1]}(y, I(x, I(y,z))) \\ &= I_{\circledast}^{[k-1]}(y, I(y, I(x,z))) \\ &= I_{\circledast}^{[k]}(y, I(x,z)), \text{for all } x,y,z \in [0,1]. \end{aligned}$$

Thus the pair $(I, I_{\circledast}^{[k]})$ satisfies (ME) an this completes the proof. □

Theorem 5.5. *If I satisfies* (EP) *(* (NP) *or* (IP) *or is self-conjugate or continuous) then the same is true of $I_{\circledast}^{[n]}$ for all $n \in \mathbb{N}$.*

Proof. We prove this theorem for (EP) only, since the proof for others is easily obtainable. Let $I \in \mathbb{I}_{\mathbf{EP}}$, i.e., I satisfies (EP). We show that $I_{\circledast}^{[n]}$ also satisfies (EP) for all $n \in \mathbb{I}$. Let $x,y,z \in [0,1]$. Then

$$\begin{aligned} I_{\circledast}^{[n]}(x, I_{\circledast}^{[n]}(y,z)) &= I\left(x, I_{\circledast}^{[n-1]}(x, I_{\circledast}^{[n]}(y,z))\right) \\ &= I(x, I_{\circledast}^{[n-1]}(x, I(y, I_{\circledast}^{[n-1]}(y,z)))) \\ &= I(x, I(y, I_{\circledast}^{[n-1]}(x, I_{\circledast}^{[n-1]}(y,z)))) \qquad \lceil\ (I, I_{\circledast}^{[n-1]}) \text{ satisfy (ME), Theorem 5.4} \\ &= I(y, I(x, I_{\circledast}^{[n-1]}(y, I_{\circledast}^{[n-1]}(x,z)))) \\ &= I(y, I_{\circledast}^{[n-1]}(y, I(x, I_{\circledast}^{[n-1]}(x,z)))) \\ &= I_{\circledast}^{[n]}(y, I_{\circledast}^{[n]}(x,z)) \ . \end{aligned}$$

□

Theorem 5.6. *Let I satisfy* (OP)*. The following statements are equivalent.*

(i) $I_{\circledast}^{[2]}$ *satisfies* (OP)*.*
(ii) $x > I(x,y)$ *for all* $x > y$*.*
(iii) $I_{\circledast}^{[n]}$ *satisfies* (OP) *for all* $n \in \mathbb{N}$*.*

Proof. (i) $\Longrightarrow$ (ii): This follows from Theorem 3.5 with $J = I$.

(ii) $\Longrightarrow$(iii): Let $x > I(x,y)$ for all $x > y$. We prove that $I_{\circledast}^{[n]}$ satisfies (OP) for n. We do this by using mathematical induction on n. Since $I_{\circledast}^{[n]}$ satisfies (OP) for $n = 1$, assume that $I_{\circledast}^{[k-1]}$ satisfies (OP). Now we show that $I_{\circledast}^{[k]}$ has also (OP).

- Let $x \leq y$. Then $I(x,y) = 1$. Hence $I_{\circledast}^{[k]}(x,y) = I_{\circledast}^{[k-1]}(x, I(x,y)) = 1$.
- Let $x > y$. Then from our assumption, we have $x > I(x,y)$. Now from (OP) of $I_{\circledast}^{[k-1]}$, it follows that $I_{\circledast}^{[k]}(x,y) = I_{\circledast}^{[k-1]}(x, I(x,y)) < 1$ and hence $I_{\circledast}^{[k]}$ satisfies (OP).

(iii) $\Longrightarrow$ (i): Follows trivially for $n = 2$. □

Corollary 5.7. *Let I satisfy* (OP). *If $I_{\circledast}^{[m]}$ satisfies* (OP) *for some $m \in \mathbb{N}$ then $I_{\circledast}^{[n]}$ satisfies* (OP) *for all $n > m \in \mathbb{N}$.*

Theorem 5.8. *If I satisfies* (LI) *w.r.t. a t-norm T, then $I_{\circledast}^{[n]}$ also satisfies* (LI) *w.r.t. the same t-norm T.*

Proof. We prove this also by using mathematical induction on n. For $n = 1$, $I_{\circledast}^{[n]}$ satisfies (LI). Assume that $I_{\circledast}^{[k-1]}$ satisfies (LI) w.r.t. the same t-norm T, i.e., $I_{\circledast}^{[k-1]}(T(x,y),z) = I_{\circledast}^{[k-1]}(x, I_{\circledast}^{[k-1]}(y,z))$ for all $x, y, z \in [0,1]$. From Theorem 5.3(i) recall that if I satisfies (LI) w.r.t. a t-norm T then $I_{\circledast}^{[k]}(x,y) = I(x_T^{[k]}, y)$ for all $x, y \in [0,1]$. Now, for any $x, y, z \in [0,1]$,

$$
\begin{aligned}
I_{\circledast}^{[k]}(T(x,y),z) &= I\left((T(x,y))_T^{[k]}, z\right) \\
&= I\left(T(T(x,y),(T(x,y))_T^{[k-1]}), z\right) \\
&= I\left(T(x,y), I((T(x,y))_T^{[k-1]}, z)\right) \\
&= I\left(T(x,y), I_{\circledast}^{[k-1]}(T(x,y),z)\right) \\
&= I\left(x, I(y, I_{\circledast}^{[k-1]}(T(x,y),z))\right) \\
&= I\left(x, I(y, I_{\circledast}^{[k-1]}(x, I_{\circledast}^{[k-1]}(y,z)))\right) \\
&= I\left(x, I_{\circledast}^{[k-1]}(x, I(y, I_{\circledast}^{[k-1]}(y,z)))\right) = I_{\circledast}^{[k]}(x, I_{\circledast}^{[k]}(y,z)) \ .
\end{aligned}
$$

This completes teh proof. □

If $I \in \mathbb{I}$ satisfies (CP) w.r.t. some strong negation N, then $I_{\circledast}^{[n]}$ may satisfy CP(N) or may not satisfy CP w.r.t. any N.

(i) Let $I(x,y) = \max(N(x), y)$, for some strong negation N. Then clearly I satisfies (CP) w.r.t. N and so does $I_{\circledast}^{[n]}$ for every $n \in \mathbb{N}$.

(ii) If we let $I = I_{\mathbf{RC}}$, then I satisfies (CP) w.r.t. $N_{\mathbf{C}}(x) = 1 - x$. Since $I_{\mathbf{RC}}$ satisfies both (NP) and (EP), from Theorem 4.11 we see that, for $I_{\circledast}^{[2]}$ to satisfy CP($N_{\mathbf{C}}$), I should be expressible as $I(x,y) = \max(N(x), y)$ for some negation N. Clearly, this is not true, since $I_{\mathbf{RC}}$ cannot be expressed as $\max(N(x), y)$ for any negation N. Note that this also means that $I_{\circledast}^{[2]}$ does not (cannot) satisfy (CP) w.r.t. any negation N.

Lemma 5.9. *Let $I \in \mathbb{I}_{\mathbf{NP}} \bigcap \mathbb{I}_{\mathbf{EP}}$ satisfy* (CP) *w.r.t. a fuzzy negation N. Then the following statements are equivalent:*

(i) $I_{\circledast}^{[n]}$ *satisfies* (CP) *for all $n \in \mathbb{N}$.*
(ii) $I(x,y) = \max(N(x), y)$.

Proof. (i) $\implies$ (ii). Let $I_{\circledast}^{[n]}$ satisfies (CP) for all $n \in \mathbb{N}$. i.e., $I_{\circledast}^{[2]} = I \circledast I$ also satisfies (CP). From Theorem 4.11, it follows that $I(x, y) = \max(N(x), y)$. Further, note that $I_{\circledast}^{[n]} = I$, in this case, and hence satisfies CP(N) for all $n \in \mathbb{N}$.

