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Pseudo-Monometrics from Fuzzy Implications

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Abstract

Several works have proposed the construction of distance functions using fuzzy logic connectives to proffer further applications of the corresponding connectives. In these works, the authors define a distance function using t-norms, t-conorms, copulas, or quasi-copulas, all of which are either associative, commutative or monotonic fuzzy logic connectives. In this work, we define a distance function, denoted d_I, from a non-associative, non-commutative, and mixed-monotonic fuzzy logic connective, viz., a fuzzy implication I, and study the above distance function along two aspects. Firstly, we investigate the necessary and sufficient conditions for d_I to be a metric, wherein the role played by a transitivity type functional inequality involving the considered fuzzy implication and the Łukasiewicz t-conorm is highlighted. In the recent past, monometrics w.r.t. a ternary relation, called the betweenness relation, have garnered a lot of attention for their important role in decision making and penalty-based data aggregation. One of the major challenges herein is that of obtaining monometrics on a given betweenness set. Our second contribution in this work is in establishing the existence of pseudo-monometrics using d_I, from whence it appears that fuzzy implications are a natural choice for obtaining pseudo-monometrics on a given betweenness set.

Keywords: Fuzzy Implication, (S, I)-transitivity, Pseudo-monometric, Betweenness Relation.

1. Introduction

The idea of constructing a distance function from fuzzy logic connectives such as t-norms, and its dual t-conorms, was originally introduced by Alsina in [1], wherein it was shown that this distance function turns out to be a metric if the t-norm is a copula. The converse, however, need not be true, i.e., t-norms that are not copulas can give rise to a metric, for instance, continuous non-strict Archimedean t-norms (see [2]).

In [3], metrics are constructed for the more general case of any t-norm and t-conorm, and a characterization of t-norms that define metrics is given in case the t-norms have the same zero region as the Łukasiewicz t-norm. There have been other works in literature that show the construction of a family of distance functions on the unit interval, induced from quasi-copulas [4] and symmetric difference functions [5] on $[0, 1]$. In [5], a complete characterisation of the triple (T, S, N) of t-norm, t-conorm, and fuzzy negation that define symmetric difference functions, which are metrics, is given.

1.1. Motivation for and Contributions of this work:

Note that all the above referenced works have considered only associative, monotonic, or commutative fuzzy logic operations on $[0, 1]$ to define distance functions. This gives us the first of the twin motivations, viz., to construct distance functions from fuzzy implications on $[0, 1]$, which are non-associative, mixed-monotonic and non-commutative.

Alsina, in his work [4] dealing with the construction of metrics obtained from quasi-copulas, has also shown that various concepts of dependencies between random variables can be expressed in terms of the

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proposed metrics. This leads us to the second of our motivations in this work, viz., to explore the context or scenarios where such constructed distance functions are both natural and of significant utility.

Keeping the above goals, we firstly propose a way of constructing distance functions d_I from fuzzy implications I on $[0,1]$ [1]. However, these distance functions are not always metrics. We explore a necessary and sufficient condition for the above distance function to be a metric. A transitivity type functional inequality involving the considered fuzzy implication and the Łukasiewicz t-conorm $S_{\mathbf{LK}}$ plays an important role herein. Investigating this functional inequality for the main families of fuzzy implications illustrates the plethora of examples that can give rise to metrics under the proposed construction.

Recently, monometrics w.r.t. a ternary relation B, called the betweenness relation, have garnered a lot of attention for their important role in decision making and penalty-based data aggregation. One of the major challenges herein is that of obtaining monometrics on a given betweenness set. Our second contribution in this work is in establishing the existence of (pseudo-)monometrics using d_I on a class of betweenness sets. We also present a complete characterisation of such a class of betweenness sets obtained from an underlying partial order.

Our work highlights both the proposed distance function and fuzzy implications as natural choices in the setting of pseudo-monometrics over betweenness sets.

2. Distance Function from Fuzzy Implications

In this section, we begin by recalling the definitions of fuzzy implication and distance function. We then construct a distance function d_I from fuzzy implications on $[0,1]$. We also give a sufficient and necessary condition for d_I to be a metric and show the important role played by the (S,I)-transitivity (SIT) in making it a metric.

Definition 1. *A function $I : [0,1]^2 \to [0,1]$ is said to be **fuzzy implication** if the following properties hold for any $x_1, x_2, y_1, y_2, x, y \in \mathcal{X}$:*

(i) $x_1 \le x_2 \implies I(x_2, y) \le I(x_1, y)$, *i.e.,* $I(\cdot, y)$ *is decreasing.*
(ii) $y_1 \le y_2 \implies I(x, y_1) \le I(x, y_2)$, *i.e.,* $I(x, \cdot)$ *is increasing.*
(iii) $I(0,0) = 1$, $I(1,1) = 1$, *and* $I(1,0) = 0$.

We shall denote the set of all fuzzy implications by $\mathbb{I}$.

A few basic examples of fuzzy implications can be seen in Table 1.

Definition 2. *A symmetric function $d : \mathcal{X} \times \mathcal{X} \to [0, \infty[$ is called a **distance function** on $\mathcal{X}$ if it satisfies the following property for any $x, y \in \mathcal{X}$:*

$$x = y \Longrightarrow d(x,y) = 0 \ . \tag{P1}$$

*Further, it is called a **metric** if the converse of* (P1) *holds, and it also satisfies the triangle inequality, i.e., for any $x, y, z \in \mathcal{X}$,*

$$d(x,z) \le d(x,y) + d(y,z) \ . \tag{P2}$$

Definition 3. *Let $I \in \mathbb{I}$ and define $d_I : [0,1] \times [0,1] \to [0,1]$ as*

$$d_I(x,y) = \begin{cases} 0, & \text{if } x = y \ , \\ I(\min(x,y), \max(x,y)), & \text{otherwise} \ . \end{cases}$$

Theorem 1. (i) *d_I is a distance function on $[0,1]$.*

[1] Note that a preliminary version of this work, containing our then nascent explorations, was presented at the 9th PREMI (Kolkata, India, 15-18 September, 2021) [6] and 19th IPMU (Milan, Italy, 11-15 July, 2022) [7].

(ii) d_I *satisfies the converse of* (P1) *iff* I *satisfies the following condition*[2]*:*

$$I(x,y) > 0, \text{ whenever } x < y, \quad x,y \in [0,1]. \tag{1}$$

Proof. (i) Since the functions max and min are symmetric, d_I is also symmetric, and clearly, $d_I(x,x) = 0$. Hence, d_I is a distance function.

(ii) I satisfies (1) $\iff I(\min(x,y), \max(x,y)) > 0$ when $x \neq y \iff d_I(x,y) > 0$ when $x \neq y \iff d_I$ satisfies the converse of (P1).

□

Example 1. *Consider the fuzzy implication* $I_{\mathbf{RC}}(x,y) = 1 - x + xy$. *Then*

$$d_{I_{\mathbf{RC}}}(x,y) = \begin{cases} 0, & \text{if } x = y \ , \\ 1 - \min(x,y) + xy, & \text{otherwise} \ . \end{cases}$$

Clearly, $d_{I_{\mathbf{RC}}}$ *is a distance function and satisfies the converse of* (P1). *Further, it can be verified that* $d_{I_{\mathbf{RC}}}$ *also satisfies the triangle inequality and hence, it is a metric.*

However, every fuzzy implication satisfying (1) need not give rise to a metric, for instance, see the example below.

Example 2. *Consider the fuzzy implication* I *defined as follows:*

$$I(x,y) = \begin{cases} 1, & \text{if } x = 0 \ , \\ \min\left(\dfrac{1+4y}{3}, 1\right), & \text{if } x < 0.11 \ , \\ y, & \text{otherwise} \ . \end{cases}$$

Then

$$d_I(x,y) = \begin{cases} 0, & \text{if } x = y \ , \\ 1, & \text{if } x = 0 \text{ or } y = 0 \ , \\ \min\left(\dfrac{1+4\max(x,y)}{3}, 1\right), & \text{if } \min(x,y) < 0.11 \ , \\ \max(x,y), & \text{otherwise} \ . \end{cases}$$

Note that d_I *doesn't satisfy the triangle inequality since*

$$d_I(0.1, 0.11) + d_I(0.11, 0.45) = 0.48 + 0.45 = 0.93 \not\geq 0.933 = d_I(0.1, 0.45) \ .$$

2.1. When is d_I *a metric?*

In this section, we discuss the necessary and sufficient condition for d_I to be a metric. The following functional inequality will play an important role in the characterisation.

Definition 4. *Given a t-conorm* S *and a fuzzy implication* I *on* $[0,1]$*, the pair* (S,I) *is said to satisfy* (S,I)***-transitivity*** *if*

$$S(I(x,y), I(y,z)) \geq I(x,z) \ , \text{ for all } x,y,z \in [0,1] \ . \tag{SIT}$$

Remark 1. *In the literature,* (S,I)*-transitivity has already been discussed in different contexts.*

[2]We would like to mention that throughout the paper, we will only consider fuzzy implications that satisfy (1) unless stated otherwise.

(i) *In [8, 9],* (SIT) *emerges as a generalisation of the triangle inequality to S-triangle inequality to show that the complement of T-equivalence fuzzy relation comes out to be an S-pseudometric i.e., a distance function that satisfies (S, d)-transitivity.*

(ii) *The functional inequality* (SIT) *appears as (S, R)-transitivity which has also been used as the dual concept of T-transitivity, see [10, 11], where R is a binary fuzzy relation. It captures the following negative transitivity - if x is not related to y and y is not related to z under a relation R, then we insist that x should also not be related to z under R, i.e., $R(x, y) = 0$ & $R(y, z) = 0 \Longrightarrow R(x, z) = 0$. In fact, this property is also named as the negative S-transitivity in the literature, see Definition 2.13 in [11]. Note that a fuzzy implication can be viewed as a fuzzy relation on* $[0, 1]$ *since it is a mapping from the unit square to the unit interval.*

Definition 5. *Let $\mathbb{A} \subset [0, 1]^3$ be given as $\mathbb{A} = \{(x, y, z) \in [0, 1]^3 \mid x < y < z\}$.*

Lemma 1. *Let S be a t-conorm and I be a fuzzy implication on* $[0, 1]$. *Then the pair (S, I) satisfies the (S, I)-transitivity for every triplet $(x, y, z) \in \mathbb{A}^c$, where $\mathbb{A}^c$ is the complement of $\mathbb{A}$ defined in Definition 5.*

Proof. Let $(x, y, z) \in \mathbb{A}^c$. Then either $x \geq y$ or $y \geq z$.

