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(U,N)-implications and their characterizations ☆

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Abstract

Recently, we have presented characterizations of (S,N)-implications generated from t-conorms and continuous (strict, strong) negations. Uninorms were introduced by Yager and Rybalov in 1996 as a generalization of t-norms and t-conorms, thus they are another fertile source based on which one can define fuzzy implications. (U,N)-implications are a generalization of (S,N)-implications, where a t-conorm S is replaced by a (disjunctive) uninorm U. In this work we present characterizations of (U,N)-implications obtained from disjunctive uninorms and continuous negations as well as (U,N)-operations defined from uninorms and continuous negations.

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1. Introduction

(U,N)-implications are a generalization of (S,N)-implications, where a t-conorm S is replaced by a uninorm U. A similar generalization of R-implications from the setting of t-norms to the setting of uninorms has been done by De Baets and Fodor [6]. Ruiz and Torrens [18,20] have investigated, quite extensively, fuzzy implications generated from uninorms [19] and their distributivity.

Despite this interest, fuzzy implications obtained from uninorms are yet to be characterized. Recently, some characterizations of (S,N)-implications were given by the authors in [2]. In this work, along similar lines, we investigate and characterize (U,N)-operations and (U,N)-implications obtained from continuous negations N.

After introducing the necessary preliminaries on the basic fuzzy logic operations, we list out some of the most desirable—but relevant to this work—properties of fuzzy implications and investigate their interdependencies. Following this, we discuss the class of (U,N)-operations and the properties they satisfy. Finally, based on the above analysis, we derive characterizations for (U,N)-operations and (U,N)-implications generated from continuous negations.

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2. Fuzzy negations and uninorms

To make this work self-contained, we briefly mention some of the concepts and results employed in the rest of the paper.

Definition 2.1 (*see Fodor and Roubens [9, p. 3]; Klement et al. [13, Definition 11.3]*). A decreasing function $N : [0,1] \to [0,1]$ is called a *fuzzy negation*, if $N(0) = 1$, $N(1) = 0$. A fuzzy negation N is called

(i) *strict*, if it is strictly decreasing and continuous,
(ii) *strong*, if it is an involution, i.e., $N(N(x)) = x$ for all $x \in [0,1]$.

It is well known that if $[a,b]$ and $[c,d]$ are two closed subintervals of the extended real line $[-\infty,+\infty]$ and $f : [a,b] \to [c,d]$ is a monotone function, then the set of discontinuous points of f is a countable subset of $[a,b]$ (see [17]). In this case we will use the pseudo-inverse $f^{(-1)} : [c,d] \to [a,b]$ of a decreasing and non-constant function f defined by (see [13, Section 3.1])

$$f^{(-1)}(y) = \sup\{x \in [a,b] \mid f(x) > y\}, \quad y \in [c,d].$$

Lemma 2.2 (*Baczyński and Jayaram [2, Proposition 3.13]*). *If N is a continuous fuzzy negation, then the function $\mathfrak{N} : [0,1] \to [0,1]$ defined by*

$$\mathfrak{N}(x) = \begin{cases} N^{(-1)}(x) & \text{if } x \in (0,1], \\ 1 & \text{if } x = 0 \end{cases}$$

is a strictly decreasing fuzzy negation. Moreover,

$$\begin{aligned} &\mathfrak{N}^{(-1)} = N, \\ &N \circ \mathfrak{N} = \mathrm{id}_{[0,1]}, \\ &\mathfrak{N} \circ N\,|_{\mathrm{Ran}(\mathfrak{N})} = \mathrm{id}_{\mathrm{Ran}(\mathfrak{N})}. \end{aligned} \tag{1}$$

Lemma 2.3 (*Baczyński and Jayaram [2, Proposition 3.8]*). *If N_1, N_2 are two fuzzy negations such that $N_1 \circ N_2 = \mathrm{id}_{[0,1]}$, then*

(i) *N_1 is a continuous fuzzy negation,*
(ii) *N_2 is a strictly decreasing fuzzy negation,*
(iii) *N_2 is a continuous fuzzy negation if and only if N_1 is a strictly decreasing fuzzy negation. In both cases $N_1 = N_2^{-1}$.*

Definition 2.4 (*see Yager and Rybalov [24], Fodor et al. [10]*). An associative, commutative and increasing operation $U : [0,1]^2 \to [0,1]$ is called a *uninorm*, if there exists $e \in [0,1]$, called the neutral element of U, such that

$$U(e,x) = U(x,e) = x, \quad x \in [0,1].$$

Remark 2.5 (*cf. Fodor et al. [10]*).

(i) If $e = 0$, then U is a t-conorm and if $e = 1$, then U is a t-norm.
(ii) The neutral element e corresponding to a uninorm U is unique.
(iii) For any uninorm U we have $U(0,1) \in \{0,1\}$.
(iv) A uninorm U such that $U(0,1) = 0$ is called *conjunctive* and if $U(0,1) = 1$, then it is called *disjunctive*.
(v) The structure of a uninorm U with the neutral element $e \in (0,1)$ is always the following. It is like a t-norm on the square $[0,e]^2$, like a t-conorm on the square $[e,1]^2$ and it takes values between the minimum and the maximum in the other cases.

There are several different classes of uninorms introduced in the literature. We only mention relevant details and results, which will be useful in the sequel, connected with the three main classes of uninorms.

Uninorms verifying that both functions $U(\cdot, 0)$ and $U(\cdot, 1)$ are continuous except at the point e, also referred to as *pseudo-continuous* uninorms, were characterized by Fodor et al. [10], as follows (see also [6]).