(ii) $\implies$ (i). Let $I(x, y) = \max(N(x), y)$. Then it follows that $I_{\circledast}^{[n]} = I$ for all $n \in \mathbb{N}$ and hence $I_{\circledast}^{[n]}$ satisfies (CP) for all $n \in \mathbb{N}$.

□

The following result follows trivially from Theorem 4.12:

Lemma 5.10. *Let $I \in \mathbb{I}$ satisfy* (NP) *and* (OP). *Then $I_{\circledast}^{[n]}$ does not satisfy* (CP) *for any $n \in \mathbb{N}$.*

6. Closures of Families of Fuzzy Implications w.r.t. the $\circledast$-composition

Among the many families of fuzzy implications, (S, N)-implications, R-implications and the Yager's families of f- and g-implications have received a lot of interest and importance from the research community due to their use in both theoretical considerations and practical applications. Further, as noted earlier, they are also representative families from the first and second type of generation methods of fuzzy implications.

In this section we study the closures of the above families of fuzzy implications w.r.t. the $\circledast$-composition. More explicitly, we investigate the solutions to the following questions: If $I, J \in \mathbb{I}$ belong to a certain family of fuzzy implications, then does $I \circledast J$ also belong to the same family? If it does not, then what are the conditions on the underlying operations such that $I \circledast J$ also belongs to the same family? Finally, we investigate the effect on any member of each of these families under self-composition w.r.t. the $\circledast$-composition, or equivalently the powers of implications from these families.

Note that, while the families of f- and g-implications have been completely characterised, see [38], the families of (S, N)- and R-implications, though two of the oldest, are yet to be characterised completely. In this work, we deal only with those sub-families of (S, N)- and R-implications for which characterisation results are available.

6.1. (S,N)-implication and the $\circledast$-composition

One of the first generalisations of a classical implication to the setting of fuzzy logic, in fact, multi-valued logic is based on the classical material implication $p \implies q \equiv \neg p \vee q$. The family of (S, N)-implications were obtained by substituting a fuzzy negation N for $\neg$ and a t-conorm S for the join / maximum operation $\vee$ in the preceeding formula and hence the nomenclature.

Definition 6.1 ([4], Definition 2.4.1). A function $I : [0, 1]^2 \longrightarrow [0, 1]$ is called an *(S,N) implication* if there exist a t-conorm S and a fuzzy negation N such that

$$I(x, y) = S(N(x), y), \qquad x, y \in [0, 1]. \tag{6}$$

If I is an (S, N)-implication then we will often denote it by $I_{S,N}$. The family of all (S, N)-implications will be denoted by $\mathbb{I}_{\mathbb{S},\mathbb{N}}$. Table 9 gives some of the basic (S,N)-implications.

S	N	(S,N)-implication $I_{S,N}$
$S_{\mathbf{M}}$	$N_{\mathbf{C}}$	$I_{\mathbf{KD}}$
$S_{\mathbf{P}}$	$N_{\mathbf{C}}$	$I_{\mathbf{RC}}$
$S_{\mathbf{LK}}$	$N_{\mathbf{C}}$	$I_{\mathbf{LK}}$
$S_{\mathbf{nM}}$	$N_{\mathbf{C}}$	$I_{\mathbf{FD}}$
any S	$N_{\mathbf{D1}}$	$I_{\mathbf{D}}$
any S	$N_{\mathbf{D2}}$	$I_{\mathbf{WB}}$

Table 9: Examples of basic (S,N)-implications. Please refer to Tables 1 and 3

Proposition 6.2 (cf. [4], Proposition 2.4.3)**.** *If $I_{S,N}$ is an (S,N)-implication, then $I_{S,N} \in \mathbb{I}$. Further, $I_{S,N}$ satisfies* (NP)*,* (EP) *and $N_{I_{S,N}} = N$.*

In the literature, the only available characterisations for (S, N)-implications are those that are obtained from continuous negations. Note that such characterisations have also been obtained based on (LI), see [34].

Theorem 6.3 ([4], Theorems 2.4.10 – 2.4.12)**.** *For a function $I\colon [0,1]^2 \longrightarrow [0,1]$ the following statements are equivalent:*

(i) *I is an (S, N)-implication with a continuous (strict, strong) fuzzy negation N.*
(ii) *I satisfies* (I1)*,* (EP) *and N_I is a continuous (strict, strong) fuzzy negation.*

Moreover, the representation of (S, N)-implication (6) *is unique in this case.*

6.1.1. Closure of $\mathbb{I}_{\mathbb{S},\mathbb{N}}$ w.r.t. the $\circledast$-composition

In general, the $\circledast$-composition of two (S, N)-implications need not be an (S, N)-implication. To see this, let $I, J \in \mathbb{I}_{\mathbb{S},\mathbb{N}}$. Then from Proposition 6.2, it follows that I, J satisfy (EP). But from Remark 3.8, it follows that $I \circledast J$ need not satisfy (EP) which implies that $I \circledast J$ need not be again an (S, N)-implication. Thus the $\circledast$-composition of two (S, N)-implications need not be again an (S, N)-implication.

Remark 6.4. (i) On the one hand, if we let $I(x, y) = S_{\mathbf{P}}(1 - x, y)$, $J(x, y) = S_{\mathbf{P}}(1 - x^2, y)$, which are (S, N)-implications, then $(I \circledast J)(x, y) = S_{\mathbf{P}}(1 - x^3, y)$, is an (S, N)-implication.

(ii) On the other hand, if $I = I_{\mathbf{RC}}, J = I_{\mathbf{KD}}$ both of which are (S,N)-implications(see Table 9), then their $\circledast$-composition $I_{\mathbf{RC}} \circledast I_{\mathbf{KD}}$, as given in Table 6, is not an (S,N)-implication since it does not satisfy (EP). To see this, let $x = 0.3, y = 0.8$ and $z = 0.5$. Then

$$(I \circledast J)(0.3, (I \circledast J)(0.8, 0.5)) = 0.91 \ ,$$

where as

$$(I \circledast J)(0.8, (I \circledast J)(0.3, 0.5)) = 0.928 \ .$$

Thus $I \circledast J$ does not satisfy (EP) and hence $I \circledast J$ does not become an (S, N)-implication.

The following result gives a sufficient condition for $I \circledast J \in \mathbb{I}_{\mathbb{S},\mathbb{N}}$.

Theorem 6.5. *Let $I(x, y) = S(N_1(x), y)$ and $J(x, y) = S(N_2(x), y)$ be two (S, N)-implications. Then $I \circledast J$ is also an (S, N)-implication.*

Proof. Let $I(x, y) = S(N_1(x), y), J(x, y) = S(N_2(x), y)$ be two (S, N)-implications. Now,

$$(I \circledast J)(x, y) = I(x, J(x, y)) = I(x, S(N_2(x), y)) = S(N_1(x), S(N_2(x), y)) = S(S(N_1(x), N_2(x)), y) \ .$$

Since for every t-conorm S and fuzzy negations N_1, N_2, the function $S(N_1(x), N_2(x))$ is again a fuzzy negation, we get $(I \circledast J)(x, y) = S(S(N_1(x), N_2(x)), y)$ is an (S, N)-implication with S being the t-conorm and $S(N_1(x), N_2(x))$ the fuzzy negation. □

Remark 6.6. Note that the converse of Theorem 6.5 is not true. For example, we know that $I_{\mathbf{WB}}(x, y) = S(N_{\mathbf{D_2}}(x), y)$ for any t-conorm S. Let $I(x, y) = S_1(N(x), y)$ be an (S, N)-implication where S_1 is a t-conorm different from S. Then $I \circledast I_{\mathbf{WB}} = I_{\mathbf{WB}} = I_{\mathbf{WB}} \circledast I$, is an (S, N)-implication but $S \neq S_1$.

