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Fuzzy Sets and Systems 160 (2009) 2063 – 2085

FUZZY sets and systems

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On special fuzzy implications

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Received 16 July 2008; received in revised form 27 October 2008; accepted 3 November 2008
Available online 20 November 2008

Abstract

Special implications were introduced by Hájek and Kohout [Fuzzy implications and generalized quantifiers, Int. J. Uncertain. Fuzziness Knowl.-Based Syst. 4 (1996) 225–233] in their investigations on some statistics on marginals. They have either suggested or only partially answered three important questions, especially related to special implications and residuals of t-norms. In this work we investigate these posers in-depth and give complete answers. Toward this end, firstly we show that many of the properties considered as part of the definition of special implications are redundant. Then, a geometric interpretation of the specialty property is given, using which many results and bounds for such implications are obtained. We have obtained a characterization of general binary operations whose residuals become special. Finally, some constructive procedures to obtain special fuzzy implications are proposed and methods of obtaining special implications from existing ones are given, showing that there are infinitely many fuzzy implications that are special but cannot be obtained as residuals of t-norms.

MSC: 03B52; 03E72; 39B99

Keywords: Fuzzy connectives; Fuzzy implication; Special implication; (S,N)-implication; Residual of conjunction; 1-Lipschitzianity

1. Introduction

A fuzzy implication is the generalization of the classical one to fuzzy logic, much the same way as a t-norm and a t-conorm are generalizations of the classical conjunction and disjunction, respectively. There exist many families of fuzzy implications, most of which are a straight-forward generalization of their classical counterparts, viz., (S,N)-, R- and QL-implications, while others are obtained rather in a novel way, for instance, the *f*- and *g*-implications proposed by Yager [27]. We shall see them in more detail later.

Special implications were introduced by Hájek and Kohout [18] in their investigations on some statistics on marginals and have shown that they are related to special GUHA-implicative quantifiers (see, for instance, [15–17]). Thus, special fuzzy implications are related to data mining. In their quest to obtain some many-valued connectives as extremal values of some statistics on contingency tables with fixed marginals, they especially focussed on special homogenous implicational quantifiers and showed that "Each special implicational quantifier determines a special implication. Conversely, each special implication is given by a special implicational quantifier".

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doi:10.1016/j.fss.2008.11.004

Table 1
Examples of fuzzy implications.

Name	Formula
Largest	$I_1(x, y) = \begin{cases} 0 & \text{if } x = 1 \text{ and } y = 0, \\ 1 & \text{otherwise.} \end{cases}$
Łukasiewicz	$I_{\mathbf{LK}}(x, y) = \min(1, 1 - x + y)$
Gödel	$I_{\mathbf{GD}}(x, y) = \begin{cases} 1 & \text{if } x \leqslant y \\ y & \text{if } x > y \end{cases}$
Goguen	$I_{\mathbf{GG}}(x, y) = \begin{cases} 1 & \text{if } x \leqslant y \\ \dfrac{y}{x} & \text{if } x > y \end{cases}$
Rescher	$I_{\mathbf{RS}}(x, y) = \begin{cases} 1 & \text{if } x \leqslant y \\ 0 & \text{if } x > y \end{cases}$
Weber	$I_{\mathbf{WB}}(x, y) = \begin{cases} 1 & \text{if } x < 1 \\ y & \text{if } x = 1 \end{cases}$
Fodor	$I_{\mathbf{FD}}(x, y) = \begin{cases} 1 & \text{if } x \leqslant y \\ \max(1 - x, y) & \text{if } x > y \end{cases}$

Definition 1.1. A fuzzy implication I is said to be *special*, if for any $\varepsilon > 0$ and for all $x, y \in [0, 1]$ such that $x + \varepsilon, y + \varepsilon \in [0, 1]$ the following condition is satisfied:

$$I(x, y) \leqslant I(x + \varepsilon, y + \varepsilon). \tag{SP}$$

The definition of a fuzzy implication will be given presently (see Definition 2.1).

For a quick overview of the context in which they were introduced see Sainio et al. [25]. In fact, [18] itself can be seen as a follow up of the much earlier work of Bandler and Kohout [5], wherein they investigate the semantics of fuzzy implication operators based on a check-list paradigm. See also the related works of Bandler and Kohout [6,7].

1.1. Motivation for this work

In [18], the authors have either suggested or only partially answered the following posers:

Problem 1. Which R-implications are special, i.e., if T is any t-norm and I_T is obtained as its residual, then what are the necessary and/or sufficient conditions for I_T to satisfy (SP)?

Problem 2. Conversely, which special implications can be expressed as residuals of a t-norm T?

Problem 3. In their conclusion, they remark “It is an interesting research program to investigate other classes of fuzzy implications (satisfying other axioms from Klir’s list or similar)”.

Hájek and Kohout [18] themselves predominantly considered fuzzy implications obtained as residuals of continuous t-norms. Recently, Sainio et al. [25] have attempted Problem 1 and shown that an R-implication from a left-continuous t-norm T is a special implication, i.e., satisfies (SP), only if T is continuous.

The authors in [18] have given a partial answer to Problem 2 by stating that “Not all special implications are R-implications”. In fact, as given therein, the least and largest special implications are the Rescher implication $I_{\mathbf{RS}}$ and I_1 (see Table 1 for their formulae), respectively. It can be seen from Remark 3.15(i) that $I_{\mathbf{RS}}$ and I_1 cannot be obtained as residuals of any t-norm (not necessarily left-continuous).

As can be seen, so far, the property (SP) has been investigated only for the fuzzy implications obtained as residuals of left-continuous t-norms, whereas there remain many other established families of fuzzy implications and hence, determining whether any of them contain sub-families of special implications, i.e., Problem 3, assumes significance.

1.2. Organization and main results of this work

In this work, we firstly show that the specialty condition on a fuzzy implication, by itself, is quite strong and imposes, independently and along with other properties, quite some restrictions on the set of all fuzzy implications. Consequently, we show that many of the conditions required in the definition of special implications, both in [18,25], are redundant. Following this, we give a geometric interpretation of the specialty condition and hence obtain bounds on special implications on different domains of $[0, 1]^2$ (Section 2.2).

Equipped with this geometric insight, we attempt the above three problems. Firstly, notice that Sainio et al. [25] have attempted Problem 1 only for R-implications obtained from left-continuous t-norms. Once again, we show that the left-continuity of the underlying t-norm T need not be assumed and that it follows from (SP). Thus the condition (SP) on an I_T becomes an alternative, though a stringent, sufficient condition for T to be continuous. In fact, in Section 4, we consider the above problem in a more general setting and obtain an equivalent condition on an underlying monotone binary operation for its residual to be special. We remark here that from this simple, but elegant, characterization and already known results on t-norms all the results in Sainio et al. [25] can be obtained. The preliminaries required for this work are covered in Section 3.

Once again, based on the obtained characterization and already available results, in Section 5, we give a complete answer to Problem 2. Following this, in Section 6, we investigate the other most established families of fuzzy implications, viz., (S,N)-, *f*- and *g*-implications. However, our studies show that these families do not seem to give rise to newer special implications. Hence, the most natural question that arises is this: *Are there any other special implications, than those that could be obtained as residuals of t-norms*?

We systematically attempt to give an answer to the above question by the following ways:

(i) investigating residuals of more generalized conjunctions than t-norms (Section 7),
(ii) proposing new construction methods for special fuzzy implications (Section 8), and
(iii) generating special implications from special implications, which would mean that one could create infinitely many special implications from a given one (Section 9).

2. Fuzzy implications

2.1. Definition and some desirable properties

In the literature, especially in the beginning, we can find several different definitions of fuzzy implications. In this work we will use the following one, which is equivalent to the definition proposed by Fodor and Roubens [13].

Definition 2.1. A function $I: [0, 1]^2 \to [0, 1]$ is called a *fuzzy implication* if it satisfies, for all $x, x_1, x_2, y, y_1, y_2 \in [0, 1]$, the following conditions:

$$\text{if } x_1 \leqslant x_2, \text{ then } I(x_1, y) \geqslant I(x_2, y), \text{ i.e., } I(\cdot, y) \text{ is decreasing,} \tag{I1}$$

$$\text{if } y_1 \leqslant y_2, \text{ then } I(x, y_1) \leqslant I(x, y_2), \text{ i.e., } I(x, \cdot) \text{ is increasing,} \tag{I2}$$

$$I(0, 0) = 1, \tag{I3}$$

$$I(1, 1) = 1, \tag{I4}$$

$$I(1, 0) = 0. \tag{I5}$$

The set of all fuzzy implications will be denoted by $\mathcal{FI}$.

The most important desirable properties of fuzzy implications are presented below (see Fodor and Roubeus [13]).

Definition 2.2. A fuzzy implication I is said to satisfy

(i) the *left neutrality property* if

$$I(1, y) = y, \quad y \in [0, 1]; \tag{NP}$$

(ii) the *exchange principle*, if

$$I(x, I(y, z)) = I(y, I(x, z)), \quad x, y, z \in [0, 1]; \tag{EP}$$

(iii) the *identity principle*, if

$$I(x, x) = 1, \quad x \in [0, 1]; \tag{IP}$$

(iv) the *ordering property*, if

$$I(x, y) = 1 \Longleftrightarrow x \leqslant y, \quad x, y \in [0, 1]. \tag{OP}$$

A fuzzy negation N is a generalization of the classical complement or negation $\neg$, whose truth table consists of the two conditions: $\neg 0 = 1$ and $\neg 1 = 0$.

Definition 2.3 (*Fodor and Roubens [13]*). A function $N: [0, 1] \to [0, 1]$ is called a *fuzzy negation* if $N(0)=1$, $N(1)=0$ and is decreasing.

Further, a fuzzy negation N is called *strong* if it is an involution, i.e., $N(N(x)) = x$ for all $x \in [0, 1]$. The classical negation, $N_{\mathbf{C}}(x) = 1 - x$, is an involutive fuzzy negation, whereas the Gödel negations, $N_{\mathbf{D1}}$ and $N_{\mathbf{D2}}$—which are the least and largest fuzzy negations—are non-strong negations:

$$N_{\mathbf{D1}}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x > 0, \end{cases} \qquad N_{\mathbf{D2}}(x) = \begin{cases} 1 & \text{if } x < 1, \\ 0 & \text{if } x = 1. \end{cases}$$

Definition 2.4. Let I be a fuzzy implication. The function N_I defined by

$$N_I(x) := I(x, 0), \quad x \in [0, 1] \tag{1}$$

is a fuzzy negation, i.e., $N_I(0) = 1$, $N_I(1) = 0$ and N_I is non-increasing. N_I is called the *natural negation* of I or the *negation induced* by I.

Proposition 2.5. *Consider an* $I: [0, 1]^2 \to [0, 1]$.