(i) **Case-1**: $x \geq y$
Since I is decreasing in the first variable, we have $I(x, z) \leq I(y, z) \leq \max(I(x, y), I(y, z))$. Hence $I(x, z) \leq S(I(x, y), I(y, z))$ since $S_{\mathbf{M}}(x, y) = \max(x, y)$ is the smallest t-conorm.

(ii) **Case-2**: $y \geq z$
Since I is increasing in the second variable, we have $I(x, z) \leq I(x, y) \leq \max(I(x, y), I(y, z))$. As above, we see that $I(x, z) \leq S(I(x, y), I(y, z))$.

□

From the above lemma, it is clear that for a given t-conorm S and a fuzzy implication I, proving (SIT) on $[0, 1]^3$ is equivalent to proving (SIT) on $\mathbb{A}$. We shall make use of this fact in the upcoming results by showing the satisfaction of (SIT) only on $\mathbb{A}$.

We now discuss the necessary and sufficient condition for d_I to be a metric, highlighting the importance of studying the functional inequality (SIT).

Theorem 2. *d_I is a metric iff I satisfies $(S_{\mathbf{LK}}, I)$-transitivity where $S_{\mathbf{LK}}(x, y) = \min(x + y, 1)$.*

Proof. ($\Longrightarrow$) Suppose d_I is a metric. Consider a triplet $(x, y, z) \in \mathbb{A}$ i.e., $x < y < z$, then $I(x, y) + I(y, z) = d_I(x, y) + d_I(y, z) \geq d_I(x, z) = I(x, z)$. Hence,

$$\min(1, I(x, y) + I(y, z)) \geq I(x, z) \implies S_{\mathbf{LK}}(I(x, y), I(y, z)) \geq I(x, z).$$

Hence, I satisfies $(S_{\mathbf{LK}}, I)$-transitivity.
($\Longleftarrow$) Since d_I is a distance function and we consider only $I \in \mathbb{I}$ satisfying (1), it suffices to show that d_I satisfies the triangle inequality. Suppose I satisfies $(S_{\mathbf{LK}}, I)$-transitivity.

(i) **Case-1**: $(x, y, z) \in \mathbb{A}^c \setminus \{(x, y, z) \mid z < y < x\}$
Triangle Inequality follows from the definition of I and d_I.

(ii) **Case-2**: $z < y < x$
It follows from $(S_{\mathbf{LK}}, I)$-transitivity, that $d_I(x, y) + d_I(y, z) = I(y, x) + I(z, y) \geq I(z, x) = d_I(x, z)$.

(iii) **Case-3**: $x < y < z$
It follows from $(S_{\mathbf{LK}}, I)$-transitivity, that $d_I(x, y) + d_I(y, z) = I(x, y) + I(y, z) \geq I(x, z) = d_I(x, z)$.

□

Corollary 1. *If I satisfies (S, I)-transitivity w.r.t. any $S \leq S_{\mathbf{LK}}$, then d_I yields a metric.*

Implication	**Type**	**Formula** $I(x,y)$	**Metric** $d_I(x,y)$
Łukasiewicz	(S,N)-	$I_{\mathbf{LK}} : \min(1, 1-x+y)$	$\begin{cases} 0, & \text{if } x = y \ , \\ 1, & \text{if } x \neq y \ . \end{cases}$
Gödel	R-	$I_{\mathbf{GD}} : \begin{cases} 1, & \text{if } x \leq y \ , \\ y, & \text{if } x > y \ . \end{cases}$	$\begin{cases} 0, & \text{if } x = y \ , \\ 1, & \text{if } x \neq y \ . \end{cases}$
Reichenbach	(S,N)-	$I_{\mathbf{RC}} : 1 - x + xy$	$\begin{cases} 0, & \text{if } x = y \ , \\ 1 - \min(x,y) + xy, & \text{if } x \neq y \ . \end{cases}$
Kleene-Dienes	(S,N)-	$I_{\mathbf{KD}} : \max(1-x, y)$	$\begin{cases} 0, & \text{if } x = y \ , \\ \max(1-x,y), & \text{if } x < y \ , \\ \max(1-y,x), & \text{if } x > y \ . \end{cases}$
Goguen	R-	$I_{\mathbf{GG}} : \begin{cases} 1, & \text{if } x \leq y \ , \\ y, & \text{if } x > y \ . \end{cases}$	$\begin{cases} 0, & \text{if } x = y \ , \\ 1, & \text{if } x \neq y \ . \end{cases}$
Rescher	R-	$I_{\mathbf{RS}} : \begin{cases} 1, & \text{if } x \leq y \ , \\ 0, & \text{if } x > y \ . \end{cases}$	$\begin{cases} 0, & \text{if } x = y \ , \\ 1, & \text{if } x \neq y \ . \end{cases}$
Yager	Yager-	$I_{\mathbf{YG}} : \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0 \ , \\ y^x, & \text{if } x > 0 \text{ and } y > 0 \ . \end{cases}$	$\begin{cases} 0, & \text{if } x = y \ , \\ y^x, & \text{if } x < y \ , \\ x^y, & \text{if } x > y \ . \end{cases}$
Weber	(S,N)-	$I_{\mathbf{WB}} : \begin{cases} 1, & \text{if } x < 1 \ , \\ y, & \text{if } x \geq 1 \ . \end{cases}$	$\begin{cases} 0, & \text{if } x = y \ , \\ 1, & \text{if } x \neq y \ . \end{cases}$
Fodor	R-	$I_{\mathbf{FD}} : \begin{cases} 1, & \text{if } x \leq y \ , \\ \max(1-x,y), & \text{if } x > y \ . \end{cases}$	$\begin{cases} 0, & \text{if } x = y \ , \\ 1, & \text{if } x \neq y \ . \end{cases}$

Table 1: Examples of metrics obtained from some fuzzy implications

Example 3. *In Table 1, we present the basic fuzzy implications as given in [12]. Note that they satisfy* $(S_{\mathbf{LK}}, I)$*-transitivity and thus yield metrics through* d_I*.*

By Corollary 2, given later, the fuzzy implications $I_{\mathbf{LK}}, I_{\mathbf{GD}}, I_{\mathbf{GG}}, I_{\mathbf{RS}}, I_{\mathbf{WB}}, I_{\mathbf{FD}}$ *satisfy the* $(S_{\mathbf{LK}}, I)$*-transitivity but only yield the discrete metric.*

It is easy to verify that Yager implication $I_{\mathbf{YG}}$ *also satisfies* $(S_{\mathbf{LK}}, I)$*-transitivity by Lemma 2 given below, since* $I_{\mathbf{YG}}(x,y) \geq 0.5$ *whenever* $x \leq y$*. By a similar reasoning, it can be seen that both the Reichenbach* $I_{\mathbf{RC}}$ *and the Kleene-Dienes* $I_{\mathbf{KD}}$ *implications also satisfy* $(S_{\mathbf{LK}}, I)$*-transitivity.*

We clearly see the importance of (S, I)-transitivity w.r.t. Łukasiewicz t-conorm, and it leads to an interesting problem of investigating the fuzzy implications that satisfy $(S_{\mathbf{LK}}, I)$-transitivity. We shall discuss the same in the next section.

Remark 2. *Note that while the metric* d_I *is obtained on* $[0,1]$*, one can easily lift it to any* $\mathcal{X} \neq \emptyset$*. Let* $f : \mathcal{X} \to [0,1]$*. Define* $d_I^* : \mathcal{X} \times \mathcal{X} \to [0,1]$ *as follows: for any* $x, y \in \mathcal{X}$*,*

$$d_I^*(x,y) = d_I(f(x), f(y)) = \begin{cases} 0, & \text{if } x = y \ , \\ I(\min(f(x), f(y)), \max(f(x), f(y))), & \text{otherwise} \ . \end{cases} \tag{2}$$

Clearly, d_I^* *is a distance function on* $\mathcal{X}$ *and it is a metric if* d_I *is a metric.*

We thus see that we can obtain a metric on any non-empty set $\mathcal{X}$ through d_I^* by mapping the elements of $\mathcal{X}$ to the unit interval. We shall see the applicability of d_I^* in the sequel.

3. $(S_{\mathbf{LK}}, I)$-transitivity

In this section, we discuss the sufficient conditions under which some families and transformations of fuzzy implications satisfy $(S_{\mathbf{LK}}, I)$-transitivity, and hence yield a metric through d_I.

3.1. $(S_{\mathbf{LK}}, I)$-transitivity of $I \in \mathbb{I}$ obtained from other FLCs

Certain families of fuzzy implications are constructed by generalising classical tautologies. Typically these are obtained from other fuzzy logic connectives (FLCs). In this section, we study the satisfaction of $(S_{\mathbf{LK}}, I)$-transitivity by some of the major families obtained through such constructions, viz., R-, (S, N)-, and QL- implications. We shall make use of the following results for the same.

Lemma 2. *An $I \in \mathbb{I}$ satisfying $I(x, x) \geq 0.5$ for every $x \in [0, 1]$, satisfies $(S_{\mathbf{LK}}, I)$-transitivity.*

Proof. Consider a triplet $(x, y, z) \in \mathbb{A}$ i.e., $x < y < z$, then

$$S_{\mathbf{LK}}(I(x, y), I(y, z)) \geq S_{\mathbf{LK}}(I(x, x), I(z, z)) \geq S_{\mathbf{LK}}(0.5, 0.5) = 1 \geq I(x, z) \ .$$

□

Corollary 2. *An $I \in \mathbb{I}$ satisfying the identity principle,*

$$I(x, x) = 1 \text{ for all } x \in [0, 1] \ , \tag{IP}$$

satisfies $(S_{\mathbf{LK}}, I)$-transitivity.