Let us denote some special subsets of $\mathbb{I}_{\mathbb{S},\mathbb{N}}$ by the following:

- $\mathbb{I}_{\mathbb{S},\mathbb{N}_{\mathbb{C}}}$ - (S, N)-implications obtained from continuous negations.
- $\mathbb{I}_{\mathbb{S}_{\mathbb{C}},\mathbb{N}_{\mathbb{C}}}$ - (S, N)-implications obtained from continuous t-conorms and continuous negations.

6.1.2. Closure of $\mathbb{I}_{\mathbb{S},\mathbb{N}_C}$ w.r.t. the $\circledast$-composition

Let us consider $I, J \in \mathbb{I}_{\mathbb{S},\mathbb{N}_C}$, i.e., $I(x,y) = S_1(N_1(x), y)$ and $J(x,y) = S_2(N_2(x), y)$, where S_1, S_2 are t-conorms and N_1, N_2 are two continuous negations. Clearly, $I \circledast J$ satisfies (I1). Note, however that, even when we consider only (S,N)-implications obtained from continuous negations, we have that the natural negation of $I \circledast J$ given by

$$N_{I\circledast J}(x) = (I \circledast J)(x,0) = I(x, J(x,0)), \qquad x \in [0,1] \ ,$$

may not be continuous. For instance, when $I(x,y) = S_{\mathbf{nM}}(1-x,y) = I_{\mathbf{FD}}(x,y)$ and $J(x,y) = S_{\mathbf{P}}(1-x,y) = I_{\mathbf{RC}}(x,y) = 1 - x + xy$, we obtain

$$N_{I\circledast J}(x) = \begin{cases} 1, & \text{if } x \leq \frac{1}{2} \\ 1-x, & \text{if } x > \frac{1}{2}. \end{cases}$$

Clearly, $N_{I\circledast J}$ is not continuous at $x = \frac{1}{2}$.

As is already shown above, $I \circledast J$ may not preserve (EP). Clearly, if the pair (I,J) satisfies (ME) then from Theorem 3.11 we know that $I \circledast J$ will satisfy (EP). While, this is neither sufficient nor necessary to ensure $I \circledast J \in \mathbb{I}_{\mathbb{S},\mathbb{N}}$, the following result shows that this is equivalent to the condition $S_1 = S_2$ of Theorem 6.5 when $I, J \in \mathbb{I}_{\mathbb{S},\mathbb{N}_C}$.

Theorem 6.7. *Let $I(x,y) = S_1(N_1(x), y), J(x,y) = S_2(N_2(x), y)$ be two (S,N)-implications such that N_1, N_2 are continuous negations. Then the following statements are equivalent.*

(i) $S_1 = S_2$.
(ii) I, J *satisfy* (ME).

Proof. (i) $\implies$ (*ii*). Let $S_1 = S_2 = S$. Then we have $I(x,y) = S(N_1(x), y), J(x,y) = S(N_2(x), y)$. Now,

$$\text{L.H.S of (ME)} = S(N_1(x), S(N_2(y), z)) = S(S(N_1(x), N_2(y)), z) \ , \text{and}$$

$$\text{R.H.S of (ME)} = S(N_2(y), S(N_1(x), z)) = S(S(N_1(x), N_2(y)), z) \ .$$

Thus I, J satisfy (ME).

(ii) $\implies$ (*i*). Let I, J satisfy (ME).i.e., $I(x, J(y,z)) = J(y, I(x,z))$ for all $x, y, z \in [0,1]$. This implies that

$$S_1(N_1(x), S_2(N_2(y), z)) = S_2(N_2(y, S_1(N_1(x), z)))$$

for all $x, y, z \in [0,1]$. Letting $z = 0$, we obtain, for any $x, y \in [0,1]$,

$$S_1(N_1(x), N_2(y)) = S_2(N_1(x), N_2(y)) \ .$$

Since the ranges of N_1, N_2 are equal to $[0,1]$, we get, $S_1(a,b) = S_2(a,b)$ for all $a, b \in [0,1]$. □

Finally, if we restrict the underlying t-conorm S also to be continuous, i.e., if we consider $\mathbb{I}_{\mathbb{S}_C,\mathbb{N}_C} \subsetneq \mathbb{I}_{\mathbb{S},\mathbb{N}}$, then the following result is immediate:

Corollary 6.8. *Let $I, J \in \mathbb{I}_{\mathbb{S}_C,\mathbb{N}_C}$. If I, J satisfy* (ME) *then $I \circledast J \in \mathbb{I}_{\mathbb{S}_C,\mathbb{N}_C}$.*

6.1.3. Powers of (S,N)-implications w.r.t. the $\circledast$-composition

Note that if $I \in \mathbb{I}_{\mathbb{S},\mathbb{N}}$, then it satisfies (EP) and hence from Theorem 5.5 it follows that $I_{\circledast}^{[n]}$ satisfies (EP) for all $n \in \mathbb{N}$.

From Theorem 6.5 we have the following:

Lemma 6.9. *Let $I \in \mathbb{I}_{\mathbb{S},\mathbb{N}}$. Then $I_{\circledast}^{[n]}$ is also an (S,N)-implication.*

6.2. R-implications and the $\circledast$-composition

A second family of fuzzy implications obtained under the first category of the generation processes listed in the Introduction, is the family of residual implications. This is a generalisation of the implication in the classical intuitionistic logic to the setting of fuzzy logic. Once again, for more details on this family regarding their properties, intersections with other families, etc., see for instance, [4].

Definition 6.10 ([4], Definition 2.5.1). A function $I : [0,1]^2 \longrightarrow [0,1]$ is called an *R- implication* if there exists a t-norm T such that

$$I(x,y) = \sup\{t \in [0,1] | T(x,t) \leq y\}, \qquad x,y \in [0,1].$$

If I is an R-implication generated from a t-norm T, then it is denoted by I_T and $I_T \in \mathbb{I}$. The family of all R-implications will be denoted by $\mathbb{I}_{\mathbb{T}}$.

t-norm T	R-implication I_T
$T_{\mathbf{M}}$	$I_{\mathbf{GD}}$
$T_{\mathbf{P}}$	$I_{\mathbf{GG}}$
$T_{\mathbf{LK}}$	$I_{\mathbf{LK}}$
$T_{\mathbf{D}}$	$I_{\mathbf{WB}}$
$T_{\mathbf{nM}}$	$I_{\mathbf{FD}}$

Table 10: Examples of basic R-implications. For detailed formulae, please see Tables 1 and 2.

As in the case of (S,N)-implications, the characterisation of R-implications is available only for R-implications obtained from *left-continuous* t-norms.

Theorem 6.11 ([4], Theorem 2.5.17)**.** *For a function $I\colon [0,1]^2 \longrightarrow [0,1]$ the following statements are equivalent:*

(i) *I is an R-implication generated from a left-continuous t-norm.*
(ii) *I satisfies* (I2)*,* (EP)*,* (OP) *and it is right-continuous with respect to the second variable.*

Moreover, the representation of R-implication, up to a left-continuous t-norms, is unique in this case.

The set of all R-implications generated from left-continuous t-norms will be denoted by $\mathbb{I}_{\mathbb{T}_{\mathbb{LC}}} \subsetneq \mathbb{I}_{\mathbb{T}}$. Once again, we focus on the problem of, if $I, J \in \mathbb{I}_{\mathbb{T}_{\mathbb{LC}}}$ then does $I \circledast J \in \mathbb{I}_{\mathbb{T}_{\mathbb{LC}}}$?