(i) *If I satisfies* (I3), (I4) and (SP) *then I satisfies* (IP).
(ii) *Let I satisfy* (I1). *I satisfies* (IP) *if and only if I satisfies* (OP′):

$$x \leqslant y \Longrightarrow I(x, y) = 1, \quad x, y \in [0, 1]. \tag{OP′}$$

Proof. (i) Since I is special, we have $1 = I(0, 0) \leqslant I(x, x) \leqslant I(1, 1) = 1$ for any $x \in (0, 1)$, i.e., I satisfies (IP).

(ii) That (OP′) implies (IP) is immediate by taking $x = y$. On the other hand, let I satisfy (IP), i.e., $I(y, y) = 1$ for any $y \in [0, 1]$. If $x \leqslant y$ then by (I1), $I(x, y) \geqslant I(y, y) = 1$, i.e., I satisfies (OP′). □

Remark 2.6. In the definition of special implications, the authors in [18] require that I satisfy (I1)–(I5), (IP) and (SP), while in [25] it is required that I satisfy (I1)–(I5), (OP′) and (SP). From Proposition 2.5, we see that properties (IP) and (OP′) are redundant in the definition of a special implication. Note that in Definition 1.1 it is only required that I is a fuzzy implication, i.e., I satisfies (I1)–(I5).

2.2. Special implications: a geometric perspective

Let I be a fuzzy implication. It is interesting to study how (SP) affects the geometry of I on $[0, 1]^2$. Firstly, from Proposition 2.5, we see that I satisfies (IP) which means that all the points above the main diagonal of the unit square are mapped to 1 by I. On the other hand, if $x > y$, by (SP), for any $\varepsilon > 0$ we have the following string of inequalities:

$$I(x, y) \leqslant I(x + \varepsilon, y + \varepsilon) \leqslant I(x + 2\varepsilon, y + 2\varepsilon) \leqslant \cdots \leqslant I(1, 1 - x + y), \tag{2}$$

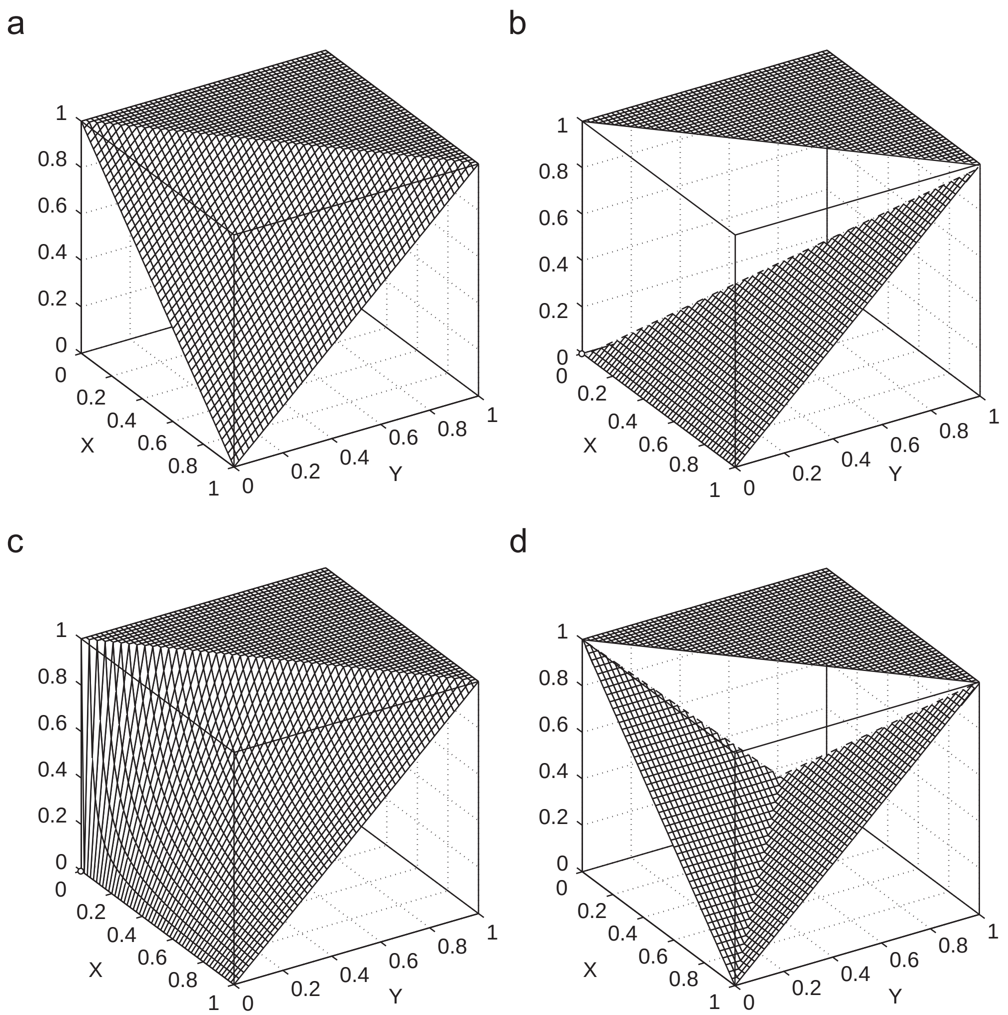

Fig. 1. Plots of Łukasiewicz, Gödel, Goguen and Fodor implications. (a) Łukasiewicz implication $I_{\mathbf{LK}}$; (b) Gödel implication $I_{\mathbf{GD}}$; (c) Goguen implication $I_{\mathbf{GG}}$ and (d) Fodor implication $I_{\mathbf{FD}}$.

i.e., (SP) essentially states that as we travel parallel to the main diagonal I should be non-decreasing. Fig. 1 amply illustrates this point: while the Łukasiewicz, Gödel and Goguen implications are special, the Fodor implication is not (see Remark 2.8).

From (2) we can easily see the following result:

Proposition 2.7. (i) *If I is a special fuzzy implication that satisfies* (NP), *then I satisfies* (OP). *Also the natural negation of I is less than the classical negation, i.e.,* $N_I \leqslant N_{\mathbf{C}}$.

(ii) *The Łukasiewicz implication* $I_{\mathbf{LK}}$ *is the largest special implication that satisfies* (NP), *while* $I_{\mathbf{GD}}$ *is the smallest special implication that satisfies* (NP).

Proof. (i) Since I is special, we know that I satisfies (OP′). Hence, it suffices to show that whenever $I(x, y) = 1 \Longrightarrow x \leqslant y$. This immediately follows from (2), since by (NP), we have

$$1 = I(x, y) \leqslant I(1, 1 - x + y) = 1 - x + y \Longrightarrow x \leqslant y.$$

Moreover, for any $x \in [0, 1]$ we have

$$N_I(x) = I(x, 0) \leqslant I(1, 1 - x) \leqslant 1 - x = N_{\mathbf{C}}(x).$$

(ii) This is obvious from (2) again, since $I(x, y) \leqslant 1 - x + y$ whenever $x > y$. Once again, that $I_{\mathbf{GD}}$ is the smallest special implication that satisfies (NP) is obvious from point (i) and property (I1) of $I_{\mathbf{GD}}$. □

Remark 2.8. We only note that the condition $I(x, y) \leqslant 1 - x + y$ whenever $x > y$ is only necessary for an I that satisfies (NP) to be special and is not sufficient. For example, consider the Fodor implication $I_{\mathbf{FD}}$, which satisfies both (NP) and (OP). For any $x > y$, we have $I_{\mathbf{FD}}(x, y) = \max(1 - x, y) \leqslant 1 - x + y$, but, as can be seen from Table 3, $I_{\mathbf{FD}}$ is the R-implication obtained from the non-continuous (but left-continuous) nilpotent minimum t-norm $T_{\mathbf{nM}}$ which, from Theorem 4.2, we see is not a special implication. In fact, for any $t \in [0, \frac{1}{2}]$ we have that $I_{\mathbf{FD}}(\frac{1}{2} + t, t) = \max(\frac{1}{2} - t, t)$, which is strictly decreasing on $t \in [0, \frac{1}{4}]$.

3. Conjunctors and their residuals

We assume that the reader is familiar with the classical results concerning basic fuzzy logic connectives, but to make this work more self-contained, we introduce basic notations used in the text and we briefly mention some of the concepts and results employed in the rest of the work.

3.1. Increasing bijections on [0, 1]

By Φ we denote the family of all increasing bijections $\varphi\colon [0, 1] \to [0, 1]$. We say that functions $f, g\colon [0, 1]^n \to [0, 1]$, where $n \in \mathbb{N}$, are Φ-conjugate (cf. [21, p. 156]), if there exists $\varphi \in \Phi$ such that $g = f_\varphi$, where

$$f_\varphi(x_1, \ldots, x_n) := \varphi^{-1}(f(\varphi(x_1), \ldots, \varphi(x_n))), \quad x_1, \ldots, x_n \in [0, 1].$$

Equivalently, g is said to be the Φ-conjugate of f.

Definition 3.1. A real function φ on [0, 1] is said to be

(i) *concave* if, for every $x, y, x + \varepsilon \in [0, 1]$, $y \leqslant x$ and $0 < \varepsilon$,

$$\varphi(x) - \varphi(y) \geqslant \varphi(x + \varepsilon) - \varphi(y + \varepsilon);$$

(ii) *convex* if, for every $x, y, x + \varepsilon \in [0, 1]$, $y \leqslant x$ and $0 < \varepsilon$,

$$\varphi(x) - \varphi(y) \leqslant \varphi(x + \varepsilon) - \varphi(y + \varepsilon).$$

3.2. Fuzzy conjunctions: triangular norms

The following definitions and results, with proofs, can be found in the literature, see for example Klement et al. [20], Nelsen [24].

Definition 3.2. A function $T\colon [0, 1]^2 \to [0, 1]$ is called a *triangular norm* (shortly *t-norm*) if it satisfies, for all $x, y, z \in [0, 1]$, the following conditions:

$$T(x, y) = T(y, x), \tag{T1}$$

$$T(x, T(y, z)) = T(T(x, y), z), \tag{T2}$$

$$\text{if } y \leqslant z, \text{ then } T(x, y) \leqslant T(x, z), \text{ i.e., } T(x, \cdot\,) \text{ is increasing}, \tag{T3}$$

$$T(x, 1) = x. \tag{T4}$$

The class of t-norms is rather large and some subclasses of t-norms have been well investigated.

Table 2
Basic t-norms.