Theorem 2.6. *For a function* $U : [0,1]^2 \to [0,1]$ *the following statements are equivalent*:

(i) *U is a conjunctive uninorm with the neutral element* $e \in (0,1)$, *such that the function* $U(\cdot, 1)$ *is continuous on* $[0, e)$.
(ii) *There exist a t-norm T and a t-conorm S such that*

$$U(x,y) = \begin{cases} e \cdot T\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } x, y \in [0,e], \\ e + (1-e)\cdot S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & \text{if } x, y \in [e,1], \\ \min(x,y) & \text{otherwise}, \end{cases} \quad x, y \in [0,1]. \tag{2}$$

Theorem 2.7. *For a function* $U : [0,1]^2 \to [0,1]$ *the following statements are equivalent*:

(i) *U is a disjunctive uninorm with the neutral element* $e \in (0,1)$, *such that the function* $U(\cdot, 0)$ *is continuous on* $(e, 1]$.
(ii) *There exist a t-norm T and a t-conorm S such that*

$$U(x,y) = \begin{cases} e \cdot T\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } x, y \in [0,e], \\ e + (1-e)\cdot S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & \text{if } x, y \in [e,1], \\ \max(x,y) & \text{otherwise}, \end{cases} \quad x, y \in [0,1]. \tag{3}$$

The class of uninorms of the form (2) is denoted by $\mathcal{U}_{\mathbf{Min}}$, while the class of uninorms of the form (3) is denoted by $\mathcal{U}_{\mathbf{Max}}$. Note that, even if a t-norm T, a t-conorm S and $e \in (0,1)$ are fixed, a pseudo-continuous uninorm is not uniquely defined—it can be conjunctive or disjunctive. If U is a conjunctive (disjunctive) uninorm, then we will write $U^{\mathbf{c}}_{T,S,e}$ ($U^{\mathbf{d}}_{T,S,e}$, respectively).

A uninorm U such that $U(x,x) = x$ for all $x \in [0,1]$ is said to be an *idempotent uninorm*. The class of all idempotent uninorms will be denoted by $\mathcal{U}_{\mathbf{Idem}}$. Martín et al. [14] have characterized all idempotent uninorms, which subsumes the results of De Baets [5], who first characterized the class of left-continuous and right-continuous idempotent uninorms.

Uninorms that can be represented as in Theorem 2.8 are called *representable uninorms* and this class will be denoted by $\mathcal{U}_{\mathbf{Rep}}$.

Theorem 2.8 (*Fodor et al. [10]*). *For a function* $U : [0,1]^2 \to [0,1]$ *the following statements are equivalent*:

(i) *U is a strictly increasing uninorm, continuous on* $(0,1)^2$ *with the neutral element* $e \in (0,1)$, *such that U is self-dual, except in points* $(0,1)$ *and* $(1,0)$, *with respect to a strong negation N with the fixed point e, i.e.,*

$$U(x,y) = N(U(N(x), N(y))), \quad x, y \in [0,1]^2 \setminus \{(0,1),(1,0)\}.$$

(ii) *U has a continuous additive generator, i.e., there exists a continuous and strictly increasing function* $h : [0,1] \to [-\infty, \infty]$, *such that* $h(0) = -\infty$, $h(e) = 0$ *for* $e \in (0,1)$ *and* $h(1) = \infty$, *which is uniquely determined up to a positive multiplicative constant, such that*

$$U(x,y) = \begin{cases} 0 & \text{if } (x,y) \in \{(0,1),(1,0)\}, \\ h^{-1}(h(x)+h(y)) & \text{if } (x,y) \in [0,1]^2 \setminus \{(0,1),(1,0)\}, \end{cases}$$

or

$$U(x,y) = \begin{cases} 1 & \text{if } (x,y) \in \{(0,1),(1,0)\}, \\ h^{-1}(h(x)+h(y)) & \text{if } (x,y) \in [0,1]^2 \setminus \{(0,1),(1,0)\}. \end{cases}$$

Particular examples of uninorms as well as the other different classes of uninorms can be found in the recent literature (see [6,7,10]).

3. Fuzzy implications

In this work the following equivalent definition proposed by Fodor and Roubens [9, Definition 1.15], (see also [12, p. 50]) is used.

Definition 3.1. A function $I : [0, 1]^2 \to [0, 1]$ is called a *fuzzy implication*, if it satisfies, for all $x, y, z \in [0, 1]$, the following conditions:

$$\text{if } x \leqslant y, \text{ then } I(x, z) \geqslant I(y, z), \tag{I1}$$

$$\text{if } y \leqslant z, \text{ then } I(x, y) \leqslant I(x, z), \tag{I2}$$

$$I(0, 0) = 1, \tag{I3}$$

$$I(1, 1) = 1, \tag{I4}$$

$$I(1, 0) = 0. \tag{I5}$$

The set of all fuzzy implications will be denoted by $\mathcal{FI}$.

Directly from the above definition we see that each fuzzy implication *I* satisfies the following *left* and *right boundary conditions*, respectively:

$$I(0, y) = 1, \quad y \in [0, 1], \tag{LB}$$

$$I(x, 1) = 1, \quad x \in [0, 1]. \tag{RB}$$

Therefore, *I* satisfies also the *normality condition*

$$I(0, 1) = 1. \tag{NC}$$

Consequently, every fuzzy implication restricted to the set $\{0, 1\}^2$ coincides with the classical implication. In the following we list out some of the desirable properties of fuzzy implications (cf. [9,21]):

Definition 3.2. Let $I \in \mathcal{FI}$ and *N* be a fuzzy negation. *I* is said to satisfy

(i) the *left neutrality property*, if

$$I(1, y) = y, \quad y \in [0, 1], \tag{NP}$$

(ii) the *exchange principle*, if

$$I(x, I(y, z)) = I(y, I(x, z)), \quad x, y, z \in [0, 1], \tag{EP}$$

(iii) the *law of left contraposition* with respect to *N*, if

$$I(N(x), y) = I(N(y), x), \quad x, y \in [0, 1], \tag{L-CP}$$

(iv) the *law of right contraposition* with respect to *N*, if

$$I(x, N(y)) = I(y, N(x)), \quad x, y \in [0, 1], \tag{R-CP}$$

(v) the *law of contraposition* with respect to *N*, if

$$I(x, y) = I(N(y), N(x)), \quad x, y \in [0, 1]. \tag{CP}$$

If *I* satisfies the law of (left, right) contraposition with respect to *N*, then we also denote this by CP(N) (respectively, by L-CP(N) and R-CP(N)).

Lemma 3.3 (*Baczyński and Jayaram [2, Lemma 3.2]*). *Let $I : [0,1]^2 \to [0,1]$ be any function which satisfies* (R-CP) *with respect to a continuous fuzzy negation N. Then I satisfies* (I1) *if and only if I satisfies* (I2).

Definition 3.4. Let $I : [0,1]^2 \to [0,1]$ be any function and $\alpha \in [0,1)$. If the function $N_I^\alpha : [0,1] \to [0,1]$ given by

$$N_I^\alpha(x) = I(x, \alpha), \quad x \in [0,1]$$

is a fuzzy negation, then it is called the *natural negation of I with respect to* α.