6.2.1. Closure of $\mathbb{I}_{\mathbb{T}_{\mathbb{LC}}}$ w.r.t. $\circledast$-composition

The composition of two R-implications from left-continuous t-norms need not be an R-implication obtained from a left-continuous t-norm.

Remark 6.12. (i) On the one hand, let $I = I_{\mathbf{GD}}, J = I_{\mathbf{LK}}$. Clearly $I, J \in \mathbb{I}_{\mathbb{T}_{\mathbb{LC}}}$ (see Table 10). However, from Remark 3.4 we know that $I_{\mathbf{GD}} \circledast I_{\mathbf{LK}}$ does not satisfy (OP) and hence from the characterisation Theorem 6.11 we see that $I_{\mathbf{GD}} \circledast I_{\mathbf{LK}} \notin \mathbb{I}_{\mathbb{T}_{\mathbb{LC}}}$.

(ii) On the other hand, consider the Goguen and Gödel implications $I_{\mathbf{GG}}, I_{\mathbf{GD}}$ which are two R-implications generated from left-continuous t-norms $T_{\mathbf{P}}, T_{\mathbf{M}}$, respectively. Then $I_{\mathbf{GG}} \circledast I_{\mathbf{GD}} = I_{\mathbf{GG}} \in \mathbb{I}_{\mathbb{T}_{\mathbb{LC}}}$.

In fact, the following result shows that the $\circledast$-composition of two R-implications, not necessarily from left-continuous t-norms, is an R-implication only if one of them is the Gödel implication $I_{\mathbf{GD}}$.

Lemma 6.13. *Let $I, J \in \mathbb{I}_{\mathbb{T}}$ and let both I, J satisfy* (OP)*. Then the following statements are equivalent.*

(i) *$I \circledast J$ satisfies* (OP)*.*
(ii) *$J = I_{\mathbf{GD}}$.*
(iii) *$I \circledast J = I$.*

Proof. (i) $\Longrightarrow$(ii). Let $I \circledast J$ satisfy (OP). Then from Theorem 3.5, it follows that $y \geq J(x,y)$ for all $x > y$. Since J is an R-implication it follows that $J(x,y) \geq y$ for all $x, y \in [0,1]$. Thus $J(x,y) = y$ for all $x > y$ and hence $J = I_{\mathbf{GD}}$.

(ii) $\Longrightarrow$(iii). This follows from a direct verification.

(iii) $\Longrightarrow$(i). This follows from our assumption. □

For more details on when an $I \in \mathbb{I}_{\mathbb{T}}$ satisfies (OP), please refer to [5], Proposition 5.8.

6.2.2. Powers of R-implications w.r.t. the $\circledast$-composition

From Lemma 6.13 the following result is obvious:

Lemma 6.14. *Let $I \in \mathbb{I}_{\mathbb{T}_{\mathbb{LC}}}$. Then the following statements are equivalent.*

(i) $I_{\circledast}^{[n]} \in \mathbb{I}_{\mathbb{T}}$ *for all* $n \in \mathbb{N}$.
(ii) $I = I_{\mathbf{GD}}$.

6.3. f-implications and the $\circledast$-composition

While the previous sections dealt with the $\circledast$-composition on the families of fuzzy implications obtained from other fuzzy logic connectives, in the following sections we deal with the Yager's families of fuzzy implications, which were proposed by Yager in [56].

In this and the following sections we present the definitions, some relevant properties and characterisations of these families and proceed along the lines similar to that of Sections 6.1 and 6.2, exploring their closures w.r.t. the operation $\circledast$.

Definition 6.15 ([4], Definition 3.1.1). Let $f : [0,1] \longrightarrow [0,\infty]$ be a strictly decreasing and continuous function with $f(1) = 0$. The function $I : [0,1]^2 \longrightarrow [0,1]$ defined by

$$I(x,y) = f^{-1}(x \cdot f(y)) \ , \qquad x, y \in [0,1] \ , \tag{7}$$

with the understanding $0 \cdot \infty = 0$, is called an *f-implication*. If I is an f-implication then it is denoted by I_f. The family of all f-impllications will be denoted by $\mathbb{I}_{\mathbb{F}}$.

It has been shown that there are only two types of f-generators depending on the value f takes at 0, i.e., either $f(0) = \infty$ or $f(0) < \infty$. In fact, when $f(0) < \infty$ it can be modified to a normed generator f' such that $f'(0) = 1$ but the f-implications generated are the same, i.e., $I_f \equiv I_{f'}$. See [4], Chapter 5 for more details.

Let us denote by

- $\mathbb{I}_{\mathbb{F},\infty}$ – the family of all f-generated implications such that $f(0) = \infty$.
- $\mathbb{I}_{\mathbb{F},1}$ – the family of all f-generated implications such that $f(0) < \infty$.

In the following, we list out some important but relevant results that give the properties and characterisations of the family of f-implications.

Theorem 6.16 ([4], Theorem 3.17)**.** *If f is an f-generator, then*

(i) I_f *satisfies* (NP) *and* (EP).
(ii) I_f *is continuous except at the point* $(0,0)$ *if and only if* $f(0) = \infty$.
(iii) I_f *is continuous if and only if* $f(0) < \infty$, *i.e.,* $I_f \in \mathbb{I}_{\mathbb{F},1}$.

Theorem 6.17 (cf. [38], Theorem 6)**.** *Let $I\colon [0,1] \longrightarrow [0,1]$ be a binary function. Then the following statements are equivalent.*

(i) I *is an f-implication with* $f(0) < \infty$, *i.e.,* $I \in \mathbb{I}_{\mathbb{F},1}$.

(ii) *I satisfies* (LI) *w.r.t. $T_{\mathbf{P}}$ and N_I is strict negation.*

Moreover f-generator is unique upto a positive multiplicative constant and it is given by $f(x) = N_I^{-1}(x)$.

Theorem 6.18 (cf. [38], Theorem 12)**.** *Let $I\colon [0,1]^2 \longrightarrow [0,1]$ be a binary function. Then the following statements are equivalent.*

(i) *I is an f-implication with $f(0) = \infty$, i.e., $I \in \mathbb{I}_{\mathbb{F},\infty}$.*
(ii) *I satisfies* (LI) *w.r.t. $T_{\mathbf{P}}$, I is continuous except $(0,0)$ and $I(x,y) = 1 \iff x = 0$ or $y = 1$.*

6.3.1. Closure of $\mathbb{I}_{\mathbb{F}}$ w.r.t. the $\circledast$-composition

Let $I, J \in \mathbb{I}_{\mathbb{F}}$. From Theorem 6.16, it follows that both I, J satisfy (EP). However, from Remark 3.8, we know that $I \circledast J$ need not satisfy (EP). Thus the $\circledast$-composition of two f-implications need not be an f-implication.

Remark 6.19. (i) If we consider the fuzzy implications $I = I_{\mathbf{YG}} \in \mathbb{I}_{\mathbb{F},\infty}$ and $J = I_{\mathbf{RC}} \in \mathbb{I}_{\mathbb{F},1}$. Their composition $I \circledast J = I_{\mathbf{YG}} \circledast I_{\mathbf{RC}}$ is as given in Remark 4.2(i). Once again, from the same remark, we see that $I_{\mathbf{YG}} \circledast I_{\mathbf{RC}}$ does not satisfy (EP) and hence cannot satisfy (LI) w.r.t. any t-norm T. Clearly, now, $I_{\mathbf{YG}} \circledast I_{\mathbf{RC}} \notin \mathbb{I}_{\mathbb{F}}$.