Name	Formula	Properties
Minimum	$T_{\mathbf{M}}(x, y) = \min(x, y)$	Idempotent, continuous, positive
Algebraic product	$T_{\mathbf{P}}(x, y) = xy$	Strict, positive
Łukasiewicz	$T_{\mathbf{LK}}(x, y) = \max(x + y - 1, 0)$	Nilpotent
Drastic product	$T_{\mathbf{D}}(x, y) = \begin{cases} 0 & \text{if } x, y \in [0, 1) \\ \min(x, y) & \text{otherwise} \end{cases}$	Archimedean, non-continuous
Nilpotent minimum	$T_{\mathbf{nM}}(x, y) = \begin{cases} 0 & \text{if } x + y \leqslant 1 \\ \min(x, y) & \text{otherwise} \end{cases}$	Left-continuous

Definition 3.3. A t-norm T is called

(i) *continuous* if it is continuous in both the arguments;
(ii) *left-continuous* if it is left-continuous in each component;
(iii) *border continuous* if it is continuous on the boundary of the unit square $[0, 1]^2$, i.e., on the set $[0, 1]^2 \setminus (0, 1)^2$;
(iv) *Archimedean*, if for all $x, y \in (0, 1)$ there exists an $n \in \mathbb{N}$ such that $x_T^{[n]} < y$;
(v) *strict*, if it is continuous and strictly monotone, i.e., $T(x, y) < T(x, z)$ whenever $x > 0$ and $y < z$;
(vi) *nilpotent*, if it is continuous and if each $x \in (0, 1)$ is a nilpotent element of T, i.e., if there exists an $n \in \mathbb{N}$ such that $x_T^{[n]} = 0$;
(vii) *positive*, if whenever $T(x, y) = 0$ then either $x = 0$ or $y = 0$ or both.

Example 3.4. Table 2 lists a few of the common t-norms along with their classification.

The following characterization theorem is based on properties of their underlying generators (see [20], Theorem 5.1) and will be useful later on in this work.

Theorem 3.5. *For a function* $T: [0, 1]^2 \to [0, 1]$ *the following statements are equivalent*:

(i) *T is a continuous Archimedean t-norm.*
(ii) *T has a continuous additive generator, i.e., there exists a continuous, strictly decreasing function $f: [0, 1] \to [0, \infty]$ with $f(1) = 0$, which is uniquely determined up to a positive multiplicative constant, such that*

$$T(x, y) = f^{(-1)}(f(x) + f(y)), \quad x, y \in [0, 1],$$

where $f^{(-1)}$ is the pseudo-inverse of f given by

$$f^{(-1)}(x) = \begin{cases} f^{-1}(x) & \text{if } x \in [0, f(0)], \\ 0 & \text{if } x \in (f(0), \infty]. \end{cases}$$

Finally, we have the following complete representation of continuous t-norms (cf. [20], Theorem 5.11).

Theorem 3.6. *For a function* $T: [0, 1]^2 \to [0, 1]$ *the following statements are equivalent*:

(i) *T is a continuous t-norm.*
(ii) *T is uniquely representable as an ordinal sum of continuous Archimedean t-norms, i.e., there exist a uniquely determined (finite or countably infinite) index set $\mathcal{A}$, a family of uniquely determined pairwise disjoint open subintervals $\{(a_\alpha, e_\alpha)\}_{\alpha \in \mathcal{A}}$ of $[0, 1]$ and a family of uniquely determined continuous Archimedean t-norms $(T_\alpha)_{\alpha \in \mathcal{A}}$*

such that

$$T(x, y) = \begin{cases} a_\alpha + (e_\alpha - a_\alpha) \cdot T_\alpha\left(\dfrac{x - a_\alpha}{e_\alpha - a_\alpha}, \dfrac{y - a_\alpha}{e_\alpha - a_\alpha}\right) & \text{if } x, y \in [a_\alpha, e_\alpha], \\ \min(x, y) & \text{otherwise}. \end{cases}$$

In this case we will write $T = (\langle a_\alpha, e_\alpha, T_\alpha\rangle)_{\alpha\in\mathcal{A}}$.

Definition 3.7. A function $F: [0,1]^2 \to [0,1]$ is said to be 1-*Lipschitz*, or said to satisfy 1-*Lipschitzianity*, in the *first* variable if

$$|F(x_1, y) - F(x_2, y)| \leqslant |x_1 - x_2|, \quad x_1, x_2, y \in [0,1]. \tag{3}$$

Similarly, one can define 1-Lipschitzianity of a binary function F in the second variable. Quite naturally, we have:

Definition 3.8. A function $F: [0,1]^2 \to [0,1]$ is said to be 1-*Lipschitz*, or said to satisfy 1-*Lipschitzianity*, if it satisfies 1-Lipschitzianity in both the variables, i.e.,

$$|F(x_1, y_1) - F(x_2, y_2)| \leqslant |x_1 - x_2| + |y_1 - y_2|, \quad x_1, x_2, y_1, y_2 \in [0,1]. \tag{4}$$

Obviously, for commutative binary operations 1-Lipschitzianity in one variable implies 1-Lipschitzianity in both the variables. Any 1-Lipschitz t-norm T is also continuous but the converse, in general, is not true (see [20], Example 1.26).

Definition 3.9 (*Cf. Durante and Sempi [11], Durante et al. [12], Genest et al. [14], Nelsen [24]*). Consider a mapping $C : [0,1]^2 \to [0,1]$.

(i) C is a *semi-copula* if it is non-decreasing in both variables and $C(1, x) = C(x, 1) = x$ for every $x \in [0,1]$.
(ii) A semi-copula C is a *quasi-copula* if it is 1-Lipschitz.
(iii) A semi-copula C is a *copula* if it is 2-increasing, viz.,

$$C(x_1, y_1) + C(x_2, y_2) \geqslant C(x_1, y_2) + C(x_2, y_1) \tag{5}$$

for all $x_1, x_2, y_1, y_2 \in [0,1]$ such that $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$.

In the case of t-norms, we have the following results, see for instance, Schweizer and Sklar [20,26].

Theorem 3.10. *For a t-norm T the following are equivalent*:

(i) *T satisfies* 1-*Lipschitzianity*.
(ii) *T is a quasi-copula*.
(iii) *T is a copula*.

Theorem 3.11. *Let T be a continuous Archimedean t-norm with continuous additive generator f. The following are equivalent*:

(i) *T is a copula*.
(ii) *f is a convex function*.

From the above two results and the representation result Theorem 3.6 of continuous t-norms, we have the following corollary:

Corollary 3.12. *Let T be a continuous t-norm which has the uniquely determined ordinal sum* $T = (\langle a_\alpha, e_\alpha, T_\alpha\rangle)_{\alpha\in\mathcal{A}}$, *where each t-norm* T_α, $\alpha \in \mathcal{A}$ *is generated by a continuous additive generator* f_α. *The following are equivalent*:

(i) *T is a copula*.
(ii) *For* $\alpha \in \mathcal{A}$, *the t-norm* T_α *is a copula*.
(iii) *For* $\alpha \in \mathcal{A}$, *the additive generator* f_α *is a convex function*.

Table 3
Examples of basic R-implications.

t-norm T	R-implication I_T
$T_{\mathbf{M}}$	$I_{\mathbf{GD}}$
$T_{\mathbf{P}}$	$I_{\mathbf{GG}}$
$T_{\mathbf{LK}}$	$I_{\mathbf{LK}}$
$T_{\mathbf{D}}$	$I_{\mathbf{WB}}$
$T_{\mathbf{nM}}$	$I_{\mathbf{FD}}$

3.3. R-implications

Definition 3.13. A function $I_T : [0, 1]^2 \to [0, 1]$ is called an *R-implication* if there exists a t-norm T such that

$$I_T(x, y) = \sup\{t \in [0, 1] | T(x, t) \leqslant y\}, \quad x, y \in [0, 1]. \tag{6}$$

I_T is also called the residual of the t-norm T.

Example 3.14. Table 3 lists few of the well-known R-implications along with their t-norms from which they have been obtained.

Remark 3.15. The following properties of R-implications can be found in many works, for example [13].

(i) For any t-norm T, not necessarily left-continuous, $I_T \in \mathcal{FI}$ and satisfies (NP), (IP).
(ii) I_T satisfies (OP) if and only if T is border continuous (see [3], Proposition 5.8).
(iii) If T is left-continuous, then I_T also satisfies (EP).
(iv) If $T_1 \leqslant T_2$ then $I_{T_1} \geqslant I_{T_2}$.
(v) If $\varphi \in \Phi$, then $(I_T)_\varphi = I_{T_\varphi}$, i.e., the Φ-conjugate of the residual of a t-norm is the residual of the Φ-conjugate of the t-norm.

If the t-norm T is left-continuous, then we have the following characterization of R-implications generated from left-continuous t-norms.

Theorem 3.16. *For a function $I : [0, 1]^2 \to [0, 1]$ the following statements are equivalent*:

(i) *I is an R-implication generated from a left-continuous t-norm.*
(ii) *I satisfies* (I2), (EP), (OP) *and it is right-continuous with respect to the second variable.*

Moreover, the representation of R-implication, up to left-continuous t-norms, is unique in this case.

Remark 3.17. It should be remarked that the mutual exclusivity of the above properties is not fully proven, i.e., there does not exist any example of a function I that satisfies (EP), (OP), but such that either I satisfies (I2) or is right-continuous with respect to the second variable, but not both.

4. Which R-implications are special, i.e., satisfy (SP)?

Remark 4.1. The following observations are immediate from Remark 3.15.

(i) $I_{\mathbf{RS}}$ and I_1 do not satisfy (NP) and hence cannot be obtained as an R-implication of any t-norm.
(ii) From Proposition 2.7(ii), we see that if I_T satisfies (SP) then $T_{\mathbf{LK}} \leqslant T \leqslant T_{\mathbf{M}}$, i.e., the Gödel and the Lukasiewicz implications, viz., $I_{\mathbf{GD}}$ and $I_{\mathbf{LK}}$, respectively, are the smallest and the largest R-implications that are also special (see Remark 3.15(iv)).

As noted already in the introduction, recently, Sainio et al. [25] have shown the following:

Theorem 4.2 (*Sainio et al. [25], Propositions 1 and 2, Corollary 2*). (i) *An R-implication I_T from a left-continuous t-norm T satisfies* (SP) *if and only if T satisfies the* 1*-Lipschitz condition*.

(ii) *An R-implication I_T from a continuous Archimedean t-norm T satisfies* (SP) *if and only if the additive generators of T is convex*.

Theorem 4.3 (*Sainio et al. [25], Theorem 2*). *Let T be a left-continuous t-norm and I_T the R-implication obtained from T. The following statements are equivalent*:

(i) *I_T satisfies* (SP).
(ii) *T has an ordinal sum representation $(\langle e_\alpha, a_\alpha, T_\alpha\rangle)_{\alpha\in\mathcal{A}}$, where each t-norm T_α, $\alpha \in \mathcal{A}$ is generated by a convex additive generator f_α*.

From Remark 3.15 (ii) it might appear that the condition of left-continuity on T can be relaxed. The following result shows that, on the contrary, the left-continuity, in fact even the continuity of T, is implied and need not even be assumed.