3.1.1. R-implication

As has been mentioned already, many families of fuzzy implications originated as a generalisation of implications from different classical logics. R-implications are a generalisation of the implication in the classical intuitionistic logic to the setting of fuzzy logic and are defined as follows.

Definition 6. *An $I \in \mathbb{I}$ is called an **R-implication**, denoted I_T, if there exists a t-norm T such that for all $x, y \in [0, 1]$,*

$$I_T(x, y) = \sup\{z \in [0, 1] \mid T(x, z) \leq y\} \ .$$

From Corollary 2 and the fact that I_T from any T satisfies the identity principle (IP), the following result follows:

Corollary 3. *An R-implication I_T always satisfies $(S_{\mathbf{LK}}, I_T)$-transitivity. Further, it generates a discrete metric i.e.,* $d_{I_T}(x, y) = \begin{cases} 0, & \text{if } x = y \ , \\ 1, & \text{otherwise} \ . \end{cases}$

Note that the fuzzy implications $I_{\mathbf{LK}}, I_{\mathbf{GG}}, I_{\mathbf{GD}}, I_{\mathbf{WB}}, I_{\mathbf{FD}}$ are in fact, R-implications, and as can be noted from Table 1, they generate the discrete metric through d_I.

3.1.2. (S, N)-implication

(S, N)-implications are a generalisation of the material implication of the classical logic to the setting of fuzzy logic and are defined as follows.

Definition 7. *An $I \in \mathbb{I}$ is called an **(S, N)-implication**, denoted $I_{S,N}$, if there exist a t-conorm S, and a fuzzy negation N such that for all $x, y \in [0, 1]$,*

$$I_{S,N}(x, y) = S(N(x), y) \ .$$

Lemma 3. *Every $I_{S,N}$ where either $S \leq S_{\mathbf{LK}}$ or $N \geq N_{\mathbf{C}}$, satisfies $(S_{\mathbf{LK}}, I_{S,N})$-transitivity, where $N_{\mathbf{C}}(x) = 1 - x$.*

Proof. Consider a triplet $(x, y, z) \in \mathbb{A}$, i.e., $x < y < z$. Then for $S \leq S_{\mathbf{LK}}$, we have

$$\begin{aligned} I_{S,N}(x,z) = S(N(x),z) &\leq S_{\mathbf{LK}}(N(x),z) \\ &= \min(1, N(x)+z) \\ &\leq \min(1, S(N(x),y) + S(N(y),z)) \qquad [\text{ Since } S(x,y) \geq \max(x,y)] \\ &= \min(1, I_{S,N}(x,y) + I_{S,N}(y,z)) \\ &= S_{\mathbf{LK}}(I_{S,N}(x,y), I_{S,N}(y,z)) \ . \end{aligned}$$

Now, consider the case where $N \geq N_{\mathbf{C}}$. Then

$$\begin{aligned} S_{\mathbf{LK}}(I_{S,N}(x,y), I_{S,N}(y,z)) &= \min(1, I_{S,N}(x,y) + I_{S,N}(y,z)) \\ &= \min(1, S(N(x),y) + S(N(y),z)) \\ &\geq \min(1, S(N_{\mathbf{C}}(x),y) + S(N_{\mathbf{C}}(y),z)) \\ &\geq \min(1, 1-x+z) = 1 \\ &\geq I_{S,N}(x,z) \ . \end{aligned}$$

□

Lemma 4. *Every $I_{S,N}$ satisfying the law of excluded middle*

$$S(N(x),x) = 1 \text{ for every } x \in [0,1] \ , \tag{LEM}$$

satisfies $(S_{\mathbf{LK}}, I_{S,N})$-transitivity. Further, such an $I_{S,N}$ will always generate a discrete metric.

Proof. Since $I_{S,N}$ satisfies (LEM), it also satisfies (IP). Hence, from Corollary 2, $I_{S,N}$ will satisfy $(S_{\mathbf{LK}}, I_{S,N})$-transitivity. □

The fuzzy implications $I_{\mathbf{LK}}, I_{\mathbf{RC}}, I_{\mathbf{KD}}, I_{\mathbf{FD}}$ are, in fact, (S,N)-implications. As can be noted from Table 1, unlike the R-implications, they can generate non-discrete metrics through d_I.

However, not every (S,N)-implication satisfies $(S_{\mathbf{LK}}, I_{S,N})$-transitivity as shown in Example 2, which is an (S,N)-implication generated from the following S and N:

$$S(x,y) = \min(x+y+xy, 1) \ , \qquad N(x) = \begin{cases} 1, & \text{if } x = 0, \\ \frac{1}{3}, & \text{if } x < 0.11, \\ 0, & \text{otherwise.} \end{cases} \ .$$

3.1.3. QL-implication

QL-implications are a generalisation of the quantum logic implication to the setting of fuzzy logic and are defined as follows.

Definition 8. *An $I \in \mathbb{I}$ is called a* ***QL-implication****, denoted $I_{T,S,N}$, if there exist a t-norm T, a t-conorm S, and a fuzzy negation N such that*

$$I_{T,S,N}(x,y) = S(N(x), T(x,y)) \ , \qquad x, y \in [0,1] \ .$$

Lemma 5. *Let S be a t-conorm such that $S \leq S_{\mathbf{LK}}$, then the QL-implication $I_{T,S,N}$ satisfies $(S_{\mathbf{LK}}, I_{T,S,N})$-transitivity.*

Implication	Formula $I(x,y)$	Metric $d_I(x,y)$
$I_{\mathbf{PC}}$	$1-(\max(x(x+xy^2-2y),0))^{\frac{1}{2}}$	$\begin{cases} 0, & \text{if } x=y\ , \\ 1-(\max(x(x+xy^2-2y),0))^{\frac{1}{2}}, & \text{if } x<y\ , \\ 1-(\max(y(y+yx^2-2x),0))^{\frac{1}{2}}, & \text{if } x>y\ . \end{cases}$
$I_{\mathbf{PR}}$	$1-(\max(x(1+xy^2-2y),0))^{\frac{1}{2}}$	$\begin{cases} 0, & \text{if } x=y\ , \\ 1-\max(x(1+xy^2-2y),0)^{\frac{1}{2}}, & \text{if } x<y\ , \\ 1-\max(y(1+yx^2-2x),0)^{\frac{1}{2}}, & \text{if } x>y\ . \end{cases}$

Table 2: Examples of metrics obtained from some QL-implications

Proof. Consider a triplet $(x,y,z)\in\mathbb{A}$ i.e., $x<y<z$. Then

$$\begin{aligned} I_{T,S,N}(x,z) = S(N(x),T(x,z)) &\le S_{\mathbf{LK}}(N(x),T(x,z)) \\ &= \min(1,N(x)+T(x,z)) \\ &\le \min(1,N(x)+T(y,z)) \\ &\le \min(1,S(N(x),T(x,y))+S(N(y),T(y,z))) \qquad [\text{ Since } S(x,y)\ge\max(x,y)] \\ &\le \min(1,I_{T,S,N}(x,y)+I_{T,S,N}(y,z)) \\ &= S_{\mathbf{LK}}(I_{T,S,N}(x,y),I_{T,S,N}(y,z)). \end{aligned}$$

□

Lemma 6. *The QL-implication $I_{T,S,N}$ satisfies $(S_{\mathbf{LK}},I_{T,S,N})$-transitivity when any of the following is true:*

(i) $T=T_{\mathbf{M}}(x,y)=\min(x,y)$,

(ii) $T=T_{\mathbf{D}}(x,y)=\begin{cases}\min(x,y), & \textit{if } \max(x,y)=1, \\ 0, & \textit{otherwise.}\end{cases}$,

(iii) *S is a positive t-conorm,*

(iv) $S=S_{\mathbf{D}}(x,y)=\begin{cases}1, & \textit{if } x,y\in(0,1], \\ \max(x,y), & \textit{otherwise.}\end{cases}$,

(v) $N=N_{\mathbf{D_2}}(x)=\begin{cases}0, & \textit{if } x=1\ , \\ 1, & \textit{otherwise}\ .\end{cases}$.

Proof. From Example 4.9, and Propositions 4.7, 4.17 in [13], it can be seen that the QL-implications $I_{T,S,N}$ obtained from any of the above t-norms T, t-conorms S or the fuzzy negation N are such that $I_{T,S,N}(x,y)=1$ whenever $x\le y$. Then, for a triplet $(x,y,z)\in\mathbb{A}$, i.e., $x<y<z$, we have

$$S_{\mathbf{LK}}(I_{T,S,N}(x,y),I_{T,S,N}(y,z)) = \min(1,I_{T,S,N}(x,y)+I_{T,S,N}(y,z)) = \min(1,2)=1=I_{T,S,N}(x,z)\ .$$

Hence, $I_{T,S,N}$ satisfies $(S_{\mathbf{LK}},I_{T,S,N})$-transitivity. □

Note that the QL-implications $I_{T,S,N}$ obtained from any of the t-norms T, t-conorms S or the fuzzy negation N given in Lemma 6 will always generate a discrete metric. Interestingly, the fuzzy implications $I_{\mathbf{LK}},I_{\mathbf{RC}},I_{\mathbf{KD}},I_{\mathbf{FD}},I_{\mathbf{WB}}$ are also QL-implications and generate a variety of metrics through d_I, see Table 1. Table 2 lists examples of metrics obtained from QL-implications, $I_{\mathbf{PC}}$ and $I_{\mathbf{PR}}$, that are not (S,N)-implications.

3.2. $(S_{\mathbf{LK}},I)$-transitivity of Transformations of Fuzzy Implications

In this section, we show the sufficient conditions under which different transformations of fuzzy implications satisfy $(S_{\mathbf{LK}},I)$-transitivity.

Let Φ denote the family of all increasing bijections $\phi : [0,1] \to [0,1]$.

Define the family of increasing bijections that are super-additive and sub-additive on admissible arguments $x, y \in [0,1]$, denoted by Φ^+ and Φ_+ respectively, as follows:

$$\Phi^+ := \{\phi \in \Phi \mid \phi(x+y) \geq \phi(x) + \phi(y)\} ,$$
$$\Phi_+ := \{\phi \in \Phi \mid \phi(x+y) \leq \phi(x) + \phi(y)\} .$$

To begin with, we have the following easy to prove result:

Lemma 7. $\phi \in \Phi^+ \iff \phi^{-1} \in \Phi_+$.