(ii) Let $I(x,y) = I_{\mathbf{RC}}(x,y) = 1 - x + xy, J(x,y) = 1 - x^2 + x^2y$. Then $N_J(x) = 1 - x^2$, a strict negation. Moreover, from Remark 4.2(iii), J satisfies (LI) w.r.t. $T_{\mathbf{P}}$. Finally from Theorem 6.17, it follows that $J \in \mathbb{I}_{\mathbb{F}}$. Now, it is analogous to see the composition $I \circledast J$ which will be given by $(I \circledast J)(x,y) = 1 - x^3 + x^3y$, also belong to $\mathbb{I}_{\mathbb{F}}$.

To begin with, the following results show that, if the $\circledast$-composition of two f-implications I_{f_1}, I_{f_2} is again an f-implication, then either both $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F},\infty}$ or both $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F},1}$.

Lemma 6.20. *Let $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F}}$ be such that $I_{f_1} \circledast I_{f_2} = I_h \in \mathbb{I}_{\mathbb{F}}$, for some f-generators f_1, f_2, h. If $f_1(0) < \infty$ and $f_2(0) < \infty$ then $h(0) < \infty$.*

Proof. Let I_{f_1}, I_{f_2} be two f-implications such that $I_{f_1} \circledast I_{f_2} = I_h$ is an f-implication. Then f_1, f_2, h satisfy the following equation.

$$f_1^{-1}\left(x \cdot f_1 \circ f_2^{-1}(x \cdot f_2(y))\right) = h^{-1}(x \cdot h(y)), \qquad x, y \in [0,1]. \tag{8}$$

Let $x > 0$ and $y = 0$. Then

$$\begin{aligned} f_2^{-1}(x \cdot f_2(0)) > 0 &\Longrightarrow f_1(f_2^{-1}(x \cdot f_2(0))) < \infty \\ &\Longrightarrow x \cdot f_1 \circ f_2^{-1}(x \cdot f_2(0)) < \infty \\ &\Longrightarrow f_1^{-1}\left(x \cdot f_1 \circ f_2^{-1}(x \cdot f_2(0))\right) > 0 \ , \text{ i.e., L.H.S. of (8)} > 0 \ . \end{aligned}$$

Thus we have R.H.S. of $(8) > 0$ or equivalently, $h^{-1}(x \cdot h(0)) > 0$. Now, if $h(0) = \infty$ then $x \cdot h(0) = \infty$ and hence $h^{-1}(x \cdot h(0)) = 0$, a contradiction. Thus $h(0) < \infty$. This completes the proof. □

Theorem 6.21. *Let $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F}}$ be such that $I_{f_1} \circledast I_{f_2} = I_h \in \mathbb{I}_{\mathbb{F}}$, for some f-generators f_1, f_2, h. Then $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F},\infty} \iff I_h \in \mathbb{I}_{\mathbb{F},\infty}$.*

Proof. Let $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F}}$ be such that $I_{f_1} \circledast I_{f_2} = I_h$ is an f-implication, for some f-generators f_1, f_2, h.

($\Longrightarrow$). Let $f_1(0) = \infty = f_2(0)$. We prove that $h(0) = \infty$. Now, $I_{f_1} \circledast I_{f_2} = I_h$ is the expression given in (8). Once again, let $x > 0$ and $y = 0$ in (8). Then

$$\begin{aligned} \text{L.H.S. of (8)} &= f_1^{-1}(x \cdot f_1 \circ f_2^{-1}(x \cdot f_2(0)) \\ &= f_1^{-1}(x \cdot f_1 \circ f_2^{-1}(x \cdot \infty)) \\ &= f_1^{-1}(x \cdot f_1 \circ f_2^{-1}(\infty)) \\ &= f_1^{-1}(x \cdot f_1(0)) \\ &= f_1^{-1}(x \cdot \infty) = f_1^{-1}(\infty) = 0. \end{aligned}$$

Now R.H.S. of (8) $= h^{-1}(x \cdot h(0)) = 0$ implies that $x \cdot h(0) = h(0)$. Since $x > 0$, either $h(0) = 0$ or $h(0) = \infty$. Now, from monotonicity of h, it follows that $h(0) = \infty$.

($\Longleftarrow$). Let $h(0) = \infty$. We prove that $f_1(0) = \infty = f_2(0)$. Now, R.H.S. of (8) $= h^{-1}(x \cdot h(0)) = 0$ and hence L.H.S. of (8) $= f_1^{-1}(x \cdot f_1 f_2^{-1}(x \cdot f_2(0))) = 0$. i.e., $x \cdot f_1 f_2^{-1}(x \cdot f_2(0)) = f_1(0)$.

Suppose that $f_1(0) < \infty$. Once again, we have the following implications:

$$f_1 \circ f_2^{-1}(x \cdot f_2(0)) < \infty \Longrightarrow f_2^{-1}(x \cdot f_2(0)) > 0 \Longrightarrow f_2(0) < \infty \ .$$

However, from Lemma 6.20, we know that if $f_1(0) < \infty$ and $f_2(0) < \infty$ then $h(0) < \infty$, a contradiction to the fact that $h(0) = \infty$. □

Similar to Theorem 6.21, we have the following result:

Corollary 6.22. *Let $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F}}$ be such that $I_{f_1} \circledast I_{f_2} = I_h \in \mathbb{I}_{\mathbb{F}}$, for some f-generators f_1, f_2, h. Then I_{f_1} or $I_{f_2} \in \mathbb{I}_{\mathbb{F},1} \Longleftrightarrow I_h \in \mathbb{I}_{\mathbb{F},1}$.*

Proof. On the one hand, if $I_h \in \mathbb{I}_{\mathbb{F},1}$ then the fact that one of I_{f_1} or I_{f_2} should belong to $\mathbb{I}_{\mathbb{F},1}$ follows from the contrapositive of Theorem 6.21.

On the other hand, if one of I_{f_1} or $I_{f_2} \in \mathbb{I}_{\mathbb{F},1}$ but $I_{f_1} \circledast I_{f_2} = I_h \in \mathbb{I}_{\mathbb{F}}$, then once again it is clear from Theorem 6.21 that I_h cannot be in $\mathbb{I}_{\mathbb{F},\infty}$ and hence is in $\mathbb{I}_{\mathbb{F}} \setminus \mathbb{I}_{\mathbb{F},\infty} = \mathbb{I}_{\mathbb{F},1}$. □

Note that what the above results show is, if two f-implications compose to give an f-implication, then where their composition would fall. However, it is not true that the composition of any arbitrary pair of f-implications from $\mathbb{I}_{\mathbb{F},\infty}$ or $\mathbb{I}_{\mathbb{F},1}$ will again be an f-implication, as the following example shows.

Example 6.23. (i) Let $I_{f_1}(x, y) = I_{\mathbf{YG}}(x, y)$ and $I_{f_2}(x, y) = \log_2(1 + (2^y - 1)^x)$ whose f-generator is $f_2(x) = -\ln(2^x - 1)$. From Example 3.1.3 (i) and (iv) in [4], it follows that both $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F},\infty}$. Now, $I_{f_1} \circledast I_{f_2}$ is given by

$$(I_{f_1} \circledast I_{f_2})(x, y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0 \ , \\ \log_2(1 + (2^{y^x} - 1)^x), & \text{otherwise} \ . \end{cases}$$

It is easy to check that

$$(I_{f_1} \circledast I_{f_2})(0.3, (I_{f_1} \circledast I_{f_2})(0.5, 0.8)) = 0.2761 \ ,$$

while

$$(I_{f_1} \circledast I_{f_2})(0.5, (I_{f_1} \circledast I_{f_2})(0.3, 0.8)) = 0.2242 \ .$$

This implies that $I_{f_1} \circledast I_{f_2}$ does not satisfy (EP) and hence $I_{f_1} \circledast I_{f_2} \notin \mathbb{I}_{\mathbb{F},\infty}$.