To this end, we present the following result, Proposition 4.4, in its most general sense. The importance of Proposition 4.4 is threefold. Firstly, note that this result characterizes those underlying operators whose residuals satisfy the special property (SP). Secondly, as will be shown in Theorem 4.6, the original results obtained in Sainio et al. [25], viz., Theorems 4.2 and 4.3, can be obtained as easy corollaries of Proposition 4.4 and known results on t-norms. Finally, the characterization also allows us to search for residuals of more general conjunctions satisfying the special property (SP) (see Section 7).

Proposition 4.4. *Let C be any binary function on* $[0, 1]$ *that is non-decreasing in both the variables. Let I_C denote the residual of C obtained using formula* (6). *The following statements are equivalent*:

(i) *The residual I_C of C satisfies* (SP).
(ii) $\sup\{t | C(x, t) \leqslant y\} \leqslant \sup\{t | C(x + \varepsilon, t) \leqslant y + \varepsilon\}$.
(iii) *C satisfies* (SPC), *for any $x, y, t \in [0, 1]$ and every suitable $\varepsilon \in (0, 1)$*,

$$C(x, t) \leqslant y \Longrightarrow C(x + \varepsilon, t) \leqslant y + \varepsilon. \tag{SPC}$$

(iv) *C is* 1*-Lipschitz in the first variable*.

Proof. (i) $\Longleftrightarrow$ (ii): By definition of I_C and the monotonicity of C.

(ii) $\Longrightarrow$ (iii): Let $\sup\{t | C(x, t) \leqslant y\} = t^*$ and $\sup\{t | C(x + \varepsilon, t) \leqslant y + \varepsilon\} = t'$. Since $t^* \leqslant t'$, for any $t < t^*$ we have $C(x, t) \leqslant y$ and since $t < t'$, this also implies $C(x + \varepsilon, t) \leqslant y + \varepsilon$.

(iii) $\Longrightarrow$ (ii): It is immediate, since C satisfies (SPC) implies that $\{t | C(x, t) \leqslant y\} \subset \{t | C(x + \varepsilon, t) \leqslant y + \varepsilon\}$ and hence (ii) follows.

(iii) $\Longrightarrow$ (iv): Let $x_1, x_2, y \in [0, 1]$ be arbitrarily fixed and let $x_1 < x_2$. Then, letting $\varepsilon = x_2 - x_1$, by (SPC) we have that, if $C(x_1, y) \leqslant z$ then $C(x_2, y) = C(x_1 + \varepsilon, y) \leqslant z + \varepsilon$ and hence we have

$$C(x_2, y) - C(x_1, y) \leqslant z + \varepsilon - z = \varepsilon = x_2 - x_1.$$

(iv) $\Longrightarrow$ (iii): The converse is obvious. □

Corollary 4.5. *Let C be any binary function on* $[0, 1]$ *that is non-decreasing in both the variables. If C satisfies* (SPC), *then C is continuous in the first variable.*

By the commutativity of a t-norm T, from the above results we see that an R-implication can be special only if the underlying T is 1-Lipschitz. Thus, from Theorems 3.10, 3.11 and Corollary 3.12, the following result, which is more general than Theorem 4.3, follows immediately.

Theorem 4.6. *Let T be any t-norm* (*not necessarily left-continuous*) *and I_T the R-implication obtained from T. The following statements are equivalent*:

(i) *I_T satisfies* (SP).
(ii) *T satisfies* (SPC).

(iii) *T satisfies the* 1*-Lipschitz condition*.
(iv) *T has an ordinal sum representation $(\langle e_\alpha, a_\alpha, T_\alpha\rangle)_{\alpha\in\mathcal{A}}$, where each t-norm T_α, $\alpha \in \mathcal{A}$ is generated by a convex additive generator f_α.*

Remark 4.7. (i) It should be emphasized that for an arbitrary function C, its residual $I_C \notin \mathcal{FI}$. In Section 7 we will deal with the necessary and sufficient conditions on C to ensure this.

Note that, if C is a t-norm T, then its residual is the R-implication $I_T \in \mathcal{FI}$. However, if C is either a t-subnorm F or a uninorm U (see Definition 7.2), which are non-decreasing in both variables, then their residuals $I_F, I_U \notin \mathcal{FI}$, in general. In the case, C is a fuzzy disjunction, for example, a t-conorm S (see Definition 6.1), their residuals $I_S \notin \mathcal{FI}$ are known as the co-implications (see [8,10]).

5. Which special implications are R-implications?

Now, we attempt the converse problem, which was also the original problem of Hájek and Kohout [18]: "*Which special implications are R-implications*?" The authors themselves have opined that "Not all special implications are R-implications". In fact, as given therein, the least and largest special implications are the Rescher implication $\boldsymbol{I_{RS}}$ and I_1 (see Table 1), respectively. It is immediate from Remark 3.15(i) that, since $\boldsymbol{I_{RS}}$ and I_1 do not satisfy (NP), they cannot be obtained as residuals of any t-norm (not necessarily left-continuous). On the other hand, it can be easily verified that $\boldsymbol{I_{LK}}$, $\boldsymbol{I_{GD}}$ and $\boldsymbol{I_{GG}}$ are all special implications which are also R-implications.

Note, firstly, that any 1-Lipschitz t-norm T is continuous, and hence left-continuous. From Theorem 4.6 we see that if a fuzzy implication I satisfies (SP) and is obtained as a residual of a t-norm T, then T satisfies 1-Lipschitzianity. From Theorem 3.10 we know that a t-norm T satisfies 1-Lipschitzianity if and only if T is a quasi-copula, whose residuals were characterized recently by Durante et al. [12].

Theorem 5.1 (*Durante et al. [12], Theorem 3.9*). *Let Q be a quasi-copula. Then the residual I_Q of Q is left-continuous with respect to the first variable, right-continuous with respect to the second variable and satisfies* (I1), (I2), (OP), (NP) *and the following properties*: *for any $x, y, \varepsilon \in (0, 1)$*

$$I(x+\varepsilon, y) \geqslant I(x, y-\varepsilon), \tag{7}$$

$$I(x, y) \geqslant I(x, y-\varepsilon) + \varepsilon. \tag{8}$$

Using Theorems 3.16 and 5.1, we obtain the following.

Theorem 5.2. *The following statements are equivalent*:

(i) *A special implication I is also a residual of some t-norm T.*
(ii) *I is right-continuous with respect to the second variable and satisfies* (I2), (EP), (OP) *and* (7).

Proof. (i) $\Longrightarrow$ (ii): Let I be a special implication and a residual of some t-norm T. From Theorems 3.10 and 4.6, we see that T is a quasi-copula and hence by Theorem 5.1, I—as its residual—is right-continuous with respect to the second variable and satisfies (I2), (OP) and (7). Since any quasi-copula is continuous and hence is left-continuous, from Theorem 3.16 we see that I also satisfies (EP).

(ii) $\Longrightarrow$ (i): If I is right-continuous with respect to the second variable and satisfies (I2), (EP), (OP), then by Theorem 3.16 there exists a left-continuous t-norm T such that $I = I_T$. Since $y, \varepsilon \in (0, 1)$ are arbitrary in (7), substituting $y+\varepsilon$ for y we see that I is a special fuzzy implication. □

6. Special implications and other families of fuzzy implications

So far we have only been concerned with fuzzy implications that were obtainable as residuals of t-norms, whereas there are many other established families of fuzzy implications, that vary both in their construction and the properties that they satisfy. In this section, we investigate whether any of these families contain some sub-families of special implications. Towards this end, we consider the following families of fuzzy implications, viz., (S,N)-, f- and g-implications. In each of these cases, we investigate whether any of them contain a subclass of special implications.

6.1. (S,N)-Implications and (SP)

The generalization of the classical binary disjunction is the fuzzy union, interpreted in many cases by the triangular conorms. The classical disjunction can be defined from the classical conjunction and negation as follows:

$$p \vee q = \neg(\neg p \wedge \neg q).$$

This duality extends to fuzzy logic operations too and the development of the theory of triangular conorms largely mirrors this duality.

Definition 6.1 (*Klement et al. [20]*). A t-conorm is a function $S: [0,1]^2 \to [0,1]$ such that it satisfies (T1), (T2), (T3) and $S(x,0) = S(0,x) = x$ for all $x \in [0,1]$.

Definition 6.2 (*Fodor and Roubens [13]*). A function $I_{S,N}: [0,1]^2 \to [0,1]$ is called an (*S,N*)-*implication* if there exist a t-conorm *S* and a fuzzy negation *N* such that

$$I_{S,N}(x,y) = S(N(x), y), \quad x, y \in [0,1]. \tag{9}$$

Proposition 6.3. *If $I_{S,N}$ is an (S,N)-implication, then*

(i) *$I_{S,N} \in \mathcal{FI}$ and $I_{S,N}$ satisfies* (NP), (EP);
(ii) *the natural negation of $I_{S,N}$ is the negation N used in its definition, i.e., $N_{I_{S,N}} = N$.*

Now, from Proposition 2.7(i) we see that an $I_{S,N}$ that satisfies (SP) must also satisfy (OP) and the negation $N \leqslant N_{\mathbf{C}}$. However, not all (S,N)-implications satisfy (OP) and we have the following result from Baczyński and Jayaram [3].

Theorem 6.4 (*Baczyński and Jayaram [3], Theorem 4.7*). *For a t-conorm S and a fuzzy negation N the following statements are equivalent*:

(i) *The (S,N)-implication $I_{S,N}$ satisfies* (OP).
(ii) *$N = N_S$ is a strong negation and $S(N_S(x), x) = 1$ for all $x \in [0,1]$, where N_S is called the* natural negation *of S or the* negation induced *by S and is given as follows*:

$$N_S(x) = \inf\{y \in [0,1] \mid S(x,y) = 1\}, \quad x \in [0,1]. \tag{10}$$

From the above discussion it is evident that $I_{S,N}$ is a function that satisfies (I2), (EP), (OP). As noted in Remark 3.17 we can only conjecture that $I_{S,N}$ also is right-continuous with respect to the second variable, and hence is an R-implication obtained from a left-continuous t-norm, in which case we do not obtain any new special implications other than those contained in Theorem 4.6.

6.2. f-Implications and (SP)

Recently, Yager [27] introduced two new classes of fuzzy implications from the additive generators of t-norms and t-conorms called *f*- and *g*-implications, respectively. For more details, we refer the readers to Yager [27], Baczyński and Jayaram [2]. We only give the relevant results here.