We now recall the definitions of certain transformations of fuzzy implications and study the conditions under which they satisfy $(S_{\mathbf{LK}}, I)$-transitivity.

Definition 9 (cf. [14])**.** *Given $\phi \in \Phi$, and an $I \in \mathbb{I}$, the following transformations of I always yield an implication:*

(i) $I^{[1]}_\phi(x, y) = \phi^{-1}(I(\phi(x), \phi(y)))$,

(ii) $I^{[2]}_\phi(x, y) = \phi(I(x, \phi^{-1}(y)))$,

(iii) $I^{[3]}_\phi(x, y) = \phi(I(\phi^{-1}(x), y))$,

(iv) $I^{[4]}_\phi(x, y) = \phi(I(x, y))$.

Lemma 8. *Let $I \in \mathbb{I}$ satisfying $I(x, x) \geq 0.5$ for every $x \in [0,1]$.*

(i) *If $\phi(0.5) \leq 0.5$ then $I^{[1]}_\phi$ satisfies $(S_{\mathbf{LK}}, I^{[1]}_\phi)$-transitivity.*

(ii) *If $\phi(0.5) = 0.5$ and $\phi(x) \leq x$ then $I^{[2]}_\phi$ satisfies $(S_{\mathbf{LK}}, I^{[2]}_\phi)$-transitivity.*

(iii) *If $\phi(x) \geq x$ then $I^{[3]}_\phi$ satisfies $(S_{\mathbf{LK}}, I^{[3]}_\phi)$-transitivity.*

(iv) *If $\phi(0.5) \geq 0.5$ then $I^{[4]}_\phi$ satisfies $(S_{\mathbf{LK}}, I^{[4]}_\phi)$-transitivity.*

Proof. (i) Consider a triplet $(x, y, z) \in \mathbb{A}$ i.e., $x < y < z$. Then

$$\begin{aligned} S_{\mathbf{LK}}(I^{[1]}_\phi(x, y), I^{[1]}_\phi(y, z)) &= \min(1, I^{[1]}_\phi(x, y) + I^{[1]}_\phi(y, z)) \\ &= \min\left(1, \phi^{-1}\left(I(\phi(x), \phi(y))\right) + \phi^{-1}\left(I(\phi(y), \phi(z))\right)\right) \end{aligned}$$

Then, by mixed-monotonicity of I,

$$S_{\mathbf{LK}}(I^{[1]}_\phi(x, y), I^{[1]}_\phi(y, z)) \geq \min\left(1, \phi^{-1}\left(I(\phi(x), \phi(x))\right) + \phi^{-1}\left(I(\phi(z), \phi(z))\right)\right)$$

Since $I(x, x) \geq 0.5$ and $\phi(0.5) \leq 0.5$, we have

$$\begin{aligned} S_{\mathbf{LK}}(I^{[1]}_\phi(x, y), I^{[1]}_\phi(y, z)) &\geq \min\left(1, \phi^{-1}(0.5) + \phi^{-1}(0.5)\right) \\ &\geq \min(1, 0.5 + 0.5) = 1 \\ &\geq I^{[1]}_\phi(x, z) . \end{aligned}$$

(ii) Consider a triplet $(x, y, z) \in \mathbb{A}$ i.e., $x < y < z$. Then

$$\begin{aligned} S_{\mathbf{LK}}(I^{[2]}_\phi(x, y), I^{[2]}_\phi(y, z)) &= \min(1, I^{[2]}_\phi(x, y) + I^{[2]}_\phi(y, z)) \\ &= \min\left(1, \phi\left(I(x, \phi^{-1}(y))\right) + \phi\left(I(y, \phi^{-1}(z))\right)\right) \end{aligned}$$

Then, by mixed-monotonicity of I,

$$\begin{aligned} S_{\mathbf{LK}}(I_\phi^{[2]}(x,y), I_\phi^{[2]}(y,z)) &\geq \min\left(1, \phi\left(I(x,\phi^{-1}(x))\right) + \phi\left(I(z,\phi^{-1}(z))\right)\right) \\ &\geq \min\left(1, \phi\left(I(x,x)\right) + \phi\left(I(z,z)\right)\right) \quad [\text{ Since } \phi(x) \leq x] \\ &\geq \min\left(1, \phi(0.5) + \phi(0.5)\right) \quad [\text{ Since } I(x,x) \geq 0.5] \\ &\geq \min(1, 0.5+0.5) = 1 \quad [\text{ Since } \phi(0.5) = 0.5] \\ &\geq I_\phi^{[2]}(x,z)\ . \end{aligned}$$

(iii) Consider a triplet $(x,y,z) \in \mathbb{A}$ i.e., $x < y < z$. Then

$$\begin{aligned} S_{\mathbf{LK}}(I_\phi^{[3]}(x,y), I_\phi^{[3]}(y,z)) &= \min(1, I_\phi^{[3]}(x,y) + I_\phi^{[3]}(y,z)) \\ &= \min\left(1, \phi\left(I(\phi^{-1}(x),y)\right) + \phi\left(I(\phi^{-1}(y),z)\right)\right) \end{aligned}$$

Then, by mixed-monotonicity of I,

$$\begin{aligned} S_{\mathbf{LK}}(I_\phi^{[3]}(x,y), I_\phi^{[3]}(y,z)) &\geq \min\left(1, \phi\left(I(\phi^{-1}(x),x)\right) + \phi\left(I(\phi^{-1}(z),z)\right)\right) \\ &\geq \min\left(1, \phi\left(I(x,x)\right) + \phi\left(I(z,z)\right)\right) \quad [\text{ Since } \phi(x) \geq x] \\ &\geq \min\left(1, 0.5+0.5\right) = 1 \quad [\text{ Since } I(x,x) \geq 0.5] \\ &\geq I_\phi^{[3]}(x,z)\ . \end{aligned}$$

(iv) Can be proven similarly as (i).

□

Lemma 9. *Let I satisfy $(S_{\mathbf{LK}}, I)$-transitivity.*

(i) *If $\phi \in \Phi^+$ then $I_\phi^{[1]}$ satisfies $(S_{\mathbf{LK}}, I_\phi^{[1]})$-transitivity.*

(ii) *If $\phi \in \Phi_+$, and $\phi(y) \geq y$, then $I_\phi^{[3]}$ satisfies $(S_{\mathbf{LK}}, I_\phi^{[3]})$-transitivity.*

(iii) *If $\phi \in \Phi_+$ then $I_\phi^{[4]}$ satisfies $(S_{\mathbf{LK}}, I_\phi^{[4]})$-transitivity.*

Proof. (i) Suppose $I_\phi^{[1]}$ does not satisfy $(S_{\mathbf{LK}}, I_\phi^{[1]})$-transitivity, i.e., there exist $x,y,z \in [0,1]$ such that $I_\phi^{[1]}(x,y) + I_\phi^{[1]}(y,z) < I_\phi^{[1]}(x,z)$. Thus,

$$\begin{aligned} &\phi^{-1}\left(I(\phi(x),\phi(y))\right) + \phi^{-1}\left(I(\phi(y),\phi(z))\right) < \phi^{-1}\left(I(\phi(x),\phi(z))\right) \\ \Longrightarrow\ &\phi\left(\phi^{-1}(I(\phi(x),\phi(y))) + \phi^{-1}(I(\phi(y),\phi(z)))\right) < \phi\left(\phi^{-1}(I(\phi(x),\phi(z)))\right) = I(\phi(x),\phi(z))\ . \end{aligned} \quad (3)$$

Since $\phi \in \Phi^+$, we have that

$$\begin{aligned} &\phi\left(\phi^{-1}\big(I(\phi(x),\phi(y))\big)\right) + \phi\left(\phi^{-1}\big(I(\phi(x),\phi(y))\big)\right) \leq \phi\left(\phi^{-1}(I(\phi(x),\phi(y))) + \phi^{-1}(I(\phi(y),\phi(z)))\right) \\ \Longrightarrow\ &\phi\left(\phi^{-1}\big(I(\phi(x),\phi(y))\big)\right) + \phi\left(\phi^{-1}\big(I(\phi(x),\phi(y))\big)\right) < I(\phi(x),\phi(z)) \qquad [\text{ By } (3)] \\ \Longrightarrow\ &I(\phi(x),\phi(y)) + I(\phi(y),\phi(z)) < I(\phi(x),\phi(z))\ , \end{aligned}$$

which is a contradiction as I satisfies $(S_{\mathbf{LK}}, I)$-transitivity.

(ii) Suppose $I_\phi^{[3]}$ does not satisfy $(S_{\mathbf{LK}}, I_\phi^{[3]})$-transitivity, i.e., there exist $x,y,z \in [0,1]$ such that $I_\phi^{[3]}(x,y) + I_\phi^{[3]}(y,z) < I_\phi^{[3]}(x,z)$. Thus,

$$\begin{aligned} &\phi\left(I(\phi^{-1}(x),y)\right) + \phi\left(I(\phi^{-1}(y),z)\right) < \phi\left(I(\phi^{-1}(x),z)\right) \\ \Longrightarrow\ &\phi^{-1}\left(\phi(I(\phi^{-1}(x),y)) + \phi(I(\phi^{-1}(y),z))\right) < \phi^{-1}\left(\phi(I(\phi^{-1}(x),z))\right) = I(\phi^{-1}(x),z)\ . \end{aligned} \quad (4)$$

Since $\phi \in \Phi_+$, we have that

$$\begin{aligned}
&\phi^{-1}\left(\phi\big(I(\phi^{-1}(x),y)\big)\right)+\phi^{-1}\left(\phi\big(I(\phi^{-1}(y),z)\big)\right) \leq \phi^{-1}(\phi(I(\phi^{-1}(x),y))+\phi(I(\phi^{-1}(y),z)))\\
\Longrightarrow\ &\phi^{-1}\left(\phi\big(I(\phi^{-1}(x),y)\big)\right)+\phi^{-1}\left(\phi\big(I(\phi^{-1}(y),z)\big)\right) < I(\phi^{-1}(x),z) \qquad [\text{ By } (4)]\\
\Longrightarrow\ &I(\phi^{-1}(x),y)+I(y,z) < I(\phi^{-1}(x),y)+I(\phi^{-1}(y),z) < I(\phi^{-1}(x),z)\ , \qquad [\text{ Since } \phi(y)\geq y]
\end{aligned}$$

which is a contradiction as I satisfies $(S_{\mathbf{LK}}, I)$-transitivity.