It should be noted that for any $I \in \mathcal{FI}$ we have (I5), so for $\alpha = 0$ we have the natural negation $N_I = N_I^0$ of I (see [2]). Also α should be less than 1 for fuzzy implications, since $I(1,1) = 1$ by (I4).

Lemma 3.5. *Let $I : [0,1]^2 \to [0,1]$ be any function and N_I^α be a fuzzy negation for an arbitrary but fixed $\alpha \in [0,1)$.*

(i) *If I satisfies* (I2), *then I satisfies* (I5).
(ii) *If I satisfies* (I2) *and* (EP), *then I satisfies* (I3) *if and only if I satisfies* (I4).
(iii) *If I satisfies* (EP), *then I satisfies* R-CP(N_I^α).

Proof. (i) Since N_I^α is a fuzzy negation and I satisfies (I2) we get

$$I(1,0) \leqslant I(1,\alpha) = N_I^\alpha(1) = 0.$$

(ii) Let I satisfy (I2) and (EP). If I satisfies (I4), then

$$1 = I(1,1) = I(1, N_I^\alpha(0)) = I(1, I(0,\alpha)) = I(0, I(1,\alpha)) = I(0, N_I^\alpha(1)) = I(0,0),$$

i.e., I satisfies (I3). The reverse implication can be shown similarly.

(iii) Since I satisfies (EP), we have

$$I(x, N_I^\alpha(y)) = I(x, I(y,\alpha)) = I(y, I(x,\alpha)) = I(y, N_I^\alpha(x)), \quad x, y \in [0,1],$$

i.e., I satisfies R-CP(N_I^α). □

Lemma 3.6. *Let $I \in \mathcal{FI}$ and N_I^α be a fuzzy negation for an arbitrary but fixed $\alpha \in [0,1)$. If N is a fuzzy negation such that $N_I^\alpha \circ N = \mathrm{id}_{[0,1]}$ and I satisfies* (EP), *then I satisfies* L-CP(N).

Proof. By our assumptions, we get

$$\begin{aligned} I(N(x), y) &= I(N(x), N_I^\alpha \circ N(y)) = I(N(x), I(N(y), \alpha)) \\ &= I(N(y), I(N(x), \alpha)) = I(N(y), N_I^\alpha \circ N(x)) = I(N(y), x), \end{aligned}$$

for all $x, y \in [0,1]$, so I satisfies L-CP(N). □

Remark 3.7. Under the assumptions of Lemma 3.6, we have:

(i) If N_I^α is a strict negation, then I satisfies L-CP($(N_I^\alpha)^{-1}$).
(ii) If N_I^α is a strong negation, then I satisfies L-CP(N_I^α) and CP(N_I^α).

4. (S,N)-implications and their characterizations

In this section, we give a brief introduction to one of the families of fuzzy implications that is very well studied in the literature.

Definition 4.1 (*cf. Trillas and Valverde [21], Fodor and Roubens [9], Alsina and Trillas [1], Baczyński and Jayaram [2]*). A function $I : [0,1]^2 \to [0,1]$ is called an (S,N)-*implication*, if there exist a t-conorm S and a fuzzy negation N such that

$$I(x, y) = S(N(x), y), \quad x, y \in [0,1].$$

If N is a strong negation, then I is called an S-*implication*.

The following characterization of some subclasses of (S,N)-implications is from [2], which is an extension of a result in [21].

Theorem 4.2 (*Baczyński and Jayaram [2]*). *For a function* $I : [0,1]^2 \to [0,1]$ *the following statements are equivalent*:

(i) *I is an* (S,N)-*implication generated from some t-conorm S and some continuous* (*strict*, *strong*) *fuzzy negation N*.
(ii) *I satisfies* (I1), (EP) *and the function* N_I *is a continuous* (*strict*, *strong*) *fuzzy negation*.

Remark 4.3 (*cf. Baczyński and Jayaram [2]*).

(i) The representation of (S,N)-implications in Theorem 4.2 is unique.
(ii) The axioms in Theorem 4.2 are independent from each other.
(iii) In Theorem 4.2, the property (I1) can be substituted by (I2).

5. (U,N)-operations and (U,N)-implications

A natural generalization of (S,N)-implications in the uninorm framework is to consider a uninorm in the place of a t-conorm in Definition 4.1.

Definition 5.1. A function $I : [0,1]^2 \to [0,1]$ is called a (U,N)-*operation*, if there exist a uninorm U and a fuzzy negation N such that

$$I_{U,N}(x, y) = U(N(x), y), \quad x, y \in [0,1]. \tag{4}$$

If I is a (U,N)-operation generated from a uninorm U and a negation N, then we will often denote it by $I_{U,N}$.

Firstly, observe that if $e = 0$, then U is a t-conorm and $I_{U,N}$, as an (S,N)-implication, is always a fuzzy implication, whose properties and characterizations are quite well known. If $e = 1$, then U is a t-norm and $I_{U,N}$ is not a fuzzy implication, since (I3) is violated. Therefore, we consider only the situation when U is a uninorm with the neutral element $e \in (0,1)$.

Proposition 5.2. *If* $I_{U,N}$ *is a* (U,N)-*operation obtained from a uninorm U with* $e \in (0,1)$ *as its neutral element*, *then*

(i) $I_{U,N}$ *satisfies* (I1), (I2), (I5), (NC) *and* (EP),
(ii) $N^e_{I_{U,N}} = N$,
(iii) $I_{U,N}$ *satisfies* R-CP(N),
(iv) *if N is strict*, *then* $I_{U,N}$ *satisfies* L-CP(N^{-1}),
(v) *if N is strong*, *then* $I_{U,N}$ *satisfies* CP(N).

Proof. (i) By the monotonicity of U and N we get that $I_{U,N}$ satisfies (I1) and (I2). Moreover, it can be easily verified that $I_{U,N}$ satisfies (I5) and (NC). Finally, from the associativity and the commutativity of U we have also (EP).

(ii) For any $x \in [0,1]$, with $e \in (0,1)$ being the identity of U, we have

$$N^e_{I_{U,N}}(x) = I_{U,N}(x, e) = U(N(x), e) = N(x).$$

(iii) Since $I_{U,N}$ satisfies (EP), from Lemma 3.5(iii) with $\alpha = e$ we have that $I_{U,N}$ satisfies R-CP(N).