(ii) Let $f_1(x) = 1 - x^2$ and $f_2(x) = \begin{cases} 1 + \frac{x(e-1)}{e}, & \text{if } x \leq e \ , \\ e + \frac{(x-e)e}{e-1}, & \text{if } x \geq e \ . \end{cases}$ where $e = \dfrac{\sqrt{5}-1}{2}$. Clearly, both f_1, f_2 are decreasing functions with $f_1(0) = f_2(0) = 1$ and $f_1(1) = f_2(1) = 0$ (in fact, both f_1, f_2 are fuzzy negations), and hence can be used as f-generators to obtain f-implications, $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F},1}$ using (7), where $f_1^{-1}(x) = \sqrt{1 - x}$ for $x \in [0, 1]$ and $f_2^{-1} = f_2$ on $[0, 1]$. Now,

$$(I_{f_1} \circledast I_{f_2})(x, y) = f_1^{-1}\left(x \cdot f_1 \circ f_2(x \cdot f_2(y))\right) \ , x, y \in [0, 1] \ .$$

Once again, it is easy to check that

$$(I_{f_1} \circledast I_{f_2})(0.6, (I_{f_1} \circledast I_{f_2})(0.7, 0)) = 0.8904 \ ,$$

while

$$(I_{f_1} \circledast I_{f_2})(0.7, (I_{f_1} \circledast I_{f_2})(0.6, 0)) = 0.9036 \ .$$

This implies that $I_{f_1} \circledast I_{f_2}$ does not satisfy (EP) and hence $I_{f_1} \circledast I_{f_2} \notin \mathbb{I}_{\mathbb{F},1}$.

In the following, we investigate the conditions under which the $\circledast$-composition of two f-implications will be an f-implication. Towards this end, we present a few small but important results.

Lemma 6.24 ([4], Lemma 3.1.8)**.** *Let I_f be an f-implication. Then $I_f(\cdot, y)$ is one-one for all $y \in (0,1)$.*

Lemma 6.25. *Let I_f be an f-implication and let $x > 0$. Then*

(i) *$I_f(x,\cdot)$ is one-one for all $x \in (0,1]$.*
(ii) *Further, if $f(0) = \infty$ then $I_f(x,\cdot)$ is an increasing bijection on $[0,1]$.*

Proof. Let $I = I_f$ be an f-implication and let $x > 0$.

(i) Let $y_1, y_2 \in [0,1]$ and $I_f(x,y_1) = I_f(x,y_2)$. Then

$$\begin{aligned} f^{-1}(x \cdot f(y_1)) = f^{-1}(x \cdot f(y_2)) &\Longrightarrow x \cdot f(y_1) = x \cdot f(y_2) \\ &\Longrightarrow f(y_1) = f(y_2) \\ &\Longrightarrow y_1 = y_2 \ . \end{aligned}$$

Thus $I_f(x,\cdot)$ is one-one for all $x \in (0,1]$.

(ii) Let $f(0) = \infty$. Now, $I_f(x,0) = f^{-1}(x \cdot f(0)) = f^{-1}(x \cdot \infty) = f^{-1}(\infty) = 0$ and $I_f(x,1) = 1$.

Since f is continuous, the range of $I_f(x,\cdot)$ is the entire $[0,1]$ and hence $I_f(x,\cdot)$ is an increasing bijection on $[0,1]$. □

Remark 6.26. In Lemma 6.25(ii), if $f(0) < \infty$ then $I_f(x,\cdot)$ need not be a bijection on $[0,1]$ for all $x > 0$. For example, let $f(x) = 1 - x$, which is the f-generator of the Reichenbach implication $I_{\mathbf{RC}} \in \mathbb{I}_{\mathbb{F},1}$. Clearly $f(0) = 1 < \infty$. When $x = 0.2$, $I_f(0.2, y) = 0.8 + 0.2y$ for all $y \in [0,1]$. Here the range of $I_f(0.2,\cdot)$ is equal to $[0.8, 1]$.

Theorem 6.27. *Let $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F},\infty}$. Then the following statements are equivalent.*

(i) *$I_{f_1} \circledast I_{f_2} \in \mathbb{I}_{\mathbb{F},\infty}$.*
(ii) *$I_{f_1} \circledast I_{f_2}$ satisfies* (LI) *w.r.t. $T_{\mathbf{P}}$.*
(iii) *I_{f_1}, I_{f_2} are mutually exchangeable, i.e., the pair (I_{f_1}, I_{f_2}) satisfies* (ME)*.*

Proof. (i) $\Longrightarrow$ (ii): Let $I_{f_1} \circledast I_{f_2} \in \mathbb{I}_{\mathbb{F},\infty}$. Then from the characterisation result, Theorem 6.18, of f-implications with $f(0) = \infty$, it follows that $I_{f_1} \circledast I_{f_2}$ satisfies (LI) w.r.t. $T_{\mathbf{P}}$.

(ii) $\Longrightarrow$ (iii): Now assume that $I_{f_1} \circledast I_{f_2}$ satisfies (LI) w.r.t. $T_{\mathbf{P}}$. From Lemma 6.25(i), it follows that I_{f_1} is one-one and from Lemma 6.25(ii) that I_{f_2} is a bijection on $[0,1]$. Now, from Theorem 4.5 it follows immediately that I_{f_1}, I_{f_2} are mutually exchangeable.

(iii) $\Longrightarrow$ (i): Let I_{f_1}, I_{f_2} be mutually exchangeable. Then from Theorem 4.5, $I_{f_1} \circledast I_{f_2}$ satisfies (LI) w.r.t. $T_{\mathbf{P}}$. Since I_{f_1}, I_{f_2} are continuous except at $(0,0)$, $I_{f_1} \circledast I_{f_2}$is also continuous except at $(0,0)$. Moreover

$$\begin{aligned} (I_{f_1} \circledast I_{f_2})(x,y) = 1 &\Longleftrightarrow I_{f_1}(x, I_{f_2}(x,y)) = 1 \\ &\Longleftrightarrow x = 0 \text{ or } I_{f_2}(x,y) = 1 \\ &\Longleftrightarrow x = 0 \text{ or } x = 0 \text{ or } y = 1 \\ &\Longleftrightarrow x = 0 \text{ or } y = 1 \ . \end{aligned}$$

Now, from Theorem 6.18 we see that $I_{f_1} \circledast I_{f_2} \in \mathbb{I}_{\mathbb{F},\infty}$. □

In the case $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F},1}$, then we have only some sufficient conditions as the following results show.