(iii) Can be proven similarly as (i).

□

Examples 4 and 5 present some transformations under which $(S_{\mathbf{LK}}, I)$-transitivity is preserved and the corresponding metrics are also presented.

Example 4. *Let $\phi(x) = x^2$. Thus, $\phi^{-1}(x) = \sqrt{x}$. Clearly, $\phi \in \Phi^+$ since*

$$\phi(x+y) = (x+y)^2 \geq x^2+y^2 \geq \phi(x)+\phi(y).$$

Let $I = I_{\mathbf{RC}}$. Then,

$$I_\phi^{[1]}(x,y) = \sqrt{I_{\mathbf{RC}}(x^2,y^2)} = \sqrt{1-x^2+x^2y^2}\ ,$$

$$d_{I_\phi^{[1]}}(x,y) = \begin{cases} 0, & \text{if } x=y\ ,\\ \sqrt{1-x^2+x^2y^2}, & \text{if } x<y\ ,\\ \sqrt{1-y^2+x^2y^2}, & \text{if } y<x\ . \end{cases}$$

Example 5. *Let $\phi(x) = \sin\left(\frac{\pi}{2}x\right)$. Thus, $\phi^{-1}(x) = \frac{2}{\pi}\sin^{-1}(x)$. Since*

$$\begin{aligned}
\phi(x+y) =& \sin\left(\frac{\pi}{2}(x+y)\right) = \sin\left(\frac{\pi}{2}x\right)\cos\left(\frac{\pi}{2}y\right)+\sin\left(\frac{\pi}{2}y\right)\cos\left(\frac{\pi}{2}x\right)\\
&\leq \sin\left(\frac{\pi}{2}x\right)+\sin\left(\frac{\pi}{2}y\right) = \phi(x)+\phi(y)\ ,
\end{aligned}$$

we see that $\phi \in \Phi_+$. Once again letting $I = I_{\mathbf{RC}}$, we obtain

$$I_\phi^{[2]}(x,y) = \sin\left(\frac{\pi}{2}I_{\mathbf{RC}}\left(x, \frac{2}{\pi}sin^{-1}(y)\right)\right) = \sin\left(\frac{\pi}{2}\left[1-x+\frac{2}{\pi}x\sin^{-1}(y)\right]\right)$$

$$d_{I_\phi^{[2]}}(x,y) = \begin{cases} 0, & \text{if } x=y\ ,\\ \sin\left(\frac{\pi}{2}[1-x+\frac{2}{\pi}x\sin^{-1}(y)]\right), & \text{if } x<y\ ,\\ \sin\left(\frac{\pi}{2}[1-y+\frac{2}{\pi}y\sin^{-1}(x)]\right), & \text{if } y<x\ . \end{cases}$$

$$I_\phi^{[3]}(x,y) = \sin\left(\frac{\pi}{2}I_{\mathbf{RC}}\left(\frac{2}{\pi}\sin^{-1}(x), y\right)\right) = \sin\left(\frac{\pi}{2}-\sin^{-1}(x)+y\sin^{-1}(x)\right)\ .$$

$$d_{I_\phi^{[3]}}(x,y) = \begin{cases} 0, & \text{if } x=y\ ,\\ \sin\left(\frac{\pi}{2}-\sin^{-1}(x)+y\sin^{-1}(x)\right), & \text{if } x<y\ ,\\ \sin\left(\frac{\pi}{2}-\sin^{-1}(y)+x\sin^{-1}(y)\right), & \text{if } y<x\ . \end{cases}$$

$$I_\phi^{[4]}(x,y) = \sin\left(\frac{\pi}{2}I_{\mathbf{RC}}(x,y)\right) = \sin\left(\frac{\pi}{2}[1-x+xy]\right)\ .$$

$$d_{I_\phi^{[4]}}(x,y) = \begin{cases} 0, & \text{if } x=y\ ,\\ \sin\left(\frac{\pi}{2}[1-x+xy]\right), & \text{if } x<y\ ,\\ \sin\left(\frac{\pi}{2}[1-x+xy]\right), & \text{if } y<x\ . \end{cases}$$

3.3. $(S_{\mathbf{LK}}, I)$-transitivity of $I \in \mathbb{I}$ obtained from Unary Generators

Yet another way to obtain fuzzy implications are through unary operations on $[0,1]$. Yager proposed this approach formally in [15] using the additive generators of Archimedean t-norms. Since then there have been many such proposals.

3.3.1. f-implications

In this section, we discuss the $(S_{\mathbf{LK}}, I)$-transitivity of f-implications using the representation theorem for the family of f-implications given in [14].

Definition 10 ([16], Definition 3.1.1)**.** *Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing and continuous function with $f(1) = 0$. With the understanding $0 \cdot (+\infty) = 0$, $I_f \in \mathbb{I}$ and is called an f-implication, when defined as follows:*

$$I_f(x,y) = f^{-1}\left(x \cdot f(y)\right), \text{ for all } x, y \in [0,1] \ .$$

Example 6. *Following are prototypical examples of f-implications and they satisfy $(S_{\mathbf{LK}}, I)$-transitivity:*

(i) *Yager:* $I_{\mathbf{YG}}(x,y) = \begin{cases} 1, & \text{if } x = y = 0 \ , \\ y^x, & \text{otherwise} \ . \end{cases}$

(ii) *Reichenbach:* $I_{\mathbf{RC}}(x,y) = 1 - x + xy$.

Remark 3. *Let us denote the family of all f-implications whose generators f satisfy $f(0) = 1$ by $I_{\mathbb{F},1}$, and those with $f(0) = +\infty$ by $I_{\mathbb{F},+\infty}$. The family of all f-implications is, in fact, $I_{\mathbb{F}} = I_{\mathbb{F},+\infty} \cup I_{\mathbb{F},1}$.*

Theorem 3 ([14], Corollary 5.9)**.** (i) *$I_f \in I_{\mathbb{F},1}$ iff $I_f = I^{[2]}_{\mathbf{RC}_\phi}$ for some $\phi \in \Phi$.*

(ii) *$I_f \in I_{\mathbb{F},+\infty}$ iff $I_f = I^{[2]}_{\mathbf{YG}_\phi}$ for some $\phi \in \Phi$.*

Lemma 10. *Let $\phi \in \Phi$ such that $\phi(x) \leq x$ for every $x \in [0,1]$, and $\phi(0.5) = 0.5$. Then*

(i) *any $I \in I^{[2]}_{\mathbf{RC}_\phi} \subseteq I_{\mathbb{F},1}$ satisfies $(S_{\mathbf{LK}}, I)$-transitivity.*

(ii) *any $I \in I^{[2]}_{\mathbf{YG}_\phi} \subseteq I_{\mathbb{F},+\infty}$ satisfies $(S_{\mathbf{LK}}, I)$-transitivity.*

3.3.2. g-implications

In this section, we discuss the $(S_{\mathbf{LK}}, I)$-transitivity of g-implications using the representation theorem for the family of g-implications given in [14].

Definition 11 ([16], Definition 3.1.1)**.** *Let $g : [0,1] \to [0,\infty]$ be a strictly increasing and continuous function with $g(0) = 0$. With the understanding $\frac{1}{0} = +\infty$ and $(+\infty).0 = +\infty$, $I_g \in \mathbb{I}$ and is called a g-implication, when defined as follows:*

$$I_g(x,y) = g^{-1}\left(\frac{1}{x} \cdot g(y)\right), \text{ for all } x, y \in [0,1] \ .$$

Example 7. *Following are prototypical examples of g-implications and they satisfy $(S_{\mathbf{LK}}, I)$-transitivity:*

(i) *Yager:* $I_{\mathbf{YG}}(x,y) = \begin{cases} 1, & \text{if } x = y = 0 \ , \\ y^x, & \text{otherwise} \ . \end{cases}$

(ii) *Goguen:* $I_{\mathbf{GG}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \ , \\ \dfrac{y}{x}, & \text{otherwise} \ . \end{cases}$

Remark 4. *Let us denote the family of all g-implications whose generators g satisfy $g(1) = 1$ by $I_{\mathbb{G},1}$ and those with $g(1) = +\infty$ by $I_{\mathbb{G},+\infty}$. The family of all g-implications is, in fact, $I_{\mathbb{G}} = I_{\mathbb{G},+\infty} \cup I_{\mathbb{G},1}$.*

Theorem 4 ([14], Corollary 5.17)**.** (i) *$I_g \in I_{\mathbb{G},1}$ iff $I_g = I^{[2]}_{\mathbf{GG}_\phi}$ for some $\phi \in \Phi$.*

(ii) *$I_g \in I_{\mathbb{G},+\infty}$ iff $I_g = I^{[2]}_{\mathbf{YG}_\phi}$ for some $\phi \in \Phi$.*

Lemma 11. *Let $\phi \in \Phi$ such that $\phi(x) \leq x$ for every $x \in [0,1]$, and $\phi(0.5) = 0.5$. Then*

(i) *any $I \in I^{[2]}_{\mathbf{GG}_\phi} \subseteq I_{\mathbb{G},1}$ satisfies $(S_{\mathbf{LK}}, I)$-transitivity.*

(ii) *any $I \in I^{[2]}_{\mathbf{YG}_\phi} \subseteq I_{\mathbb{G},+\infty}$ satisfies $(S_{\mathbf{LK}}, I)$-transitivity.*

4. Pseudo-monometrics from Fuzzy Implications

Recently, (pseudo-)monometrics w.r.t. betweenness relations have garnered a lot of attention, mainly due to their application in decision making and penalty based data aggregation (see [17, 18, 19]). In this section, we begin by taking a look at the usefulness of (pseudo-)monometrics in applications. We then recall the definitions of these notions and show that d_I and d_I^* are essentially pseudo-monometrics on appropriate betweenness sets.