(iv) If N is a strict negation, then from Remark 3.7(i) we see that $I_{U,N}$ satisfies L-CP(N^{-1}).

(v) If N is a strong negation, then from Remark 3.7(ii) we see that $I_{U,N}$ satisfies CP(N). □

If $e \in (0,1)$, then not for every uninorm U the (U,N)-operation is a fuzzy implication. The next result characterizes those (U,N)-operations, which satisfy (I3) and (I4).

Theorem 5.3 (*cf. De Baets and Fodor [6, p. 98]*). *For a uninorm U with neutral element* $e \in (0,1)$ *the following statements are equivalent*:

(i) *The function* $I_{U,N}$ *as defined in* (4) *is a fuzzy implication*.
(ii) *U is a disjunctive uninorm*, *i.e.*, $U(0,1) = U(1,0) = 1$.

Proof. (i) $\Longrightarrow$ (ii) If $I_{U,N}$ as defined in (4) is a fuzzy implication, then from (I3) we have

$$U(0,1) = U(1,0) = I_{U,N}(0,0) = 1.$$

(ii) $\Longrightarrow$ (i) Assume that $U(0,1) = 1$. From Proposition 5.2 it is enough to show only (I3) and (I4):

$$I_{U,N}(0,0) = U(N(0),0) = U(1,0) = 1, \qquad I_{U,N}(1,1) = U(N(1),1) = U(0,1) = 1. \qquad \square$$

Remark 5.4. Following the terminology used by Mas et al. [16] for the QL-implications, only if the (U,N)-operation $I_{U,N}$ is a fuzzy implication we use the term (U,N)-*implication*.

Example 5.5. In the following, we give examples of (U,N)-implications obtained using the classical strong negation $N_{\mathbf{C}}(x) = 1 - x$ for all $x \in [0,1]$, and for different uninorms. Note that $I_{\mathbf{KD}}$ is the Kleene–Dienes implication given by

$$I_{\mathbf{KD}}(x,y) = \max(1-x, y), \quad x, y \in [0,1].$$

(i) Let us consider the disjunctive uninorm $U_{\mathbf{LK}}$ from the class $\mathcal{U}_{\mathbf{Max}}$ generated by the triple $(T_{\mathbf{LK}}, S_{\mathbf{LK}}, 0.5)$, where $T_{\mathbf{LK}}$ denotes the Łukasiewcz t-norm

$$T_{\mathbf{LK}}(x,y) = \max(x+y-1, 0), \quad x, y \in [0,1],$$

and $S_{\mathbf{LK}}$ denotes the Łukasiewcz t-conorm

$$S_{\mathbf{LK}}(x,y) = \min(x+y, 1), \quad x, y \in [0,1].$$

Then

$$I_{U_{\mathbf{LK}},N_{\mathbf{C}}}(x,y) = \begin{cases} \max(y-x+0.5, 0) & \text{if } \max(1-x,y) \leqslant 0.5, \\ \min(y-x+0.5, 1) & \text{if } \max(1-x,y) > 0.5, \\ I_{\mathbf{KD}}(x,y) & \text{otherwise}, \end{cases} \quad x, y \in [0,1].$$

Fig. 1(a) gives the plot of $I_{U_{\mathbf{LK}},N_{\mathbf{C}}}$.

(ii) Let us consider the disjunctive uninorm $U_{\mathbf{P}}$ from the class $\mathcal{U}_{\mathbf{Max}}$ generated by the triple $(T_{\mathbf{P}}, S_{\mathbf{P}}, 0.5)$, where $T_{\mathbf{P}}$ denotes the algebraic product t-norm

$$T_{\mathbf{P}}(x,y) = xy, \quad x, y \in [0,1],$$

and $S_{\mathbf{P}}$ denotes the probabilistic sum t-conorm

$$S_{\mathbf{P}}(x,y) = x + y - xy, \quad x, y \in [0,1].$$

Then

$$I_{U_{\mathbf{P}},N_{\mathbf{C}}}(x,y) = \begin{cases} 2y - 2xy & \text{if } \max(1-x,y) \leqslant 0.5, \\ 1 - 2x + 2xy & \text{if } \max(1-x,y) > 0.5, \\ I_{\mathbf{KD}}(x,y) & \text{otherwise}, \end{cases} \quad x, y \in [0,1].$$

Fig. 1(b) gives the plot of $I_{U_{\mathbf{P}},N_{\mathbf{C}}}$.

(iii) Let us consider the disjunctive uninorm $U_{\mathbf{M}}$ from the class $\mathcal{U}_{\mathbf{Max}}$ generated from the triple $(T_{\mathbf{M}}, S_{\mathbf{M}}, 0.5)$, where $T_{\mathbf{M}}$ denotes the minimum t-norm

$$T_{\mathbf{M}}(x,y) = \min(x,y), \quad x, y \in [0,1],$$

and $S_{\mathbf{M}}$ denotes the maximum t-conorm

$$S_{\mathbf{M}}(x,y) = \max(x,y), \quad x, y \in [0,1].$$

Observe, that $U_{\mathbf{M}}$ is also an idempotent uninorm. Then

$$I_{U_{\mathbf{M}},N_{\mathbf{C}}}(x,y) = \begin{cases} \min(1-x,y) & \text{if } \max(1-x,y) \leqslant 0.5, \\ I_{\mathbf{KD}}(x,y) & \text{otherwise}, \end{cases} \quad x, y \in [0,1].$$

Fig. 1(c) gives the plot of $I_{U_{\mathbf{M}},N_{\mathbf{C}}}$.