Definition 6.5 (*Yager [27]*). Let $f: [0,1] \to [0,\infty]$ be a strictly decreasing and continuous function with $f(1) = 0$. The function $I: [0,1]^2 \to [0,1]$ defined by

$$I(x,y) = f^{-1}(x \cdot f(y)), \quad x, y \in [0,1], \tag{11}$$

with the understanding $0 \cdot \infty = 0$, is called an *f-generated implication*. The function *f* itself is called an *f*-generator of the *I* generated as in (11). In such a case, to emphasize the apparent relation we will write I_f instead of *I*.

Proposition 6.6. *If f is an f-generator, then*

(i) $I_f \in \mathcal{FI}$;
(ii) I_f *satisfies* (NP);
(iii) I_f *does not satisfy* (OP).

From the above result and Proposition 2.7 we see that no *f*-implication can be a special implication.

6.3. g-Implications and (SP)

Definition 6.7 (*Yager [27, p. 202]*). Let $g: [0, 1] \to [0, \infty]$ be a strictly increasing and continuous function with $g(0) = 0$. The function $I: [0, 1]^2 \to [0, 1]$ defined by

$$I(x, y) = g^{(-1)}\left(\frac{1}{x} \cdot g(y)\right), \quad x, y \in [0, 1], \tag{12}$$

with the understanding $\frac{1}{0} = \infty$ and $\infty \cdot 0 = \infty$, is called a *g-generated implication*, where the function $g^{(-1)}$ in (12) is the pseudo-inverse of g given by

$$g^{(-1)}(x) = \begin{cases} g^{-1}(x) & \text{if } x \in [0, g(1)], \\ 1 & \text{if } x \in [g(1), \infty]. \end{cases}$$

The function g itself is called a g-generator of the I generated as in (12). Once again, we will often write I_g instead of I.

Proposition 6.8. *If g is a g-generator, then*

(i) $I_g \in \mathcal{FI}$;
(ii) I_g *satisfies* (NP);
(iii) I_g *satisfies* (OP) *if and only if* I_g *is the Goguen implication* $I_{\mathbf{GG}}$.

From the above result and Proposition 2.7 we see that the only g-implication that is a special implication is the Goguen implication $I_{\mathbf{GG}}$.

6.4. Are there any other special implications?

From the above discussion, it can be seen that the families of (S,N)-, *f*- and *g*-implications do not lead to any new special implications. Then the most natural question that arises is this: *Are there any other special implications, than those that could be obtained as residuals of t-norms*?

Consider the Baczyński implication [1]:

$$I_{\mathbf{BZ}}(x, y) = \min(\max(0.5, \min(1 - x + y, 1)), 2 - 2x + 2y).$$

From the plot of the Baczyński implication $I_{\mathbf{BZ}}$ in Fig. 4(a) it can be easily seen that $I_{\mathbf{BZ}}$ satisfies (SP), a fact that can also be verified from the formula. Interestingly, $I_{\mathbf{BZ}}$ does not belong to any of the families of fuzzy implications considered so far (see Baczyński and Jayaram [4]).

In the rest of this work, we systematically attempt to give an answer to the above question by the following ways:

1. Investigating residuals of more generalized conjunctions than t-norms,
2. Proposing new construction methods for special fuzzy implications, and
3. Generating special implications from special implications, which would mean that one could create infinitely many special implications from a given one.

7. Residuals of generalized conjunctions

T-norms are only one particular generalization of conjunctions on {0, 1} to the interval [0, 1]. Other notable generalizations are the t-subnorms and uninorms. In this section, we investigate whether the residuals of these more generalized conjunctions satisfy (SP). Finally, we determine the class of binary operations with minimal properties whose residuals satisfy (SP). Moreover, we also show that these operations are in a one-to-one bijection with their residuals.

However, note firstly, that from Remark 4.7, we know that not for every arbitrary function C, its residual $I_C \in \mathcal{FI}$. The following result can be easily obtained:

Theorem 7.1 (*cf. Demirli and De Baets [9], Theorem 4.1*). *Let C be a function from* $[0,1]^2$ *to* $[0,1]$ *and let* I_C *denote its residual, using formula* (6). *Then we have the following equivalences*:

(i) I_C *satisfies* (I1) *if and only if C is non-decreasing in the first variable.*
(ii) I_C *satisfies* (I2) *if and only if C is non-decreasing in the second variable.*
(iii) I_C *satisfies* (I3) *if and only if* $C(0,1)=0$.
(iv) I_C *satisfies* (I4) *always.*
(v) I_C *satisfies* (I5) *if and only if* $C(1,x)>0$ *for all* $x \in (0,1]$.

7.1. Residuals of T-subnorms

Definition 7.2 (*Jenei [19]*). A t-subnorm is a function $M:[0,1]^2 \to [0,1]$ such that it satisfies (T1), (T2), (T3) and $M(x,y) \leqslant \min(x,y)$ for all $x,y \in [0,1]$.

Obviously, every t-norm is a t-subnorm, however, the converse is not true. Usually, for emphasis, a t-subnorm that is not a t-norm is called a *proper* t-subnorm (see [23]).

Note that it is not insisted that $M(1,x)>0$ for all $x \in (0,1]$. It is clear now, from Theorem 7.1, that residuals obtained from t-subnorms M using formula (6) need not satisfy (I5) and hence need not be fuzzy implications. However, such residuals, if required, can be suitably redefined at the point $(1,0)$ to make them fuzzy implications. Hence, without loss of generality, we consider the residuals of t-subnorms to be fuzzy implications.

We now investigate which subclass of t-subnorms give rise to residuals that satisfy (SP). Since M is non-decreasing in both variables, by Proposition 4.4, we see that a necessary condition on a t-subnorm M for the corresponding I_M to satisfy (SP) is that M be 1-Lipschitz in both variables and hence continuous. In the following, we list some of the relevant results on continuous t-subnorms, see [22,23].

Theorem 7.3 (*Mesiar and Mesiarová [23], Theorem 2*). *For a function* $M:[0,1]^2 \to [0,1]$ *the following statements are equivalent*:

(i) *M is a continuous t-subnorm.*
(ii) *M is an ordinal sum of continuous Archimedean t-norms and a continuous Archimedean t-subnorm, i.e., there exist a uniquely determined (finite or countably infinite) index set* $\mathcal{K}$, *a family of uniquely determined pairwise disjoint open sub-intervals* $\{(a_k,b_k)\}_{k\in\mathcal{K}}$ *of* $[0,1]$ *with* $b_{k_0}=1$ *for some* $k_0 \in \mathcal{K}$, M_{k_0} *is a continuous Archimedean proper t-subnorm and* M_k *is a continuous Archimedean t-norm for all* $k \neq k_0$ *such that*

$$M(x,y)=\begin{cases} a_k+(b_k-a_k)\cdot M_k\left(\dfrac{x-a_k}{b_k-a_k},\dfrac{y-a_k}{b_k-a_k}\right) & \text{if } x,y \in (a_k,b_k],\\ \min(x,y) & \text{otherwise.}\end{cases}$$

In this case we will write $M=(\langle a_k,b_k,M_k\rangle)_{k\in\mathcal{K}}$.

In other words, the above result states that a continuous t-subnorm is made up of ordinal summands all of which are continuous Archimedean t-norms, except for the summand at the top-right of the unit square, which is a continuous Archimedean t-subnorm. Although, a representation of continuous Archimedean t-subnorms, in general, is not yet known, the following result is available.

Theorem 7.4 (*Mesiar and Mesiarová [22], Theorem 3*). *For a function* $M:[0,1]^2 \to [0,1]$ *the following statements are equivalent*:

(i) *M is a continuous strictly monotone Archimedean t-subnorm.*

(ii) *There is a continuous strictly decreasing mapping* $m:[0,1]\to[0,\infty]$ *with* $m(0)=\infty$, *such that*

$$M(x,y)=m^{-1}(m(x)+m(y)),\quad x,y\in[0,1]. \tag{13}$$

Theorem 7.5 (*Mesiar and Mesiarová [23], Theorem 3*). *For a continuous Archimedean t-subnorm M the following statements are equivalent*:

(i) *A continuous non-increasing mapping* $m:[0,1]\to[0,\infty]$ *is an additive generator of M, i.e.,* $M(x,y)=m^{(-1)}(m(x)+m(y))$ *for all* $x,y\in[0,1]$, *where* $m^{(-1)}$ *is the pseudo-inverse of m given as follows*:

$$m^{(-1)}(x)=\{z\in[0,1]\,|\,m(z)>x\},\quad x\in[0,1]. \tag{14}$$

(ii) *m is such that* $m(1)>0$ *and m is strictly monotone on the interval* $[0,m^{(-1)}(2m(1))]$.

The proof of the following result is a straight-forward calculation based on the convexity of the generators and their pseudo-inverses.

Theorem 7.6. *Let M be a continuous t-subnorm such that its last summand has an additive generator and* I_M *its residual. If M has an ordinal sum representation as given in Theorem 7.3 and each of the summands is generated by a convex additive generator* m_k, *then* I_M *satisfies* (SP).

In the following Table 4 we give some examples of t-subnorms along with their residuals. The fact that these residuals are fuzzy implications and also satisfy (SP) can be easily seen from the geometric interpretation of the speciality property given in Section 2.2 and their plots given in Fig. 2(a)–(c). Note that $M_2(1,y)=0$ for all $y\leqslant a$ and hence the need to suitably redefine its residual at $(1,0)$. However, this redefinition does not affect I_{M_2} from satisfying (SP), since the point $(1,0)$ is, in a sense, "a one-point line" that is parallel to the main diagonal and can be redefined independently. Although all the t-subnorms M_1–M_3 are continuous, M_1 is Archimedean and strictly monotone, while M_2 is only Archimedean, whereas M_3 is not Archimedean. However, the t-subnorms M_1–M_3 can be generated from convex generators (see [22], Example 1).

Remark 7.7. The convexity of the additive generator of a t-subnorm is not necessary for the obtained residual to satisfy (SP). Consider, for example, the following continuous but non-convex additive generator and its pseudo-inverse:

$$m_4(x)=\begin{cases}3-4x & \text{if } x\in\left[0,\frac14\right],\\ 2 & \text{if } x\in\left[\frac14,\frac12\right],\\ 4-4x & \text{if } x\in\left[\frac12,\frac34\right],\\ 1 & \text{if } x\in\left[\frac34,1\right],\end{cases}\qquad m_4^{(-1)}(x)=\begin{cases}1 & \text{if } x\in[0,1),\\ 1-\dfrac{x}{4} & \text{if } x\in[1,2),\\ \dfrac{3-x}{4} & \text{if } x\in[2,3],\\ 0 & \text{if } x\in[3,\infty].\end{cases}$$

Then the continuous t-subnorm M_4 generated from m_4 and its residual I_{M_4} are given as follows (note that $I_{M_4}(1,0)$ has been suitably redefined):

$$M_4(x,y)=\begin{cases}x+y-\frac54 & \text{if } x,y\in[\frac12,\frac34] \text{ and } y+x\geqslant\frac54,\\ x-\frac12 & \text{if } x\in[\frac12,\frac34] \text{ and } y\in[\frac34,1],\\ y-\frac12 & \text{if } y\in[\frac12,\frac34] \text{ and } x\in[\frac34,1],\\ \frac14 & \text{if } x,y\in[\frac34,1],\\ 0 & \text{otherwise},\end{cases}$$

$$I_{M_4}(x,y)=\begin{cases}\dfrac54-x+y & \text{if } y\in[0,\frac14], x\in[\frac12,\frac34] \text{ and } x-y\geqslant\frac12,\\ y+\frac12 & \text{if } y\in[0,\frac14], x\in[\frac34,1] \text{ and } (x,y)\neq(1,0),\\ 0 & \text{if } (x,y)=(1,0),\\ 1 & \text{otherwise}.\end{cases}$$

From the plot of I_{M_4}, given in Fig. 2(d), it is clear that I_{M_4} indeed is a special fuzzy implication.