Lemma 6.28. *If $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F},1}$, then $N_{I_{f_1} \circledast I_{f_2}}$ is a strict negation.*

Proof. Let $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F},1}$. Then from Theorem 6.17 we know that $N_{I_{f_1}}, N_{I_{f_2}}$ are strict negations. Now,

$$(N_{I_{f_1} \circledast I_{f_2}})(x) = (I_{f_1} \circledast I_{f_2})(x, 0) = f_1^{-1}(x \cdot f_1 f_2^{-1}(x \cdot f_2(0))) = f_1^{-1}(x \cdot f_1(N_{I_{f_2}}(x))).$$

Clearly, $N_{I_{f_1} \circledast I_{f_2}}$ being the composition of continuous functions is continuous. To show that it is a strict negation, it suffices to show that it is strictly decreasing. From the antitonicity of f_1, f_2 we have the following implications:

$$\begin{aligned} x_1 < x_2 &\Longrightarrow N_{I_{f_2}}(x_1) > N_{I_{f_2}}(x_2) \\ &\Longrightarrow f_1(N_{I_{f_2}}(x_1)) < f_1(N_{I_{f_2}}(x_2)) \\ &\Longrightarrow x_1 \cdot f_1(N_{I_{f_2}}(x_1)) < x_1 \cdot f_1(N_{I_{f_2}}(x_2)) < x_2 \cdot f_1(N_{I_{f_2}}(x_2)) \\ &\Longrightarrow f_1^{-1}(x_1 \cdot f_1(N_{I_{f_2}}(x_1))) > f_1^{-1}(x_2 \cdot f_1(N_{I_{f_2}}(x_2))) \\ &\text{i.e., } (N_{I_{f_1} \circledast I_{f_2}})(x_1) > (N_{I_{f_1} \circledast I_{f_2}})(x_2)\ . \end{aligned}$$

This completes the proof. □

Theorem 6.29. *Let $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F},1}$. If I_{f_1}, I_{f_2} are mutually exchangeable then $I_{f_1} \circledast I_{f_2} \in \mathbb{I}_{\mathbb{F},1}$.*

Proof. Let $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F},1}$. Then from Theorem 6.17, I_{f_1}, I_{f_2} satisfy (LI) w.r.t. $T_{\mathbf{P}}$ and $N_{I_{f_1}}, N_{I_{f_2}}$ are strict negations. Now from Lemma 4.3, if I_{f_1}, I_{f_2} are mutually exchangeable then $I_{f_1} \circledast I_{f_2}$ satisfies (LI) w.r.t. $T_{\mathbf{P}}$. Moreover from Lemma 6.28, it follows directly that $N_{I_{f_1} \circledast I_{f_2}}$ is a strict negation. Again from Theorem 6.17 it follows that $I_{f_1} \circledast I_{f_2} \in \mathbb{I}_{\mathbb{F},1}$. □

Note, however, it is not clear whether the converse of Lemma 6.29 is true, i.e., whether the mutual exchangeability of $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F},1}$ is also necessary for $I_{f_1} \circledast I_{f_2} \in \mathbb{I}_{\mathbb{F},1}$ and hence we have only the following result, the proof of which follows from Theorem 6.17 and Lemma 6.29.

Corollary 6.30. *Let $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F},1}$. Let us consider the following statements:*

(i) $I_{f_1} \circledast I_{f_2} \in \mathbb{I}_{\mathbb{F},1}$.
(ii) $I_{f_1} \circledast I_{f_2}$ *satisfies* (LI) *w.r.t.* $T_{\mathbf{P}}$.
(iii) I_{f_1}, I_{f_2} *are mutually exchangeable.*

Then, the following implications are true: (i) $\Longleftrightarrow$ (ii) and (iii) $\Longrightarrow$ (i).

6.3.2. Powers of f-implication w.r.t. $\circledast$.

Theorem 6.31. *If $I_f \in \mathbb{I}_{\mathbb{F},\infty}$ then $(I_f)_{\circledast}^{[n]} \in \mathbb{I}_{\mathbb{F},\infty}$ for all $n \in \mathbb{N}$.*

Proof. Let $I_f \in \mathbb{I}_{\mathbb{F},\infty}$. The proof is by induction on n.

Note that $(I_f)_{\circledast}^{[2]} = (I_f \circledast I_f)(x, y) = I_f(x, I_f(x, y))$, since I_f satisfies (EP), the (repeated) pair (I_f, I_f) satisfies (ME) and from Theorem 6.27(iii) we see that $I_f \circledast I_f \in \mathbb{I}_{\mathbb{F},\infty}$.

Now, let us assume that $(I_f)_{\circledast}^{[k-1]} \in \mathbb{I}_{\mathbb{F},\infty}$. Since $(I_f)_{\circledast}^{[k-1]} \in \mathbb{I}_{\mathbb{F},\infty}$, $(I_f)_{\circledast}^{[k-1]}$ satisfies (LI) w.r.t. $T_{\mathbf{P}}$ and is continuous except at $(0, 0)$ and $(I_f)_{\circledast}^{[k-1]}(x, y) = 1 \Longleftrightarrow x = 0$ or $y = 1$. Since $I_f, (I_f)_{\circledast}^{[k-1]}$ satisfy (EP) from Theorem 6.16 and then from Lemma 5.4, we see that $I_f, (I_f)_{\circledast}^{[k-1]}$ satisfy (ME). Now, from Theorem 6.27, it follows that $(I_f)_{\circledast}^{[k]} = I_f \circledast (I_f)_{\circledast}^{[k-1]} \in \mathbb{I}_{\mathbb{F},\infty}$. □

The proof of the following result is similar to that of Theorem 6.31.

Theorem 6.32. *If $I_f \in \mathbb{I}_{\mathbb{F},1}$ then $(I_f)_{\circledast}^{[n]} \in \mathbb{I}_{\mathbb{F},1}$ for all $n \in \mathbb{N}$.*

Corollary 6.33. *If $I_f \in \mathbb{I}_{\mathbb{F}}$ then $(I_f)_{\circledast}^{[n]} \in \mathbb{I}_{\mathbb{F}}$ for all $n \in \mathbb{N}$.*

Lemma 6.34. *Let $I = I_f$ be an f-implication. Then $I_{\circledast}^{[n]}(x, y) = I(x^n, y)$ for all $x, y \in [0, 1], n \in \mathbb{N}$.*

Proof. Let $I = I_f$ be an f-implication. Since we know I satisfies (LI) w.r.t. the product t-norm $T_{\mathbf{P}}(x, y) = xy$, the result follows immediately from Theorem 5.3.

□

Lemma 6.35. *Let $I \in \mathbb{I}_{\mathbb{F}}$. Then $\mathcal{O}(I) = \infty$.*

Proof. Suppose for some $m \in \mathbb{N}$, $I_{\circledast}^{[m]} = I_{\circledast}^{[m+1]}$. Let $x, y \in (0, 1)$ be arbitrarily chosen. Then

$$\begin{aligned} I_f(x^m, y) = I_f(x^{m+1}, y) &\Longrightarrow f^{-1}\left(x^m \cdot f(y)\right) = f^{-1}(x^{m+1} \cdot f(y)) \\ &\Longrightarrow x^m \cdot f(y) = x^{m+1} \cdot f(y) \\ &\Longrightarrow x = 0 \text{ or } y = 1 \text{ or } x = 1 \ , \end{aligned}$$

which is a contradiction. Thus $\mathcal{O}(I) = \infty$. □

Corollary 6.36. *No f-implication satisfies the idempotent equation*

$$I(x, I(x, y)) = I(x, y), \qquad x, y \in [0, 1]. \tag{9}$$

6.4. g-implications and the $\circledast$-composition

In this section, we discuss the closure of the $\circledast$-composition w.r.t. the second family of fuzzy implications proposed by Yager, viz., the g-implications. The results and proofs in this section largely mirror those that were given in the earlier section that dealt with f-implications (Section 6.3) and hence only a sketch of the proof is given wherever necessary.