4.1. Applications of (Pseudo-)Monometrics

Monometrics(or pseudo-monometrics) play an integral role in the applications dealing with rationalisation of ranking rules, penalty-based aggregation, and binary classification.

In the problem of aggregation of rankings, given a profile of rankings, the aim is to obtain a single ranking that best represents the nature of this given profile. The aggregated rankings can be characterised as minimizing the distance from a consensus state using a distance function. In [18], it was proposed that the distance function should be replaced by a monometric, which essentially preserves the betweenness relation under consideration.

The study of penalty-based aggregation has been mainly confined to the domain of real numbers. In [17], the definition of penalty-based function was extended to accommodate more general structures and expand its scope beyond real numbers by demanding compatibility with a betweenness relation. It was shown that penalty-based functions could be constructed using monometrics on the given betweenness relation.

A distance function appears in almost every DA/ML algorithm, either explicitly as a metric or a norm, or implicitly as its dual, similarity measure, for instance, in the form of an inner product. That the general purpose distances may not be appropriate for all situations is well-known, see for instance, an excellent articulation of the same in [20]. In [21], authors have claimed that the distance functions compatible with the relational structure(pseudo-monometrics) present in the data are the most appropriate in the problem of binary classification, especially in nearest-neighbor classification.

4.2. Pseudo-monometric on Betweenness Relations

We shall now discuss some preliminary order-theoretic concepts before presenting the definitions of betweenness and pseudo-monometrics.

Definition 12. *Let $\mathbb{P} \neq \emptyset$. A* ***partial order*** *on $\mathbb{P}$ is a binary relation $\leq$ on $\mathbb{P}$ such that, for all $a, b, c \in \mathbb{P}$, the following properties hold:*

- *Reflexivity: $a \leq a$,*
- *Antisymmetry: If $a \leq b$ and $b \leq a$, then $a = b$,*
- *Transitivity: If $a \leq b$ and $b \leq c$, then $a \leq c$.*

Definition 13. *Let $(\mathbb{P}, \leq)$ be a partially ordered set or poset. An element a in $\mathbb{P}$ is said to be*

- *the* ***least element*** *(minimum element) if for every element b in $\mathbb{P}$, we have that $a \leq b$.*
- *the* ***greatest element*** *(maximum element) if for every element b in $\mathbb{P}$, we have that $a \geq b$.*

Definition 14. *A poset $(\mathbb{P}, \leq)$ is called*

- ***bounded below*** *if there exists a least element.*
- ***bounded above*** *if there exists a greatest element.*
- ***bounded*** *if it is both bounded below and above.*

Definition 15. *Let* B *be a ternary relation on an* $\mathcal{X} \neq \emptyset$*. Then* B *is said to be a* ***betweenness*** *relation if* B *satisfies the following for any* $o, x, y, z \in \mathcal{X}$*:*

$$(x, y, z) \in \mathrm{B} \iff (z, y, x) \in \mathrm{B} \ , \tag{BS}$$

$$(x, y, z) \in \mathrm{B} \wedge (x, z, y) \in \mathrm{B} \iff y = z \ , \tag{BU}$$

$$(o, x, y) \in \mathrm{B} \ \wedge \ (o, y, z) \in \mathrm{B} \Longrightarrow (o, x, z) \in \mathrm{B} \ . \tag{BT}$$

Remark 5. (i) $(\mathcal{X}, \mathrm{B})$ *is known as a Betweenness set or a Beset. Also,* $(x, y, z) \in \mathrm{B}$ *is read as 'y is in between x and z'.*

(ii) *The minimal betweenneess relation* B_0 *on* $\mathcal{X}$ *is defined as follows:*

$$\mathrm{B}_0 = \{(x, y, z) \in \mathcal{X}^3 \mid x = y \vee y = z\} \ .$$

(iii) *For an arbitrary but fixed* $o \in \mathcal{X}$*, the following is a partial order on* $\mathcal{X}$ *[22]:*

$$x \preceq y \text{ iff } (o, x, y) \in \mathrm{B} \ . \tag{5}$$

(iv) *Conversely, a betweenness relation* B *can be defined from a partial order* $\preceq$ *on* $\mathcal{X}$ *as follows [22]:*

$$\mathrm{B}_{\preceq} = \mathrm{B}_0 \cup \{(x, y, z) \in \mathcal{X}^3 \mid x \preceq y \preceq z \ \vee \ z \preceq y \preceq x\} \ . \tag{6}$$

Example 8. *We present below a few examples of betweenness relations.*

(i) *(cf. [19]) Consider a metric space* $(\mathcal{X}, d)$*. Then the ternary relation* B_d*, defined on* $\mathcal{X}$*, as*

$$\mathrm{B}_d := \{(a, b, c) \in \mathcal{X}^3 \mid d(a, b) + d(b, c) = d(a, c)\} \ . \tag{BD}$$

is a betweenness relation.

(ii) *([cf. [23]) Let* $(L, \wedge, \vee)$ *be a lattice. Then the ternary relation* B_L*, defined on* L*, as*

$$\mathrm{B}_L := \{(a, b, c) \in L^3 \mid (a \wedge b) \vee (b \wedge c) = b = (a \vee b) \wedge (b \vee c)\} \ . \tag{BL}$$

is a betweenness relation.

(iii) *([cf. [23]) Let* V *be a vector space. Then the ternary relation* B_A*, defined on* V*, as*

$$\mathrm{B}_A := \{(a, b, c) \in V^3 \mid b = \lambda a + (1 - \lambda)c, \lambda \in [0, 1]\} \ . \tag{BA}$$

is a betweenness relation.

Now, we present the definition of pseudo-monometric as given in [24].

Definition 16. *Consider a betweenness set* $(\mathcal{X}, \mathrm{B})$*. A distance function* $d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ *is called a pseudo-monometric (w.r.t.* B*) if for every* $(x, y, z) \in \mathrm{B}$*, it holds that:*

$$\max(d(x, y), d(y, z)) \leq d(x, z) \ . \tag{MC}$$

Note that if a function d satisfies (P1) and its converse, along with (MC), we shall refer to it as a monometric [17].

Often, in applications, one is either given or able to determine the beset $(\mathcal{X}, \mathrm{B})$. However, it is not well known how to construct pseudo-monometrics on it. In the following sections, we show that the construction detailed in the previous section offers us a solution to the problem of constructing pseudo-monometrics in the case when betweenness relation is obtained from a partial order.

4.2.1. Pseudo-monometric on the beset obtained from a totally ordered set

In this section, we begin by showing that d_I is indeed a natural choice for obtaining pseudo-monometric w.r.t. the betweenness set obtained from the usual total order on $[0,1]$. Note that in this section I need not satisfy (1).

Let us define a betweenness relation $\mathrm{B}_{\leq}$ on $[0,1]$ where $\leq$ denotes the usual total order on it:

$$\mathrm{B}_{\leq} = \{(x,y,z) \in [0,1]^3 \mid x \leq y \leq z \ \vee \ z \leq y \leq x\}.$$

Lemma 12. *d_I is a pseudo-monometric on $([0,1], \mathrm{B}_{\leq})$.*

Proof. Since $d_I(x,x) = 0$ and $d_I(x,y) = d_I(y,x)$, it is a distance function. Suppose $(x,y,z) \in \mathrm{B}_{\leq}$. Without loss of generality, assume $x \leq y \leq z$. By the definition of I, we have $I(x,z) \geq I(x,y)$, and $I(x,z) \geq I(y,z)$. Thus

$$I(x,z) \geq \max(I(x,y), I(y,z)) \implies d_I(x,z) \geq \max(d_I(x,y), d_I(y,z)) \ . \tag{7}$$

Hence, d_I is a pseudo-monometric on $([0,1], \mathrm{B}_{\leq})$. □

From (7), it can be clearly seen that fuzzy implications - due to their mixed monotonicity - are both a natural choice and a rich source for construction of pseudo-monometrics.

Now, we see that by mapping $\mathcal{X}$ to $[0,1]$, the above distance function can be lifted to a pseudo-monometric on $\mathcal{X}$ for a suitably defined betweenness relation. The proof follows along similar lines as the proof of Lemma 12.

Lemma 13. *Let $([0,1], \mathrm{B}_{\leq})$ be the beset obtained from the usual ordering on $[0,1]$ and $f : \mathcal{X} \to [0,1]$ be any mapping. Let us define a betweenness relation B on $\mathcal{X}$ as follows:*

$$(x,y,z) \in \mathrm{B} \iff (f(x), f(y), f(z)) \in \mathrm{B}_{\leq} \ .$$

Then the distance d_I^*, as defined in (2), is a pseudo-monometric on the beset $(\mathcal{X}, \mathrm{B})$.

Further, if we are given a totally ordered set $(\mathcal{X}, \leq)$, and let $(\mathcal{X}, \mathrm{B}_{\leq})$ be the corresponding beset. Once again, if there exists an $f : \mathcal{X} \to [0,1]$ which is an order-preserving mapping, i.e., $x \preceq y \implies f(x) \leq f(y)$, then clearly, one can easily show that d_I^* defined on $\mathcal{X}$ is a pseudo-monometric on $(\mathcal{X}, \mathrm{B}_{\leq})$.

We now see that the above results can be extended to a poset, which is not a chain. In the rest of the section, we shall see results pertaining to partially ordered sets.

4.2.2. Pseudo-monometric on the beset obtained from a partially ordered set

We begin by discussing the case where the betweenness set is obtained from a partially ordered set as in (6) and show the existence of a monometric on it. In our quest to prove it, we shall make use of the following result which shows the existence of an order-preserving map from a partially ordered set to a totally ordered set.