Table 4
Examples of some t-subnorms whose residuals satisfy (SP).

T-subnorm M	Residual I_M
$M_1 : \dfrac{x \cdot y}{2}$	$I_{M_1} : \begin{cases} 1 & \text{if } (x, y) = (0, 0) \\ \min\left(1, \dfrac{2 \cdot y}{x}\right) & \text{otherwise} \end{cases}$
$M_2 : \max(0, \min(x + y - 1, x - a, y - a, 1 - 2a)), a \in [0, 0.5]$	$I_{M_2} : \begin{cases} 0 & \text{if } (x, y) = (1, 0) \\ y + \max(a, 1 - x) & \text{if } y \leqslant \min(x - a, 1 - 2a) \\ 1 & \text{otherwise} \end{cases}$
$M_3 : \max(0, \min(x + y - 0.5, x, y, 0.5))$	$I_{M_3} : \begin{cases} 1 & \text{if } \min(x, 0.5) \leqslant y \\ \max(y, y - x + 0.5) & \text{otherwise} \end{cases}$

7.2. Residuals of uninorms

Definition 7.8 (*Yager and Rybalov [28]*). A uninorm is a function $U: [0, 1]^2 \to [0, 1]$ such that it satisfies (T1), (T2), (T3) and there exists an $e \in [0, 1]$ such that $U(x, e) = U(e, x) = x$ for all $x \in [0, 1]$.

Remark 7.9. (i) It is easy to see that if $e = 1$ then U is a t-norm.

(ii) Any uninorm U with neutral element $e \in (0, 1)$ behaves like a t-norm on the square $[0, e]^2$ and as a t-conorm on the square $[e, 1]^2$. In fact, for every uninorm U there exists a t-conorm S such that, on $[e, 1]^2$, U can be expressed as

$$U(x, y) = 1 - e + S\left(\frac{x - e}{1 - e}, \frac{y - e}{1 - e}\right). \tag{15}$$

(iii) In the case of residuals from uninorms obtained as in (6) with a uninorm U instead of a t-norm T (let us denote them as I_U), we only remark that, once again, an I_U is a fuzzy implication if and only if $U(x, 0) = 0$ for all $x \in (0, 1)$.

Unfortunately, when we generalize the t-norm to a uninorm we do not obtain any special implications as their residuals, as shown below.

Proposition 7.10. *If U is a uninorm, then its residual I_U does not satisfy* (SP).

Proof. Let U be a uninorm with neutral element $e \in (0, 1)$. Firstly, we show that I_U does not satisfy (IP). Let us fix an arbitrary $x \in (e, 1)$. Clearly, $x = U(e, x) \leqslant U(x, x) \leqslant U(x, 1) = 1$. For any $1 > t > x$, from (15), we have that

$$U(x, t) = 1 - e + S\left(\frac{x - e}{1 - e}, \frac{t - e}{1 - e}\right) \geqslant 1 - e + \frac{t - e}{1 - e} = t,$$

i.e., $I_U(x, x) = \sup\{t | U(x, t) \leqslant x\} \lneq 1$, i.e., I_U does not satisfy (IP). The result now follows from Proposition 2.5(i). □

7.3. Residuals of semi-copulas

From Theorem 7.1 and Proposition 4.4 the following result easily follows:

Theorem 7.11. *Let $C: [0, 1]^2 \to [0, 1]$ be such that it is non-decreasing in both variables, 1-Lipschitz in the first variable and*

$$C(1, x) > 0, \quad x \in (0, 1], \tag{16}$$

$$C(x, 1) \leqslant x, \quad x \in [0, 1]. \tag{17}$$

Then the residual I_C of C is a special fuzzy implication.

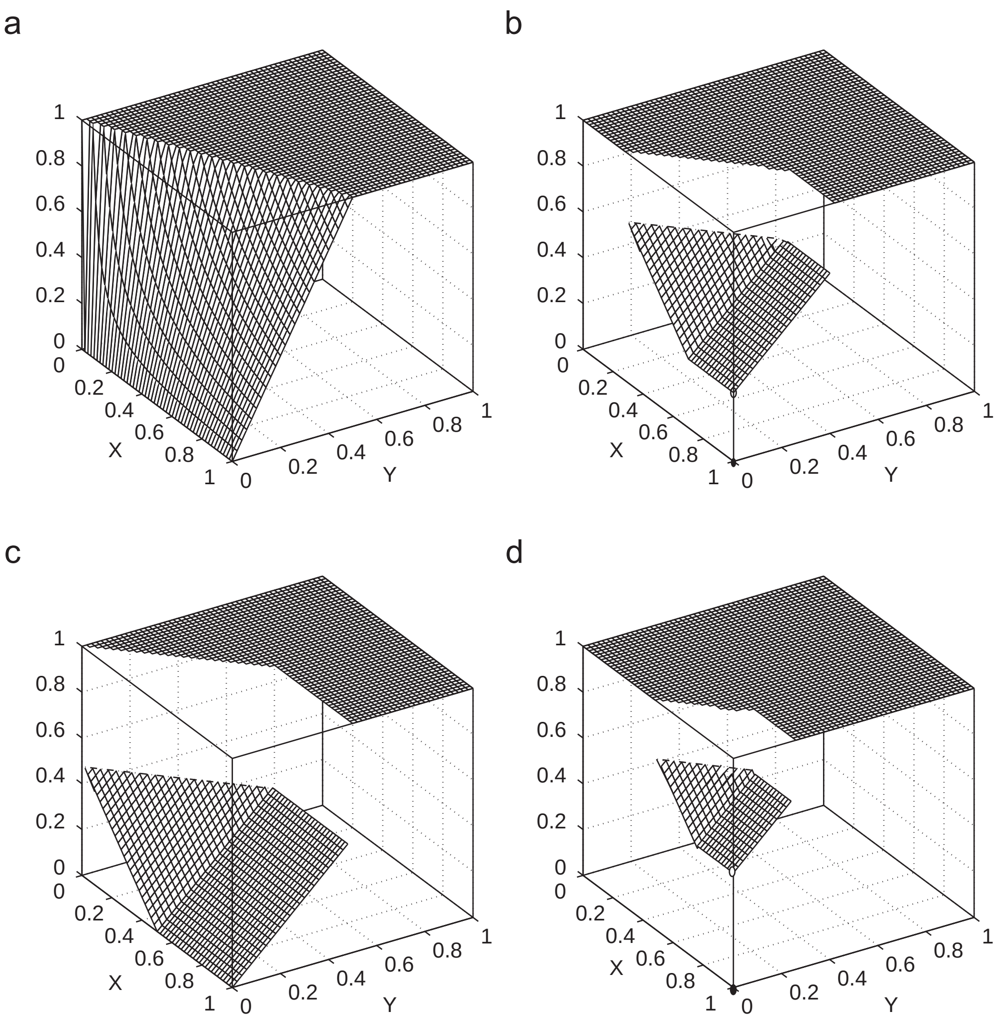

Fig. 2. Plots of the residuals of the t-subnorms (see Example 4 and Remark 7.7). (a) I_{M_1}, (b) I_{M_2}, (c) I_{M_3}, (d) I_{M_4}.

Since for any $I \in \mathcal{FI}$ we have that $I(x, 1) = 1$ for any $x \in [0, 1]$ the following definition is valid.

Definition 7.12. For an $I \in \mathcal{FI}$ we define a mapping $C_I : [0, 1]^2 \to [0, 1]$ as

$$C_I(x, y) = \inf\{t \in [0, 1] \mid I(x, t) \geqslant y\}, \quad x, y \in [0, 1]. \tag{18}$$

C_I is also called the deresiduum of I (see [12]).

It is well-known that for certain classes of binary operations on [0, 1], in fact, conjunctors there exists a one-to-one bijection between them and their residuals. Hence, we now investigate the minimal assumptions required on a binary operation C on [0, 1] so as to obtain, on the one hand, special fuzzy implications as their residuals, and on the other hand, a one-to-one bijection between them and such residuals. We see that semi-copulas that are 1-Lipschitz in the first variable are exactly the class of operations that we are looking for.

Theorem 7.13. *Let C be a semi-copula that is* 1*-Lipschitz in the first variable. Then the residual I_C of C is a special fuzzy implication that also satisfies* (NP), *i.e.*, $I_C \in \mathcal{FI}$ *and satisfies* (NP) *and* (SP).

Conversely, if $I \in \mathcal{FI}$ and satisfies (NP) and (SP), then its deresiduum C_I is a semi-copula that is 1-Lipschitz in the first variable.

Proof. If C is a semi-copula that is 1-Lipschitz in the first variable, then from Theorem 7.11 we see that I_C is a special fuzzy implication. Note also that I_C satisfies (NP), since

$$I_C(1, y) = \sup\{t | C(1, t) \leqslant y\} = y, \quad y \in [0, 1]. \tag{19}$$

Conversely, let $I \in \mathcal{FI}$ and satisfy (NP) and (SP). Firstly, from Proposition 2.7(i) we have that I satisfies (OP). Let C_I be defined as in (18). From Theorem 7.1 we see that C_I is monotonic in both the variables. The following equalities show that C_I indeed is a semi-copula:

$$C_I(x, 1) = \inf\{t \in [0, 1] \mid I(x, t) \geqslant 1\} = x \quad \text{by (OP)},$$

$$C_I(1, x) = \inf\{t \in [0, 1] \mid I(1, t) \geqslant x\} = x \quad \text{by (NP)}.$$

From Proposition 4.4 it follows that C_I is 1-Lipschitz in the first variable. □

8. Special implications: some constructions

Thus far we have investigated known families of fuzzy implications for sub-families that satisfy (SP). In this section, we propose specific methods to construct special implications, often with some additional desirable property.