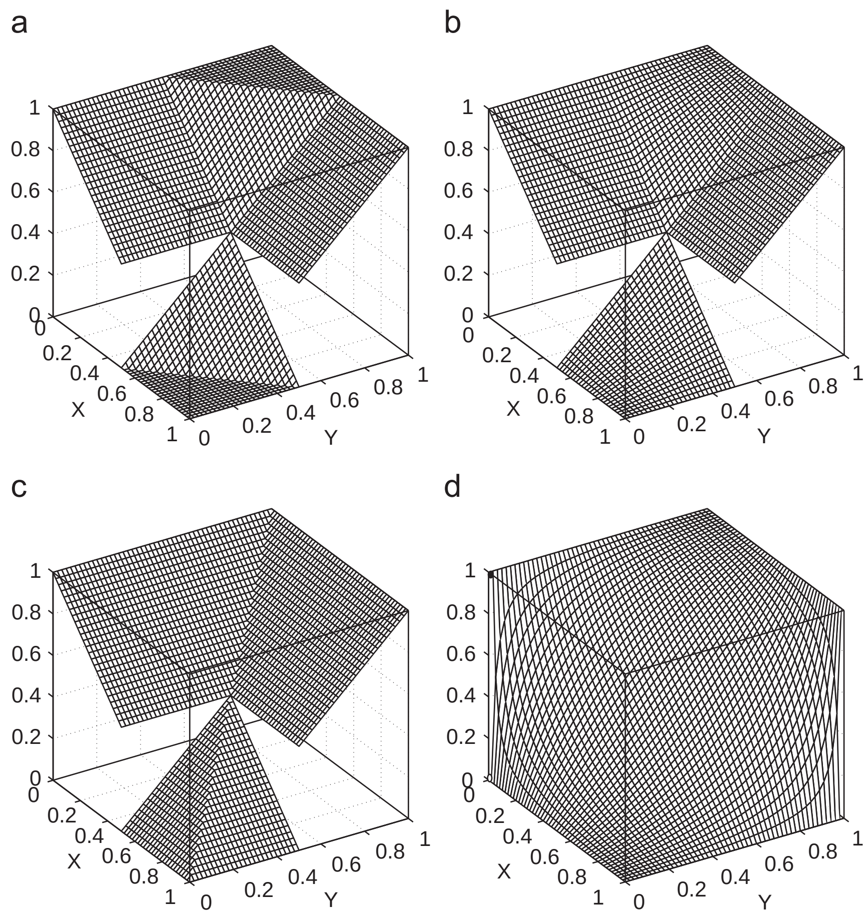

Fig. 1. Plots of (U,N)-implications from Example 5.5. (a) $I_{U_{\mathbf{LK}},N_{\mathbf{C}}}$, (b) $I_{U_{\mathbf{P}},N_{\mathbf{C}}}$, (c) $I_{U_{\mathbf{M}},N_{\mathbf{C}}}$ and (d) $I_{U^{\mathbf{d}}_{h_1},N_{\mathbf{C}}}$.

(iv) If we consider the additive generator $h_1(x) = \ln(x/(1-x))$, then we get the following disjunctive representable uninorm:

$$U^{\mathbf{d}}_{h_1}(x, y) = \begin{cases} 1 & \text{if } (x, y) \in \{(0, 1), (1, 0)\}, \\ \dfrac{xy}{(1-x)(1-y)+xy} & \text{otherwise,} \end{cases} \qquad x, y \in [0, 1].$$

In this case $e = \frac{1}{2}$. Now, we have

$$I_{U^{\mathbf{d}}_{h_1},N_{\mathbf{C}}}(x, y) = \begin{cases} 1 & \text{if } (x, y) \in \{(0, 0), (1, 1)\}, \\ \dfrac{(1-x)y}{x+y-2xy} & \text{otherwise,} \end{cases} \qquad x, y \in [0, 1].$$

Fig. 1(d) gives the plot of $I_{U^{\mathbf{d}}_{h_1},N_{\mathbf{C}}}$.

Lemma 5.6. *Let $I_{U,N}$ be a* (U,N)-*implication obtained from a uninorm U with $e \in (0, 1)$ as its neutral element and a continuous negation N. Let $\alpha \in (0, 1)$ be an arbitrary but fixed number. Then the following statements are equivalent*:

(i) $N^{\alpha}_{I_{U,N}} = N$.
(ii) $\alpha = e$.

Proof. Let $e \in (0, 1)$ be the neutral element of U and $\alpha \in (0, 1)$ be an arbitrary but fixed number.

(i) $\Longrightarrow$ (ii) If $N^{\alpha}_{I_{U,N}} = N$, then since N is continuous there exists e' such that $e = N(e')$ and hence $N^{\alpha}_{I_{U,N}}(e') = I_{U,N}(e', \alpha) = U(N(e'), \alpha) = N(e') = e$. But $U(N(e'), \alpha) = U(e, \alpha) = \alpha$, because e is the neutral element of U, thus $\alpha = e$.

(ii) $\Longrightarrow$ (i) This implication is just Proposition 5.2(ii). □

Remark 5.7. It is interesting to note that, unlike (S,N)-implications, (U,N)-implications do not satisfy the left neutrality property (NP). To see this, let U be a uninorm with the neutral element $e \in (0, 1)$ and N be a fuzzy negation. Then, $I_{U,N}(1, e) = U(N(1), e) = U(0, e) = 0 \neq e$.

6. Characterizations of (U,N)-implications and (U,N)-operations

We start our presentation with the following result.

Proposition 6.1. *Let $I \in \mathcal{FI}$ and N be a fuzzy negation. If we define a binary operation $U_{I,N}$ on $[0, 1]$ as follows*

$$U_{I,N}(x, y) = I(N(x), y), \quad x, y \in [0, 1], \tag{5}$$

then

(i) $U_{I,N}(x, 1) = U_{I,N}(1, x) = 1$ *for all* $x \in [0, 1]$, *in particular* $U_{I,N}(0, 1) = 1$,
(ii) $U_{I,N}$ *is increasing in both the variables*,
(iii) $U_{I,N}$ *is commutative if and only if I satisfies* L-CP(N).

In addition, if I satisfies L-CP(N) with a continuous fuzzy negation N, *then*

(iv) $U_{I,N}$ *is associative if and only if I satisfies* (EP),
(v) *an arbitrary* $\alpha \in (0, 1)$ *is the neutral element of* $U_{I,N}$ *if and only if* $N^{\alpha}_{I} \circ N = \mathrm{id}_{[0,1]}$.

Proof. (i) Let $x \in [0, 1]$. By the boundary condition (RB) of I we have $U_{I,N}(x, 1) = I(N(x), 1) = 1$. Also, $U_{I,N}(1, x) = I(N(1), x) = I(0, x) = 1$ again by (LB) of I.

(ii) That $U_{I,N}$ is increasing in both the variables is a direct consequence of the monotonicity of I and N.

(iii) If $U_{I,N}$ is commutative, then $I(N(x), y) = U_{I,N}(x, y) = U_{I,N}(y, x) = I(N(y), x)$ for all $x, y \in [0, 1]$, i.e., I satisfies L-CP(N). The reverse implication can be obtained by retracing the above steps.