Theorem 5. *Let $(\mathcal{X}, \preceq)$ be a partially ordered set. Then there always exists a non-constant order-preserving map $f : \mathcal{X} \to [0,1]$, i.e., $x \preceq y$ implies $f(x) \leq f(y)$.*

Proof. Let $\alpha \in \mathcal{X}$ be arbitrary but fixed and $\alpha\downarrow$ denote the downset of α in $(\mathcal{X}, \preceq)$, i.e., $\alpha\downarrow = \{z \in \mathcal{X} \mid z \leq \alpha\}$. Define $f : \mathcal{X} \to [0,1]$ as

$$f(x) = \begin{cases} 0.2, & \text{if } x \in \alpha\downarrow \ , \\ 0.4, & \text{otherwise} \ . \end{cases}$$

Suppose $a \preceq b$. If $a \in \alpha\downarrow$, $f(a) = 0.2 \leq f(b)$, and if $a \notin \alpha\downarrow$, then $b \notin \alpha\downarrow$, and $f(a) = f(b) = 0.4$.

□

In the above proof, we provided an explicit construction for an order-preserving f, essentially showing that an order-preserving f exists from $(\mathcal{X}, \preceq)$ to the unit interval endowed with the usual order. While the proof offers one such construction of f, there can be various other constructions depending on the cardinality of $\mathcal{X}$. We provide some alternate constructions in the following remark.

Remark 6. (i) *Note that if f is a constant function, we will get the discrete metric through d_I^*, which is always a trivial monometric on $(\mathcal{X}, \mathrm{B}_{\preceq})$.*

(ii) *If the cardinality of $\mathcal{X}$ is either finite or countable then the following f is one such order-preserving map:*

$$f(x) = 1 - \frac{1}{h(x)+1} \ ,$$

where $h(x)$ gives the maximum of the heights of x from the minimal element of each of its chains. Example 9 provides yet another mapping with a clear visualisation of such a projection in Fig. 1 (a).

(iii) *Some constructions for examples where $\mathcal{X}$ is of infinite cardinality, are provided in Examples 10, 11 and 12.*

Using any such f, we can obtain a distance function on $(\mathcal{X}, \preceq)$ through d_I^* defined as in (2), and by using the mixed-monotonicity property of I, we can prove that it is a pseudo-monometric on $(\mathcal{X}, \mathrm{B}_{\preceq})$.

Now, we are ready to prove one of the main results of this work - that of showing that if a betweenness relation B is obtained from an underlying partial order $\preceq$ on $\mathcal{X}$, then there always exists a pseudo-monometric on $\mathcal{X}$. In fact, the proof of the result is not only existential in nature but also constructive.

Theorem 6. *Let $(\mathcal{X}, \preceq)$ be a poset, and $(\mathcal{X}, \mathrm{B}_{\preceq})$ the beset obtained as given in* (6)*. Then there exists a non-trivial, i.e., a non-discrete, pseudo-monometric on $(\mathcal{X}, \mathrm{B}_{\preceq})$.*

Proof. Given $(\mathcal{X}, \preceq)$, from Theorem 5, we know there exists an order-preserving map $f : \mathcal{X} \to [0,1]$. Let $(x,y,z) \in (\mathcal{X}, \mathrm{B}_{\preceq})$. Without loss of generality, assume that $x \preceq y \preceq z$. Thus $f(x) \leq f(y) \leq f(z)$, and by the definition of a fuzzy implication I, $I(f(x), f(y)) \leq I(f(x), f(z))$, and $I(f(y), f(z)) \leq I(f(x), f(z))$. Thus we see that the d_I^* as defined in (2) satisfies the following inequalities,

$$d_I^*(x,y) \leq d_I^*(x,z) \ ,$$
$$d_I^*(y,z) \leq d_I^*(x,z) \ ,$$

and hence is a pseudo-monometric on $(\mathcal{X}, \mathrm{B}_{\preceq})$. □

The following is an example of a (pseudo-)monometric on $(\mathcal{X}, \mathrm{B}_{\preceq})$ where the cardinality of $\mathcal{X}$ is finite.

Example 9. *Consider $\mathcal{X} = \{o, x, y, z\}$. Let $(\mathcal{X}, \preceq)$ be a partially ordered set as given in Fig. 1 (a). Then*

$$\mathrm{B}_{\preceq} = \mathrm{B}_0 \cup \{(o,x,z), (z,x,o), (o,y,z), (z,y,o)\} \ .$$

Now, define a mapping $f : \mathcal{X} \to [0,1]$ as in Table 3 (a):

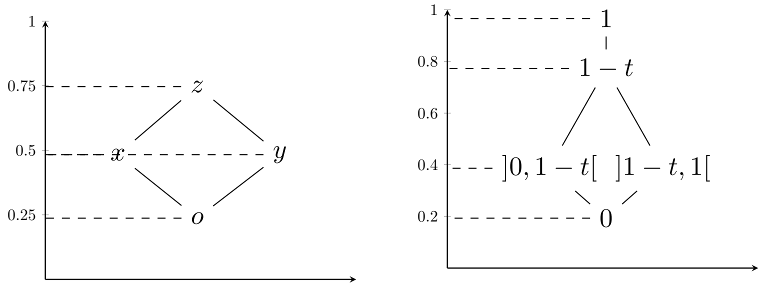
$\mathcal{X}$	o	x	y	z
f	0.25	0.5	0.5	0.75

d_I^*	o	x	y	z
o	0	0.875	0.875	0.9375
x	0.875	0	0.75	0.875
y	0.875	0.75	0	0.875
z	0.9375	0.875	0.875	0

Table 3: (a) The mapping f (b) The pairwise distance matrix on $\mathcal{X}$ under d_I^*

A geometric visualisation of the mapping f is given in Fig. 1 (a).

Consider $I = I_{\mathbf{RC}}$. We obtain the distance function d_I^* on $\mathcal{X}$ as given in Table 3 (b). Clearly, d_I^* is a pseudo-monometric w.r.t. $\mathrm{B}_{\preceq}$. In fact, it is a monometric on $\mathrm{B}_{\preceq}$ as $I_{\mathbf{RC}}$ satisfies (1)*.*



(a) Finite poset (b) Uncountable poset

Figure 1: Hasse Diagrams of the posets in Examples 9 and 11

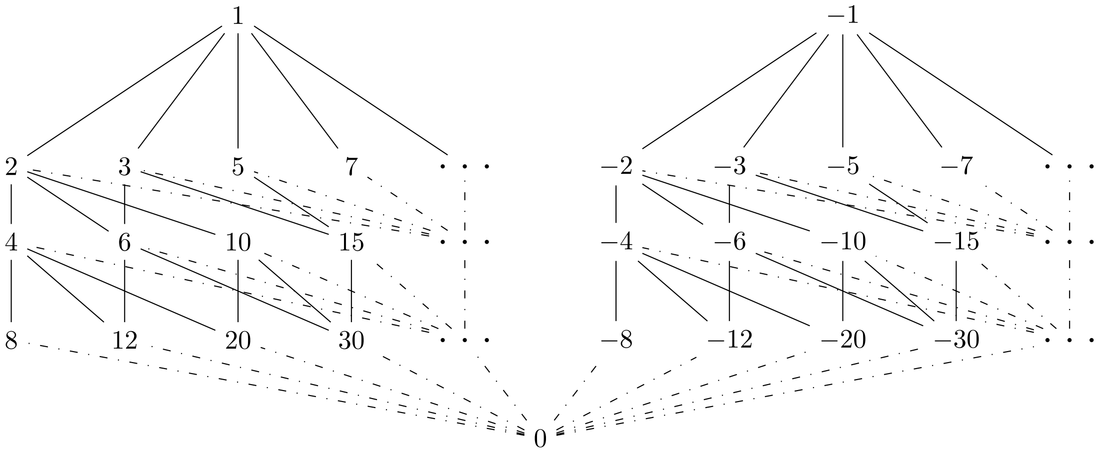


Figure 2: Hasse Diagram of the poset in Example 10.

The following is an example, once again, of a pseudo-monometric on $(\mathcal{X}, \mathrm{B}_{\preceq})$. However, the considered set $\mathcal{X}$ is now a countable one.

Example 10. *Consider the set of all integers $\mathbb{Z}$. Let $(\mathbb{Z}, \preceq)$ be a partially ordered set defined as $x \preceq y \iff$ there exists a $z \in \mathbb{Z}$ such that $|z|.y = x$. The Hasse diagram of the partially ordered set is given in Fig. 2. Now, define an order-preserving mapping $f : \mathbb{Z} \to [0, 1]$ as follows:*

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \ , \\ 1, & \text{if } x = -1 \text{ or } x = 1 \ , \\ 1 - \sum_{n=1}^{|I_x|} \dfrac{1}{2^n}, & \text{otherwise} \ . \end{cases}$$

where $|I_x|$ denotes the sum of powers of primes in the prime factorisation of x. For instance, $|I_{12}| = 2+1 = 3$ since $12 = 2^{\mathbf{2}} \times 3^{\mathbf{1}}$, and $|I_{-30}| = 1+1+1 = 3$ since $-30 = (-1) \times 2^{\mathbf{1}} \times 3^{\mathbf{1}} \times 5^{\mathbf{1}}$.

Consider the fuzzy implication $I(x, y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 1 \ , \\ y - x, & \text{if } x < y \ , \\ 0, & \text{otherwise} \ . \end{cases}$.

Then d_I^* on $\mathbb{Z}$ is given by

$$d_I^*(x,y) = \begin{cases} 0, & \text{if } x = y \ , \\ 1, & \text{if } x \ \text{ or } \ y \in \{1,-1,0\} \ , \\ \sum\limits_{n=\min(|I_x|,|I_y|)+1}^{\max(|I_x|,|I_y|)} \frac{1}{2^n}, & \text{otherwise} \ . \end{cases} .$$

By Theorem 6, d_I^* is indeed a pseudo-monometric w.r.t. $\mathrm{B}_{\preceq}$. *In fact, it is a monometric on* $\mathrm{B}_{\preceq}$ *as I satisfies* (1).

The following are examples of pseudo-monometrics on $(\mathcal{X}, \mathrm{B}_{\preceq})$, where $\mathcal{X}$ is an uncountable set.

Example 11. *Let $\mathcal{X} = [0,1]$ and let $t_0 \in]0,1[$ be arbitrary but fixed. Consider the poset $(\mathcal{X}, \preceq)$ whose Hasse diagram is given in Fig. 1 (b). Let $(\mathcal{X}, \mathrm{B}_{\preceq})$ be the beset obtained from $\preceq$ through* (6).