8.1. Neutral special implications with a given negation

Other than the special implications obtained as residuals of t-norms, the special implications seen so far in this work, viz., as residuals of t-subnorms or the Baczyński implication $I_{\mathbf{BZ}}$, do not satisfy the neutrality property (NP). Note, firstly, that if a special implication I also satisfies (NP), then its natural negation N_I is less than the classical negation $N_{\mathbf{C}}$. Given a negation $N \leqslant N_{\mathbf{C}}$, we now propose a method to construct special fuzzy implications that also satisfy (NP).

Let a fuzzy negation $N \leqslant N_{\mathbf{C}}$ be given. Define the function $I_N: [0, 1]^2 \to [0, 1]$ as follows:

$$I_N(x, y) = \begin{cases} 1 & \text{if } x \leqslant y, \\ y + \dfrac{N(x-y)(1-x)}{1-x+y} & \text{if } x > y, \end{cases} \quad x, y \in [0, 1]. \tag{20}$$

It can be easily verified that I_N is a special fuzzy implication and that it satisfies (NP). In fact, the implications obtained from the negations $N_{\mathbf{C}}$ and $N_{\mathbf{D1}}$ using (20) are the Łukasiewicz and the Gödel implications, respectively, i.e., $I_{N_{\mathbf{C}}} = I_{\mathbf{LK}}$ and $I_{N_{\mathbf{D1}}} = I_{\mathbf{GD}}$. Moreover, these are the only fuzzy implications obtained using (20) and are residuals of a t-norm. Fig. 3 gives the plots of special implications constructed from (20) for the strict negation $N_1(x) = 1 - \sqrt{x}$ and the strong negation $N_2(x) = (1 - \sqrt{x})^2$.

However, for a given negation N the special implications I_N may not be the minimal fuzzy implications, in the sense of the underlying order on $[0, 1]^2$. In the next subsection, we propose a method whose construction will ensure this.

8.2. Minimal special implications with a given negation

Definition 8.1. Let $f: [-1, 1] \to [0, 1]$ be any non-increasing function such that $f(u) = 1$ whenever $u \leqslant 0$ and $f(1) = 0$. Define the function $I^{(f)}: [0, 1]^2 \to [0, 1]$ as follows:

$$I^{(f)}(x, y) = f(x - y), \quad x, y \in [0, 1]. \tag{21}$$

Remark 8.2. (i) Once again, it can be easily verified that $I^{(f)}$ is a special fuzzy implication.

(ii) It is also immediately clear that the natural negation of $I^{(f)}$ is $N_{I^{(f)}} = f|_{[0,1]}$.

(iii) $I^{(f)}$ is the minimal special implication with respect to the negation $N = N_{I^{(f)}} = f|_{[0,1]}$. Indeed, note that for every point on the line parallel to the diagonal, viz., on the line joining $(x, 0)$ and $(1, 1 - x)$ for any arbitrary $x \in (0, 1)$, we have that $I^{(f)}$ is a constant value equal to the function value, i.e., $I^{(f)}(x, 0) = I^{(f)}(1, 1 - x) = f(x)$.

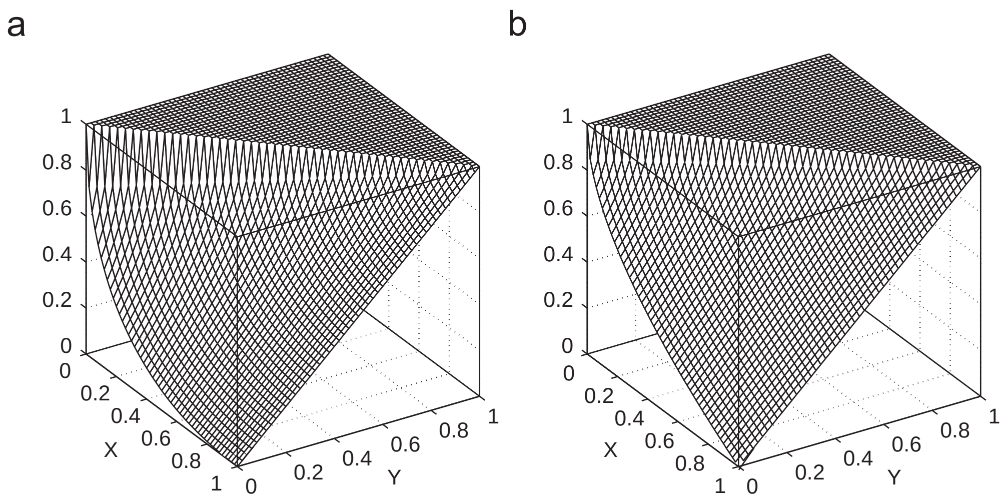

Fig. 3. Plots of some special fuzzy implications constructed using (20). (a) I_{N_1} with $N_1(x) = 1 - \sqrt{x}$, (b) I_{N_2} with $N_2(x) = (1 - \sqrt{x})^2$.

(iv) Obviously, an $I^{(f)}$ satisfies (NP) if and only if $f(x) = 1 - x$ on [0, 1], in which case we get the Łukasiewicz implication.

8.3. A specific method

Finally, we give a very specific way of obtaining special implications from existing ones. To this end, let us consider the following generalization of $I_{\mathbf{BZ}}$:

$$I_{\mathbf{BZ}}^{\langle\alpha,\beta\rangle}(x, y) = \min(\max(\alpha, \min(1 - x + y, 1)), \beta - \beta \cdot x + \beta \cdot y),$$

where $\alpha \in (0, 1)$; $\beta \geqslant 1$. It can be easily verified that $I_{\mathbf{BZ}}^{\langle\alpha,\beta\rangle}$ satisfies (SP). In fact, the value $\alpha \in (0, 1)$ determines the "height" at which the graph of the function becomes "flat". In the case when $\beta = 1$, $I_{\mathbf{BZ}}^{\langle\alpha,1\rangle}$ is nothing but the Łukasiewicz implication $I_{\mathbf{LK}}$ for any $\alpha \in (0, 1)$. In the case when $\beta > 1$ is fixed and varying $\alpha \in (0, 1)$, the plot of $I_{\mathbf{BZ}}^{\langle\alpha,\beta\rangle}$ is nothing but cutting the Łukasiewicz implication $I_{\mathbf{LK}}$ flat at the level of α. For a fixed α, varying $\beta > 1$ determines the "tapering" of the plot of $I_{\mathbf{BZ}}^{\langle\alpha,\beta\rangle}$ towards the point (1, 0) as Fig. 4(a)–(c) illustrate below. Note that the same process can also be applied to other special implications. Fig. 4(d) gives the plot of the Gödel implication $I_{\mathbf{GD}}$ "cut" at the value 0.5.

9. Generating special implications from special implications

Let us consider the set of all fuzzy implications denoted by $\mathcal{FI}$. Since these are basically functions, many operations performed on functions to obtain newer functions with similar properties can also be applied to a fuzzy implication. For example, three of the most popular and general ways of obtaining newer fuzzy implications from existing or given fuzzy implications are as follows:

- From a pair of fuzzy implications I, J we can consider the lattice meet and join operations, viz., $I \vee J$, $I \wedge J$.
- One can also obtain their convex combinations.
- Yet another typical way of generating newer fuzzy implications from a given fuzzy implication is by means of automorphisms of [0, 1].

Interestingly, all the above operations turn out to be fuzzy implications. In this section, we investigate whether these operations preserve (SP), i.e., if the original fuzzy implications satisfy (SP), will the new fuzzy implications obtained as above also satisfy (SP).

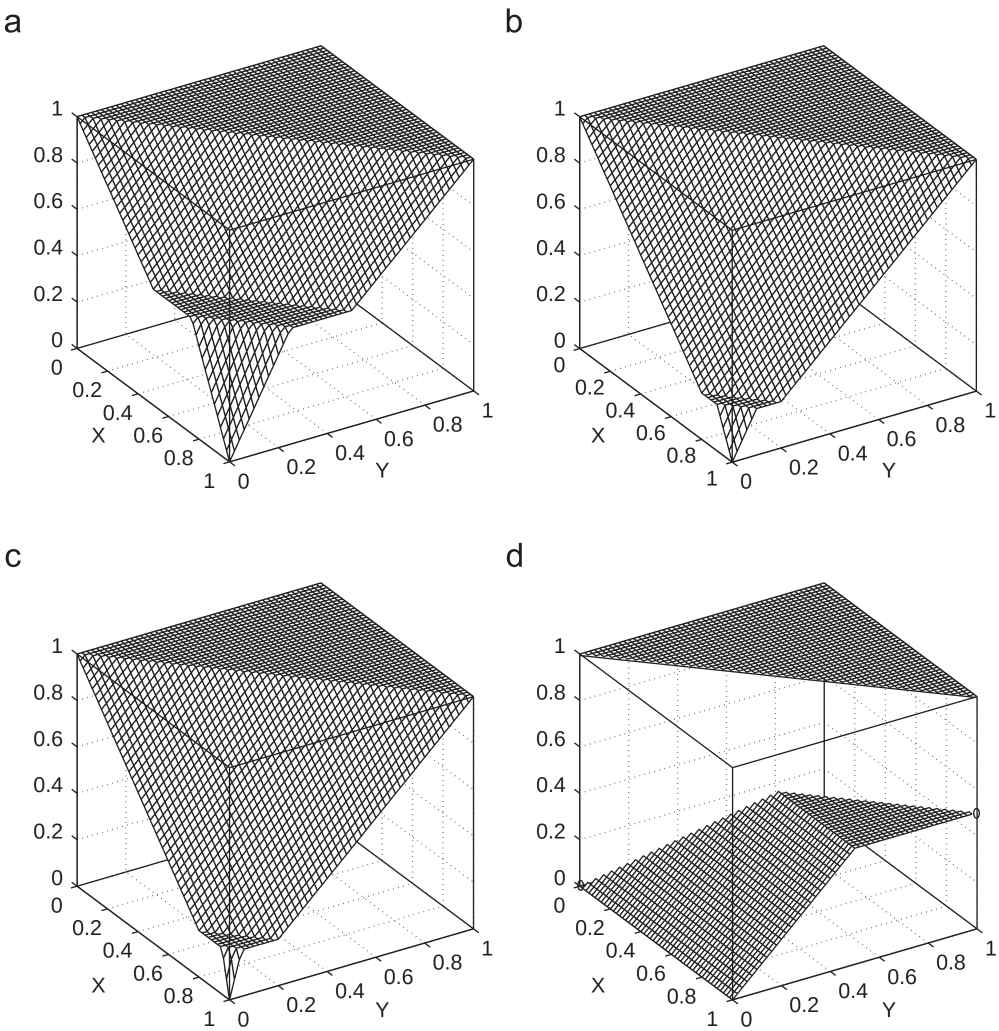

Fig. 4. Plots of the generalization of Baczyński implication, viz., $I_{\mathbf{BZ}}^{\langle\alpha,\beta\rangle}$, for different values of α, β: (a)–(c); (d) the Gödel implication $I_{\mathbf{GD}}$ "cut" at the value 0.5 (see Section 8.3 for more details). (a) $I_{\mathbf{BZ}} = I_{\mathbf{BZ}}^{\langle 0.5,2\rangle}$, (b) $I_{\mathbf{BZ}}^{\langle 0.2,2\rangle}$, (c) $I_{\mathbf{BZ}}^{\langle 0.2,4\rangle}$, (d) $I_{\mathbf{GD}}$ "cut" at 0.5.