(iv) Let $x, y, z \in [0, 1]$. If I satisfies (EP), then

$$\begin{aligned} U_{I,N}(x, U_{I,N}(y, z)) &= I(N(x), I(N(y), z)) = I(N(x), I(N(z), y)) \\ &= I(N(z), I(N(x), y)) = I(N(I(N(x), y)), z) \\ &= I(N(U_{I,N}(x, y)), z) = U_I(U_{I,N}(x, y), z). \end{aligned}$$

Conversely, if $U_{I,N}$ is associative and N is continuous, then there exists $x', y', z' \in [0, 1]$ such that $x = N(x')$, $y = N(y')$ and $z = N(z')$. Now we obtain

$$\begin{aligned} I(x, I(y, z)) &= I(N(x'), I(N(y'), N(z'))) = U_{I,N}(x', U_{I,N}(y', N(z'))) = U_{I,N}(U_{I,N}(x'y'), N(z')) \\ &= U_{I,N}(U_{I,N}(y', x'), N(z')) = U_{I,N}(y', U_{I,N}(x', N(z'))) = I(y, I(x, z)). \end{aligned}$$

(v) Let $\alpha \in (0, 1)$ be arbitrary but fixed. If α is the neutral element of $U_{I,N}$, then for all $x \in [0, 1]$ we have $x = U_{I,N}(x, \alpha) = I(N(x), \alpha) = N^{\alpha}_{I}(N(x))$. Conversely, if $N^{\alpha}_{I} \circ N = \mathrm{id}_{[0,1]}$, then for any $x \in [0, 1]$ we get $U_{I,N}(\alpha, x) = U_{I,N}(x, \alpha) = I(N(x), \alpha) = N^{\alpha}_{I}(N(x)) = x$, so α is the neutral element of $U_{I,N}$. □

If N^{α}_{I} is a continuous fuzzy negation for an arbitrary but fixed $\alpha \in (0, 1)$, then by Lemma 2.2 and previous results we can consider the modified pseudo-inverse $\mathfrak{N}^{\alpha}_{I}$ given by

$$\mathfrak{N}^{\alpha}_{I}(x) = \begin{cases} \left(N^{\alpha}_{I}\right)^{(-1)}(x) & \text{if } x \in (0, 1], \\ 1 & \text{if } x = 0, \end{cases} \tag{6}$$

as the potential candidate for the fuzzy negation N in (5). Hence, from Lemma 3.6 with $N = \mathfrak{N}_I^\alpha$, we obtain the following result.

Corollary 6.2. *If $I \in \mathcal{FI}$ satisfies* (EP) *and N_I^α, the natural negation of I with respect to an arbitrary but fixed $\alpha \in (0,1)$, is a continuous fuzzy negation, then I satisfies* (L-CP) *with $\mathfrak{N}_I^\alpha$ from* (6).

Hence, if a fuzzy implication I satisfies (EP) and N_I^α is a continuous fuzzy negation for some $\alpha \in (0,1)$, then we conclude that formula (5) can be considered with the modified pseudo-inverse of the natural negation of I given by (6).

Corollary 6.3. *If $I \in \mathcal{FI}$ satisfies* (EP) *and N_I^α is a continuous fuzzy negation with respect to an arbitrary but fixed $\alpha \in (0,1)$, then the function U_I defined by*

$$U_I(x,y) = I(\mathfrak{N}_I^\alpha(x), y), \quad x, y \in [0,1] \tag{7}$$

is a disjunctive uninorm with the neutral element α, where $\mathfrak{N}_I^\alpha$ is as defined in (6).

Now we are ready to formulate the main result of this work.

Theorem 6.4. *For a function $I : [0,1]^2 \to [0,1]$ the following statements are equivalent*:

(i) *I is an* (U,N)*-operation generated from some disjunctive uninorm U with the neutral element $e \in (0,1)$ and some continuous fuzzy negation N.*
(ii) *I is an* (U,N)*-implication generated from some uninorm U with the neutral element $e \in (0,1)$ and some continuous fuzzy negation N.*
(iii) *I satisfies* (I1), (I3), (EP) *and the function N_I^e is a continuous negation for some $e \in (0,1)$.*

Moreover, the representation (4) *of* (U,N)*-implication is unique in this case.*

Proof. That (i) is equivalent to (ii) follows immediately from Theorem 5.3.

(ii) $\Longrightarrow$ (iii) Assume that I is an (U,N)-implication based on a uninorm U with the neutral element $e \in (0,1)$ and a continuous negation N. Since every (U,N)-implication is a fuzzy implication, I satisfies (I1) and (I3). Moreover, by Proposition 5.2, it satisfies (EP) and $N_I^e = N$. In particular N_I^e is continuous.

(iii) $\Longrightarrow$ (ii) Firstly, see that from Lemma 3.5(iii) it follows that I satisfies (R-CP) with respect to the continuous negation N_I^e. Next, Lemma 3.3 implies that I satisfies (I2). Once again from Lemma 3.5(i) and (ii) we have that I satisfies (I3)–(I5), and hence $I \in \mathcal{FI}$. Further, by virtue of Lemmas 2.2 and 3.6, the implication I satisfies L-CP($\mathfrak{N}_I^e$). Because of Corollary 6.3 the function U_I defined by (7) is a disjunctive uninorm with the neutral element e. We will show that $I_{U_I, N_I^e} = I$. Fix arbitrarily $x, y \in [0,1]$. If $x \in \mathrm{Ran}(\mathfrak{N}_I^e)$, then by (1) we have

$$I_{U_I,N_I^e}(x,y) = U_I(N_I^e(x), y) = I(\mathfrak{N}_I^e \circ N_I^e(x), y) = I(x,y).$$

If $x \notin \mathrm{Ran}(\mathfrak{N}_I^e)$, then from the continuity of N_I^e there exists $x_0 \in \mathrm{Ran}(\mathfrak{N}_I^e)$ such that $N_I^e(x) = N_I^e(x_0)$. Firstly, see that $I(x,y) = I(x_0, y)$ for all $y \in [0,1]$. Indeed, let us fix arbitrarily $y \in [0,1]$. From the continuity of N_I^e there exists $y' \in [0,1]$ such that $N_I^e(y') = y$, so

$$I(x,y) = I(x, N_I^e(y')) = I(y', N_I^e(x)) = I(y', N_I^e(x_0)) = I(x_0, N_I^e(y')) = I(x_0, y).$$

From the above fact we get

$$I_{U_I,N_I^e}(x,y) = U_I(N_I^e(x), y) = U_I(N_I^e(x_0), y) = I(x_0, y) = I(x,y),$$

so I is a (U,N)-implication.