Definition 6.37 ([4], Definition 3.2.1). Let $g : [0, 1] \longrightarrow [0, \infty]$ be a strictly increasing and continuous function with $g(0) = 0$. The function $I : [0, 1]^2 \to [0, 1]$ defined by

$$I(x, y) = g^{(-1)}\left(\frac{1}{x} \cdot g(y)\right), \qquad x, y \in [0, 1] \ ,$$

with the understanding $\frac{1}{0} = \infty$ and $\infty \cdot 0 = \infty$, is called a *g-generated implication*, where the function $g^{(-1)}$ is the pseudo inverse of g given by

$$g^{(-1)}(x) = \begin{cases} g^{-1}(x), & \text{if } x \in [0,\ g(1)] \ , \\ 1, & \text{if } x \in [g(1),\ \infty] \ . \end{cases}$$

The family of all g-generated implications is denoted by $\mathbb{I}_{\mathbb{G}}$. Once again, it can be shown that it is sufficient to consider two types of g-generators, viz., those with $g(1) = \infty$ and $g(1) = 1$. Let us denote by

- $\mathbb{I}_{\mathbb{G},\infty}$ – the family of all g-generated implications such that $g(1) = \infty$.
- $\mathbb{I}_{\mathbb{G},1}$ – the family of all g-generated implications such that $g(1) < \infty$.

For more details, please see Chapter 3 of [4] and [38].

Proposition 6.38 ([4], Proposition 4.4.1)**.** *The following equalities are true:*

$$\begin{aligned} \mathbb{I}_{\mathbb{F},1} \cap \mathbb{I}_{\mathbb{G}} &= \emptyset \ , \\ \mathbb{I}_{\mathbb{F}} \cap \mathbb{I}_{\mathbb{G},1} &= \emptyset \ , \\ \mathbb{I}_{\mathbb{F},\infty} &= \mathbb{I}_{\mathbb{G},\infty} \ . \end{aligned}$$

Lemma 6.39. *Let $I_{g_1}, I_{g_2} \in \mathbb{I}_{\mathbb{G}}$ be such that $I_{g_1} \circledast I_{g_2} = I_h \in \mathbb{I}_{\mathbb{G}}$. Then $I_{g_1}, I_{g_2} \in \mathbb{I}_{\mathbb{G},\infty} \Longleftrightarrow I_h \in \mathbb{I}_{\mathbb{G},\infty}$.*

Proof. Proof follows from Proposition 6.38 and Theorem 6.21. □

Corollary 6.40. *Let $I_{g_1}, I_{g_2} \in \mathbb{I}_{\mathbb{G}}$ be such that $I_{g_1} \circledast I_{g_2} = I_h \in \mathbb{I}_{\mathbb{G}}$. Then I_{g_1} or $I_{g_2} \in \mathbb{I}_{\mathbb{G},1} \iff I_h \in \mathbb{I}_{\mathbb{G},1}$.*

Theorem 6.41. *Let $I_{g_1}, I_{g_2} \in \mathbb{I}_{\mathbb{G},\infty}$. Then the following are equivalent:*

(i) $I_{g_1} \circledast I_{g_2} \in \mathbb{I}_{\mathbb{G},\infty}$.
(ii) $I_{g_1} \circledast I_{g_2}$ *satisfies* (LI) *w.r.t.* $T_{\mathbf{P}}$.
(iii) I_{g_1}, I_{g_2} *are mutually exchangeable.*

Proof. Proof follows from Proposition 6.38 and Theorem 6.27. □

Theorem 6.42. *Let $I_{g_1}, I_{g_2} \in \mathbb{I}_{\mathbb{G},1}$. If I_{g_1}, I_{g_2} are mutually exchangeable then $I_{g_1} \circledast I_{g_2} \in \mathbb{I}_{\mathbb{G},1}$.*

6.4.1. Powers of g-implication w.r.t. $\circledast$.

The proofs of the following results are anologous to those in Section 6.3.2.

Theorem 6.43. *If $I_g \in \mathbb{I}_{\mathbb{G},\infty}$ then $(I_g)_{\circledast}^{[n]} \in \mathbb{I}_{\mathbb{G},\infty}$ for all $n \in \mathbb{N}$.*

Theorem 6.44. *If $I_g \in \mathbb{I}_{\mathbb{G},1}$ then $(I_g)_{\circledast}^{[n]} \in \mathbb{I}_{\mathbb{G},1}$ for all $n \in \mathbb{N}$.*

Corollary 6.45. *If $I_g \in \mathbb{I}_{\mathbb{G}}$ then $(I_g)_{\circledast}^{[n]} \in \mathbb{I}_{\mathbb{G}}$ for all $n \in \mathbb{N}$.*

Lemma 6.46. *Let $I = I_g$ be an g-implication. Then $I_{\circledast}^{[n]}(x, y) = I(x^n, y)$ for all $x, y \in [0, 1], n \in \mathbb{N}$.*

Lemma 6.47. *Let $I \in \mathbb{I}_{\mathbb{G}}$. Then $\mathcal{O}(I) = \infty$.*

Corollary 6.48. *No g-implication satisfies the idempotent equation* (9).

7. Concluding Remarks

Recently, in [50], the authors had proposed a novel generative method to obtain a fuzzy implication from a given pair of fuzzy implications. The $\circledast$-composition proposed in [50] not only gave rise to new fuzzy implications but also imposed a monoid structure on $\mathbb{I}$, the set of all fuzzy implications. The algebraic aspects of the $\circledast$-composition were explored in [53] and based on the results some hitherto unknown representations of some families of fuzzy implications were obtained.

In this work, we have investigated the analytical aspects of the $\circledast$-composition. We have shown that the $\circledast$-composition carries over most of the properties of the underlying fuzzy implications. Further, we have investigated the preservation of the law of importation (LI) and contraposition w.r.t. a strong negation N under the $\circledast$-composition. The choice of these functional equations were not only dictated by their centrality but also due to their relevance in further exploration, as shown by the dependence of many of the results in Sections 5 and 6 on them. In future works, we intend to study other functional equations involving fuzzy implications like distributivity, T-conditionality, etc.

This study also has necessitated the introduction of the concept of mutual exchangeability (ME) between a pair of fuzzy implications, which plays a central role in our investigation. In fact, (ME) can be seen as one generalisation of the exchange principle (EP) and hence deserves further exploration.

It is also heartening to note that due to the associativity of the $\circledast$-composition, one can define powers of fuzzy implications w.r.t. the $\circledast$-composition. Exploring this, we have shown that it is possible to obtain infinitely many new fuzzy implications from a single given fuzzy implication by self-composition with the $\circledast$-composition, often carrying all the desirable properties. Towards this end, once again, a new concept of order of a fuzzy implication w.r.t. the $\circledast$-composition was proposed.

Finally, since our proposed method falls under the third category of generating fuzzy implications (under the broad classification espoused in Section 1), we have also studied its effect on fuzzy implications obtained from the other two categories. Specifically, we considered the families of (S, N)- and R-implications (Sections 6.1 and 6.2) which are representative of the first type of generation methods and the Yager's families of f- and g-implications (Sections 6.3 and 6.4) which are representative of the second type of generation methods. It was shown that these families are not completely closed w.r.t. the $\circledast$-composition.

We believe that determining the closures of these families, though a highly non-trivial task, can be immensely beneficial in obtaining newer perspectives on the set of fuzzy implications, as was similarly shown in [53] with group actions on the monoid $(\mathbb{I}, \circledast)$ that gave rise to hitherto unknown representations for the Yager's families of f- and g-implications. For instance, fuzzy implications that fall within the closure but outside of the families of, say (S, N)- or R-implications, can be seen as a $\circledast$-composition of two (S, N)- or R-implications for an appropriate pair of fuzzy implications.

Note that while discussing the powers of an $I \in \mathbb{I}$ w.r.t. the $\circledast$-composition, one can also discuss the periodicity of I w.r.t. the $\circledast$-composition, i.e., the question of when $I_{\circledast}^{[n]} = I$ for an $n > 2$ and not for any $n_0 < n$. Of course, if $\mathcal{O}(I) = 1$ then the periodicity of I is 1 and, further, I is idempotent w.r.t. the $\circledast$-composition. We intend to take up exploration along the above lines in the near future.

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