9.1. Convex combinations and (SP)

Definition 9.1 (*Baczynski and Jayaram [4]*). Let $I, J \in \mathcal{FI}$ and $\lambda \in [0,1]$. The convex combination of I, J is the function

$$K = \lambda \cdot I + (1-\lambda) \cdot J,$$

which is again a fuzzy implication, i.e., $K \in \mathcal{FI}$.

Proposition 9.2. *Let $I, J \in \mathcal{FI}$ both satisfy* (SP). *Then their convex combination also satisfies* (SP).

Proof. Let $I, J \in \mathcal{FI}$ both satisfy (SP). Let the convex combination of I, J be the function $K = \lambda \cdot I + (1-\lambda) \cdot J$, which, as we already know, is again a fuzzy implication. Hence it only remains to show that K satisfies (SP). Let $\varepsilon > 0$ be arbitrary and let $x, y \in [0,1]$ be such that $x+\varepsilon, y+\varepsilon \in [0,1]$. The result is now immediate from the following inequalities:

$$K(x,y) = \lambda \cdot I(x,y) + (1-\lambda) \cdot J(x,y)$$

$$\leqslant \lambda \cdot I(x+\varepsilon, y+\varepsilon) + (1-\lambda) \cdot J(x+\varepsilon, y+\varepsilon)$$
$$= K(x+\varepsilon, y+\varepsilon).$$

Thus, we see that convex combinations of special implications is again a special implication.

9.2. Lattice operations and (SP)

In the family $\mathcal{FI}$ of all fuzzy implications we can consider the partial order induced from the unit interval [0, 1]. It is interesting and important to note that incomparable pairs of fuzzy implications generate new fuzzy implications by using the standard min (inf) and max (sup) operations. This is another method of generating new fuzzy implications from the given ones.

Definition 9.3 (*Baczynski and Jayaram [4]*). Let $I, J \in \mathcal{FI}$. The meet and join of I, J, defined as below:

$$(I \vee J)(x, y) := \max(I(x, y), J(x, y)), \quad x, y \in [0, 1], \tag{22}$$

$$(I \wedge J)(x, y) := \min(I(x, y), J(x, y)), \quad x, y \in [0, 1], \tag{23}$$

again become fuzzy implications, i.e., both $I \vee J, I \wedge J \in \mathcal{FI}$.

Similar to Proposition 9.2 we have the following result, which again can be proven in a straight-forward way.

Proposition 9.4. *Let $I, J \in \mathcal{FI}$ both satisfy* (SP). *Then both $I \vee J, I \wedge J$ satisfy* (SP).

Thus, we see that the meet and join of a pair of special implications is again a special implication.

9.3. Conjugacy and (SP)

Definition 9.5 (*Baczynski and Jayaram [4]*). Let $I \in \mathcal{FI}$ and φ be any increasing bijection on [0, 1]. Then the φ-conjugate of *I* is given by

$$I_\varphi(x, y) = \varphi^{-1}(I(\varphi(x), \varphi(y))), \quad x, y \in [0, 1]$$

and $I_\varphi \in \mathcal{FI}$.

Once again, the interesting question is whether conjugacy preserves (SP), i.e., if *I* satisfies (SP) then is it true that I_φ also satisfies (SP)? Unfortunately, in general this need not be the case.

To see this, let $I = I_{\mathbf{LK}}$, the Łukasiewicz implication which does satisfy (SP). If we consider the set of increasing bijections $\varphi_p(x) = x^p$ for any $p \in \mathbb{N}$; $p > 1$, then $(I_{\mathbf{LK}})_{\varphi_p}$ are exactly the R-implications given in Sainio et al. [25] after Corollary 1, which, as has been shown therein, do not satisfy (SP).

In the following, we give a necessary and sufficient condition on the bijection φ so that the Φ-conjugate preserves (SP).

Theorem 9.6. *Let φ be any increasing bijection on* [0, 1]. *Then the following are equivalent*:

(i) *For each special fuzzy implication I, I_φ is a special fuzzy implication.*
(ii) *φ is concave.*

Proof. (i) $\Longrightarrow$ (ii): Let φ be any increasing bijection on [0, 1] and I_φ be a special fuzzy implication for any special fuzzy implication *I*. Then, this is also true when $I = I_{\mathbf{LK}}$, Łukasiewicz implication, which is a special fuzzy implication. Hence, for this φ, $(I_{\mathbf{LK}})_\varphi$ is also a special fuzzy implication. However, $(I_{\mathbf{LK}})_\varphi$ is a residual implication obtained from a nilpotent t-norm with an additive generator $t: [0, 1] \to [0, 1]$ such that $t = 1 - \varphi$. From Theorem 4.6(iv) we know that *t* is convex and thus φ is concave.

(ii) $\Longrightarrow$ (i): Conversely, let φ be any increasing bijection that is concave and let *I* be a special fuzzy implication. Clearly, I_φ is a fuzzy implication which satisfies (IP). Hence, it suffices to consider the case when $x > y$, i.e., we only need to show that $I_\varphi(x, y) \leqslant I_\varphi(x + \varepsilon, y + \varepsilon)$ for each $x > y$ and $\varepsilon > 0$. Since I satisfies (SP), we know that $I(\varphi(x), \varphi(y)) \leqslant I(\varphi(x) + \delta, \varphi(y) + \delta)$ for any $\delta > 0$ such that $\varphi(x) + \delta, \varphi(y) + \delta \in [0, 1]$. Fixing $\delta = \varphi(x + \varepsilon) - \varphi(x)$ we have

$$I(\varphi(x), \varphi(y)) \leqslant I(\varphi(x + \varepsilon), \varphi(y) + \varphi(x + \varepsilon) - \varphi(x)).$$

Now, by the concavity of φ and the monotonicity (I2) of I, we have

$$\begin{aligned} I(\varphi(x), \varphi(y)) &\leqslant I(\varphi(x + \varepsilon), \varphi(y) + \varphi(x + \varepsilon) - \varphi(x)) \\ &\leqslant I(\varphi(x + \varepsilon), \varphi(y + \varepsilon)), \end{aligned}$$

i.e., I_φ is a special fuzzy implication. □

9.4. Yet another transformation of special implications

Finally, in this section, we propose yet another transformation of a special fuzzy implication which preserves (SP). As the final result in this section shows, we can characterize all special fuzzy implications that obey the equality in (SP) based on this transformation.

Definition 9.7. Let φ be any real function on [0, 1] and $I \in \mathcal{FI}$. Let us define the following transformation:

$$\varphi(I)(x, y) = \varphi(I(x, y)), \quad x, y \in [0, 1]. \tag{24}$$

The following result is immediate from the properties of an $I \in \mathcal{FI}$:

Proposition 9.8. *Let φ be any real function on* [0, 1]. *The following are equivalent*:

(i) *For any special fuzzy implication I, the transformation $\varphi(I)$, as defined in* (24), *is also a special fuzzy implication*.
(ii) *φ is non-decreasing and $\varphi(0) = 0$; $\varphi(1) = 1$.*

Remark 9.9. The following observations can be made about the transformation $\varphi(I)$:

(i) $\varphi(I)$ preserves (NP) only for $\varphi = id$.
(ii) $\varphi(I)$ preserves (OP) only if φ is such that $\varphi(x) = 1 \Longleftrightarrow x = 1$.
(iii) Consider the function f as given in Section 8.2. Then $I^{(f)} = \varphi(I_{\mathbf{LK}})$, where $\varphi(u) = f(1 - u)$ for $u \in [0, 1]$.

Proposition 9.10. *For an $I \in \mathcal{FI}$ the following are equivalent*:

(i) *$I(x, y) = I(x + \varepsilon, y + \varepsilon)$ for all admissible $x, y, \varepsilon \in [0, 1]$.*
(ii) *There exists a non-decreasing function h: [0, 1] → [0, 1] such that $h(0) = 0$; $h(1) = 1$ and $I = h(I_{\mathbf{LK}})$, the transformation defined in* (24).

Proof. Firstly, note that for any non-decreasing function f we have $f(\min(x, y)) = \min(f(x), f(y))$.

(i) ⟹ (ii): Let a fuzzy implication I be such that $I(x, y) = I(x + \varepsilon, y + \varepsilon)$ for all admissible $x, y, \varepsilon \in [0, 1]$. Then by (2), we have that $I(x, y) = I(1, 1 - x + y)$ for all $x, y \in [0, 1]$. Let us define $h(x) = I(1, x)$ for all $x \in [0, 1]$. It is immediate that $h: [0, 1] \to [0, 1]$ is such that $h(0) = 0$; $h(1) = 1$.

Now, the result follows from the following equalities:

$$\begin{aligned} h(I_{\mathbf{LK}}(x, y)) &= h(\min(1, 1 - x + y)) \\ &= I(1, \min(1, 1 - x + y)) \\ &= \min(I(1, 1), I(1, 1 - x + y)) \\ &= I(1, 1 - x + y) \\ &= I(x, y), \quad x, y \in [0, 1]. \end{aligned}$$

(ii) ⟹ (i): This follows directly from the fact that the Łukasiewicz implication $I_{\mathbf{LK}}$ obeys the equality in (SP).

10. Concluding remarks

In this work, we have investigated in-depth fuzzy implications and the *special* property (SP). We have shown that 1-Lipschitzianity in the first variable characterizes general monotonic binary operations whose residuals satisfy (SP). Following this, we have shown that not all special implications are residuals of t-norms, but that there are special implications which are residuals of more general conjunctions. We have investigated the minimal assumptions required on a binary operation on [0, 1] so as to obtain a one-to-one bijection between them and their residuals which are special fuzzy implications. We have investigated the well-known families of fuzzy implications, viz., (S,N)-, *f*- and *g*-implications, and shown that they do not seem to give rise to any hitherto unknown special implications. Finally, some constructive procedures to obtain special fuzzy implications are proposed and methods of obtaining special implications from existing ones are given, showing that there are infinitely many fuzzy implications that are special but cannot be obtained as residuals of t-norms.

Acknowledgments

This work was done when the first author was a visiting researcher at the Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology, Bratislava. He gratefully acknowledges the grant obtained from SAIA under the National Scholarship Programme (NSP) for the mobility of Teachers and Researchers that made this visit possible. The second author was supported by the Grants VEGA 1/3006/06 and MSM VZ 6198898701.

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