Finally, assume that there exist two continuous fuzzy negations N_1, N_2 and two uninorms U_1, U_2 with neutral elements $e, e' \in (0,1)$, respectively, such that $I(x,y) = U_1(N_1(x), y) = U_2(N_2(x), y)$ for all $x, y \in [0,1]$. Fix arbitrarily $x_0, y_0 \in [0,1]$. Observe now that from Proposition 5.2 we get $N_1 = N_2 = N_I^e = N_I^{e'}$. By virtue of Lemma 5.6 we get that $e' = e$. Moreover, since N_I^e is a continuous negation, there exists $x_1 \in [0,1]$ such that

$N_I^e(x_1) = x_0$. Thus $U_1(x_0, y_0) = U_1(N_I^e(x_1), y_0) = U_2(N_I^e(x_1), y_0) = U_2(x_0, y_0)$, i.e., $U_1 = U_2$. Hence N and U are uniquely determined. In fact, $U = U_I$ defined by (7). □

From the above proof the following result easily follows.

Theorem 6.5. *For a function* $I : [0, 1]^2 \to [0, 1]$ *the following statements are equivalent*:

(i) *I is a* (U,N)-*implication generated from some disjunctive uninorm U with the neutral element* $e \in (0, 1)$ *and some strict* (*strong*) *fuzzy negation N*.
(ii) *I satisfies* (I1), (I3), (EP) *and the function* N_I^e *is a strict* (*strong*) *negation*.

Remark 6.6.

(i) In Theorems 6.4 and 6.5 the property (I1) can be substituted by (I2) and the property (I3) can be substituted by (I4).
(ii) In Table 1, we show the mutual independence of the properties from Theorem 6.4. The same examples can be considered for the mutual independence of axioms in Theorem 6.5. We recognize that the verification of Table 1, vis-á-vis, the functions and the corresponding properties indicated, may not be obvious. Hence, in the following, we show that the presented examples in Table 1 are correct. Firstly, observe that if a function F satisfies (NP), then $F(1, \alpha) = \alpha$ for all $\alpha \in (0, 1)$, i.e., $N_F^\alpha(1) \neq 0$ and hence N_F^α is not a fuzzy negation for any $\alpha \in (0, 1)$.
   (a) It is clear that F_1 satisfies (I1), but neither does it satisfy (I3) nor is $N_{F_1}^\alpha$ a continuous fuzzy negation for any $\alpha \in (0, 1)$. Moreover, F_1 does not satisfy (EP), since
   $$F_1(0.4, F_1(0.6, 0.5)) = 0 \neq 1 = F_1(0.6, F_1(0.4, 0.5)).$$
   (b) F_2 does not satisfy (I1), since $F_2(0, y) = 0 < y = F_2(1, y)$ for any $y \in (0, 1)$. From the same equality we see that $N_{F_2}^\alpha$ is not a fuzzy negation for any $\alpha \in (0, 1)$. Since
   $$F_2(0, F_2(0.5, 1)) = 1 \neq 0 = F_2(0.5, F_2(0, 1)),$$
   F_2 does not satisfy (EP).
   (c) F_3 does not satisfy (I1), since $F_3(0, y) = 0 < y = F_3(1, y)$ for any $y \in (0, 1)$. From the same equality we see that $N_{F_3}^\alpha$ is not a fuzzy negation for any $\alpha \in (0, 1)$. Further, $F_3(0, 0) = 0$, so F_3 does not satisfy (I3). Finally, the minimum satisfies (EP).
   (d) F_4 does not satisfy (I1), since $F_4(0, 0.3) = 0 < 1 = F_4(1, 0.3)$. It is easy to see that $N_{F_4}^{0.5} = N_{\mathbf{C}}$ and also that F_4 does not satisfy (I3). F_4 does not satisfy (EP), since
   $$F_4(0.5, F_4(0.6, 0.5)) = 1 \neq 0.4 = F_4(0.6, F_4(0.5, 0.5)).$$
   (e) It is clear that F_5 satisfies both (I1) and (I3). F_5 does not satisfy (EP), since
   $$F_5(0.5, F_5(0.8, 0.5)) = 1 \neq 0 = F_5(0.8, F_5(0.5, 0.5)).$$
   Moreover, $N_{F_5}^\alpha$ is not a continuous fuzzy negation for any $\alpha \in (0, 1)$.
   (f) Function F_6 is self-explanatory.
   (g) Clearly, the function F_7 satisfies (I1), $N_{F_7}^{0.5} = N_{\mathbf{C}}$, but it does not satisfy (I3). Moreover, since
   $$F_7(0.5, F_7(0.4, 0.6)) = 0.5 \neq 0.6 = F_7(0.4, F_7(0.5, 0.6)),$$
   F_7 does not satisfy (EP).
   (h) F_8 does not satisfy (I1), since $F_8(0, 0.6) = 0 < 1 = F_8(1, 0.6)$. $N_{F_8}^\alpha$ is not a continuous negation for any $\alpha \in (0, 1)$. However, it can be easily verified that F_8 does satisfy both (I3) and (EP).
   (i) F_9 does not satisfy (I1), since $F_9(0, 0.6) = 0 < 1 = F_9(1, 0.6)$. It is easy to see that $N_{F_9}^{0.5} = N_{\mathbf{C}}$ and also that F_9 does satisfy (I3). F_9 does not satisfy (EP), since
   $$F_9(0.5, F_9(0.6, 0.5)) = 0 \neq 0.4 = F_9(0.6, F_9(0.5, 0.5)).$$