Now, define a mapping $f : \mathcal{X} \to]0,1[$ as in Table 4 (a). Note that the second column should be read as f mapping the entire open interval $]0, 1-t_0[$ to 0.4 and similarly $f(]1-t_0, 1[) = 0.4$.

$\mathcal{X}$	0	$]0,1-t_0[$	$]1-t_0,1[$	$1-t_0$	1
f	0.2	0.4	0.4	0.8	1

d_I^*	0	$y \in]0,1-t_0[$	$y \in]1-t_0,1[$	$1-t_0$	1
0	0	0.88	0.88	0.96	1
$x \in]0,1-t_0[$	0.88	0	0.76	0.92	1
$x \in]1-t_0,1[$	0.88	0.76	0	0.92	1
$1-t_0$	0.96	0.92	0.92	0	1
1	1	1	1	1	0

Table 4: (a) The mapping f (b) The pairwise distance matrix on $\mathcal{X}$ under d_I^*

Consider $I = I_{\mathbf{RC}}$. We obtain the distance function d_I^* on $\mathcal{X}$ as given in Table 4 (b). Clearly, d_I^* is a pseudo-monometric w.r.t. $\mathrm{B}_{\preceq}$. *In fact, it is a monometric on* $\mathrm{B}_{\preceq}$ *as $I_{\mathbf{RC}}$ satisfies* (1).

Example 12. *Consider the partially ordered set $(\mathcal{P}(\mathbb{N}), \subseteq)$, where $\mathcal{P}(\mathbb{N})$ denotes the power set of the set of natural numbers, partially ordered through inclusion. The Hasse diagram of the corresponding partially ordered set is given in Fig. 3. Now, define an order-preserving mapping $f : \mathcal{P}(\mathbb{N}) \to [0,1]$ as follows:*

$$f(X) = \begin{cases} 0, & \text{if } X = \phi \ , \\ 1, & \text{if } X = \mathbb{N} \ , \\ \sum\limits_{n=1}^{|X|} \frac{1}{2^n}, & \text{otherwise} \ . \end{cases}$$

Consider the fuzzy implication $I(x,y) = \begin{cases} 1, & \text{if } x = 0 \ \text{or } y = 1 \ , \\ y - x, & \text{if } x < y \ , \\ 0, & \text{otherwise} \ . \end{cases} .$

Then d_I^* on $\mathcal{P}(\mathbb{N})$ is given by

$$d_I^*(X,Y) = \begin{cases} 0, & \text{if } X = Y \ , \\ 1, & \text{if } X \ \text{ or } Y \in \{\phi, \mathbb{N}\} \ , \\ \sum\limits_{n=\min(|X|,|Y|)+1}^{\max(|X|,|Y|)} \frac{1}{2^n}, & \text{otherwise} \ . \end{cases} .$$

Once again, by Theorem 6, d_I^* is a pseudo-monometric w.r.t. $\mathrm{B}_{\subseteq}$. *In fact, it is a monometric on* $\mathrm{B}_{\preceq}$ *as $I_{\mathbf{RC}}$ satisfies* (1).

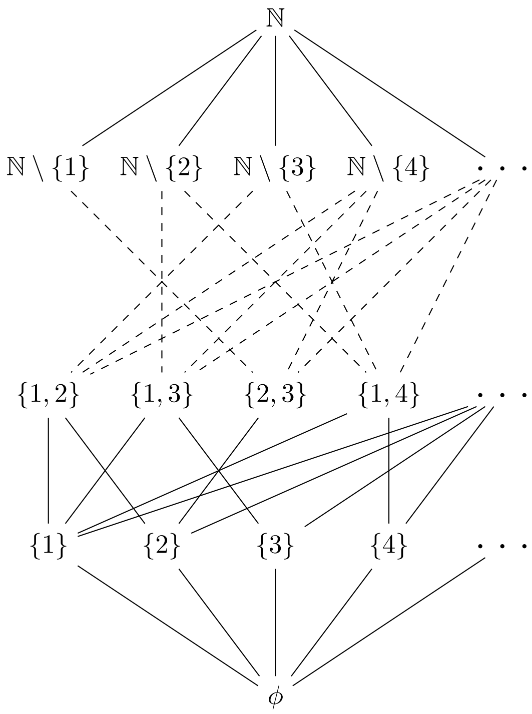


Figure 3: Hasse Diagram of the poset in Example 12

5. Besets Obtainable from a Poset: A Characterisation

While Theorem 6 depicts the existence of pseudo-monometrics on besets obtained from partially ordered sets, the existence of a pseudo-monometric on an arbitrary beset is not clear.

In the following results, by providing a characterisation of betweenness sets obtained from a bounded below poset, we illustrate the scope and applicability of Theorem 6.

Theorem 7. *Let* $(\mathcal{X}, \mathrm{B})$ *be a betweenness set such that* B *satisfies the following property with a special element* $e \in \mathcal{X}$*: whenever* $x \neq y \neq z \in \mathcal{X}$*,*

$$(x, y, z) \in \mathrm{B} \iff \begin{cases} & \{(e, x, y), (e, y, z)\} \subset \mathrm{B} \\ or & \{(e, y, x), (e, z, y)\} \subset \mathrm{B} \ . \end{cases} \quad (8)$$

Let us define the relation $x \preceq_e y \iff (e, x, y) \in \mathrm{B}$*. Then the following are true:*

(i) $(\mathcal{X}, \preceq_e)$ *is a poset bounded below by* e*.*

(ii) *The natural betweenness obtained from* $\preceq_e$ *coincides with* B*, i.e.,* $\mathrm{B} = \mathrm{B}_{\preceq_e}$*.*

Proof. (i) From Remark 5 (iii), we see that $\preceq_e$ is a partial order and since $(e, e, x) \in \mathrm{B}$ for any $x \in \mathcal{X}$, $e \preceq_e x$. Hence, the poset $(\mathcal{X}, \preceq_e)$ is bounded below by e.

(ii) Let us assume that $(x, y, z) \in \mathrm{B}$. Then, by (8), $(e, x, y), (e, y, z) \in \mathrm{B}$ or $(e, y, x), (e, z, y) \in \mathrm{B}$. This implies $x \preceq_e y \preceq_e z$ or $z \preceq_e y \preceq_e x$. Hence, $(x, y, z) \in \mathrm{B}_{\preceq_e}$ which implies $\mathrm{B} \subseteq \mathrm{B}_{\preceq_e}$.
Now, let us assume that $(x, y, z) \in \mathrm{B}_{\preceq_e}$. Then, $(e, x, y), (e, y, z) \in \mathrm{B}_{\preceq_e}$ or $(e, y, x), (e, z, y) \in \mathrm{B}_{\preceq_e}$, which implies $x \preceq_e y$ and $y \preceq_e z$ or $y \preceq_e x$ and $z \preceq_e y$, respectively. Hence, $(e, x, y), (e, y, z) \in \mathrm{B}$ or $(e, y, x), (e, z, y) \in \mathrm{B}$, clearly implying by (8) that $(x, y, z) \in \mathrm{B}$.

□

The final result of this section completely characterises the betweenness relations that can be obtained through (6) from a given partial order.

Theorem 8. *Let $(\mathcal{X}, \mathrm{B})$ be a betweenness set. The betweenness relation* B *is induced from a bounded below poset iff there exists an* $e \in \mathcal{X}$ *such that* (8) *is true.*

Proof. ($\Longrightarrow$) Let us assume that B is induced from a poset $(\mathcal{X}, \preceq)$ that is bounded below by e. Consider $x \neq y \neq z$.

Let $(x, y, z) \in \mathrm{B} = \mathrm{B}_{\preceq}$. Then, $x \preceq y \preceq z$ or $z \preceq y \preceq x$. Since e is the bottom element, we have $e \preceq x \preceq y \preceq z$ or $e \preceq z \preceq y \preceq x$. This implies (e, x, y) and $(e, y, z) \in \mathrm{B}$ or (e, z, y) and $(e, y, x) \in \mathrm{B}$.

Conversely, let us assume that $\{(e, x, y), (e, y, z)\} \subset \mathrm{B}$. This implies $e \preceq_e x \preceq_e y$ and $e \preceq_e y \preceq_e z \implies x \preceq_e y \preceq_e z$. Hence, $(x, y, z) \in \mathrm{B}$. Similarly, we can show that $\{(e, y, x), (e, z, y)\} \subset \mathrm{B} \implies (x, y, z) \in \mathrm{B}$.

($\Longleftarrow$) It follows from Theorem 7. □

Since Theorem 6 asserts the existence of pseudo-monometric on the betweenness relation induced from partially ordered sets, we have the following corollary.

Corollary 4. *Let $(\mathcal{X}, \mathrm{B})$ be a betweenness set such that* B *satisfies* (8) *with a special element* $e \in \mathcal{X}$. *Then there exists a pseudo-monometric on $(\mathcal{X}, \mathrm{B})$.*

6. Concluding Remarks:

In this submission, we introduced and investigated the distance function d_I, defined as in 3, using a non-associative, non-symmetric, and mixed-monotonic operator: fuzzy implication. The role of (S, I)-transitivity in the characterisation of fuzzy implications that yield a metric through d_I has been noted in this work. However, note that, so far the authors were unable to find a QL-implication that did not satisfy $(S_{\mathbf{LK}}, I)$-transitivity, leading one to suspect if this functional inequality could somehow quintessentially capture the geometry and flavour of this family of fuzzy implications, whose characterisation has remained both open and challenging. For these reasons, we believe that it is worthwhile to study the (SIT) functional inequality in its own right.

By demonstrating the role of d_I in defining pseudo-monometrics on certain betweenness relations, fuzzy implications, and indeed d_I, suggest themselves to be both a natural choice and a rich source for obtaining such distance functions. While our result shows the existence of pseudo-monometrics on betweenness relations obtained precisely from partially ordered sets, it would be worthwhile to investigate the applicability of d_I on other betweenness relations. Some initial steps in this direction are already underway.

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