Table 1
The mutual independence of the properties in Theorem 6.4

Function F	(I1)	(I3)	(EP)	N_F^α is continuous for some $\alpha \in (0,1)$
$F_1(x,y) = \begin{cases} 1 & \text{if } x < y \\ 0 & \text{otherwise} \end{cases}$	✓	×	×	×
$F_2(x,y) = \begin{cases} 1 & \text{if } x = 0 \text{ and } y = 0 \\ y & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$	×	✓	×	×
$F_3 = T_{\mathbf{M}}$	×	×	✓	×
$F_4(x,y) = \begin{cases} 0 & \text{if } (x,y) \in [0,0.5)^2 \cup (0.5,1]^2 \\ 1-x & \text{if } y = 0.5 \\ 1 & \text{otherwise} \end{cases}$	×	×	×	✓
$F_5(x,y) = \begin{cases} 1 & \text{if } x, y \in [0,0.5] \\ 0 & \text{otherwise} \end{cases}$	✓	✓	×	×
$F_6 = 0$	✓	×	✓	×
$F_7(x,y) = \begin{cases} 1-x & \text{if } y = 0.5 \\ \min(1-x,y) & \text{otherwise} \end{cases}$	✓	×	×	✓
$F_8(x,y) = \begin{cases} 1 & \text{if } (x,y) \in [0,0.5)^2 \cup (0.5,1]^2 \\ 0 & \text{otherwise} \end{cases}$	×	✓	✓	×
$F_9(x,y) = \begin{cases} 1 & \text{if } (x,y) \in [0,0.5)^2 \cup (0.5,1]^2 \\ 1-x & \text{if } y = 0.5 \\ 0 & \text{otherwise} \end{cases}$	×	✓	×	✓
$F_{10}(x,y) = \begin{cases} 0 & \text{if } (x,y) \in [0,0.5)^2 \cup (0.5,1]^2 \\ 1-x & \text{if } y = 0.5 \\ y & \text{if } x = 0.5 \\ 1 & \text{otherwise} \end{cases}$	×	×	✓	✓
$F_{11}(x,y) = \begin{cases} 1 & \text{if } (x,y) \in [0,0.5)^2 \cup (0.5,1]^2 \\ 1-x & \text{if } y = 0.5 \\ y & \text{if } x = 0.5 \\ 0 & \text{otherwise} \end{cases}$	×	✓	✓	✓
$F_{12}(x,y) = \begin{cases} \min(1-x,y) & \text{if } x \leqslant y \\ \max(1-x,y) & \text{if } x > y \end{cases}$	✓	×	✓	✓
$F_{13}(x,y) = 1-x$	✓	✓	×	✓
$F_{14}(x,y) = \begin{cases} 1 & \text{if } x = 0 \text{ and } y = 0 \\ y^x & \text{if } x > 0 \text{ or } y > 0 \end{cases}$	✓	✓	✓	×

(j) Although $N_{F_{10}}^{0.5} = N_{\mathbf{C}}$ and F_{10} satisfies (EP), it does not satisfy either (I3) or (I1), since

$$F_{10}(0, 0.3) = 0 < 1 = F_{10}(1, 0.3).$$

(k) F_{11} does not satisfy (I1), since $F_{11}(0, 0.6) = 0 < 1 = F_{11}(1, 0.6)$. Interestingly, $N_{F_{11}}^{0.5} = N_{\mathbf{C}}$ and F_{11} satisfies both (EP) and (I3).

(l) Clearly, the function F_{12} does not satisfy (I3), but it satisfies both (EP) and (I1). Moreover, $N_{F_{12}}^{0.5} = N_{\mathbf{C}}$.

(m) F_{13} satisfies both (I1) and (I3), and once again, $N_{F_{13}}^{\alpha} = N_{\mathbf{C}}$ for any $\alpha \in [0, 1)$. It does not satisfy (EP), since

$$F_{13}(0, F_{13}(0.5, 0)) = 1 \neq 0.5 = F_{13}(0.5, F_{13}(0, 0)).$$

(n) Finally, F_{14} is the Yager implication (see [22]), which is a fuzzy implication that satisfies both (EP) and (NP) (cf. [2,3]).

The following characterization of (U,N)-operations can now be obtained along similar lines as above.

Theorem 6.7. *For a function* $I : [0,1]^2 \to [0,1]$ *the following statements are equivalent*:

(i) *I is a* (U,N)-*operation generated from some uninorm U with the neutral element* $e \in (0,1)$ *and some continuous fuzzy negation N*.
(ii) *I satisfies* (I1), (EP) *and the function* N_I^e *is a continuous negation for some* $e \in (0,1)$.

Once again, in the above theorem, the property (I1) can be substituted by (I2) and all properties in point (ii) are mutually independent from each other.

7. Concluding remarks

In this work, we have characterized (U,N)-operations and (U,N)-implications obtained from uninorms and continuous negations. Toward this end, we have investigated some desirable algebraic properties of fuzzy implications and obtained some characterization results. Unfortunately, the characterization of (U,N)-implications obtained from non-continuous negations is still unavailable. It should be noted that (U,N)-implications are closely related to *e-implications* investigated by Khaledi et al. [11], whose characterization is also still unknown.

Similar to uninorms, *t-operators*, usually denoted by *F*, were proposed by Mas et al. [15]. They are currently known as *nullnorms* (see [4]), since like uninorms, these are commutative, associative and increasing binary operations on the unit interval [0, 1], but unlike uninorms where the neutral element gets the focus, here the emphasis is on the annihilator $k \in [0,1]$ such that $F(1,0) = F(0,1) = k$. It immediately follows that if $k = 0$, then *F* is a t-norm, while $k = 1$ implies that *F* is a t-conorm. Unfortunately, nullnorms do not exactly turn out to be a fertile field for generating fuzzy implications the usual way. For example, consider the generalization of (S,N)-implications to the setting of nullnorms, with any fuzzy negation *N*, defined as

$$I_{F,N}(x,y) = F(N(x),y), \quad x,y \in [0,1].$$

Then

$$I_{F,N}(0,0) = F(N(0),0) = F(1,0) = k.$$

Now, if $I_{F,N}$ were to satisfy (I3), i.e., $I_{F,N}(0,0) = 1$, it would fix *F* to be a t-conorm. Hence $I_{F,N}$ reduces to an (S,N)-implication.

Recently, there has been a lot of interest on non-commutative fuzzy conjunctions and disjunctions. One of the earliest studies along these lines was done by Fodor and Keresztfalvi [8]. Such operations again have proven to be a fertile ground for obtaining fuzzy implications. For example, (S,N)-type implications from copulas/co-copulas were obtained by Yager [23]. Characterization of this family of fuzzy implications seems worthy of an attempt and will be taken up in our future endeavors.

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