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QL-implications: Some properties and intersections

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Abstract

In this paper, we attempt a systematic study of QL-implications. Towards this end, firstly, we investigate the conditions under which a QL-operation becomes a fuzzy implication without imposing any conditions on the underlying operations. Following this, we discuss the conditions under which this family satisfies some desirable algebraic properties. Based on the obtained results and existing characterization results, the intersections between QL-implications and the two most established families of fuzzy implications, viz., (S,N)- and R-implications are determined. It is shown that QL-implications contain the set of all R-implications obtained from left-continuous t-norms that are also (S,N)-implications. Finally, the overlaps between QL-implications and the recently proposed *f*- and *g*-implications are also studied.

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1. Introduction

Fuzzy implications were introduced and studied in the literature as a generalization of the classical implication operation that obeys the truth table provided in Table 1. Following are the two main ways of defining an implication in the Boolean lattice $(L, \wedge, \vee, \neg)$:

$$p \to q \equiv \neg p \vee q, \tag{1}$$

$$p \to q \equiv \max\{t \in L \mid p \wedge t \leqslant q\}, \tag{2}$$

where $p, q \in L$ and the relation $\leqslant$ is defined in the usual way, i.e., $p \leqslant q$ iff $p \vee q = q$, for every $p, q \in L$. Implication (1) is usually called the material implication, while (2) is from the intuitionistic logic framework, where the implication is obtained as the residuum of the conjunction, and is often called as the pseudo-complement of p relative to q (see [6]). It is important to note that, despite their different formulas, expressions (1) and (2) are equivalent in the Boolean lattice $(L, \wedge, \vee, \neg)$. Interestingly, in the fuzzy logic framework, where the truth values can vary in the unit interval [0, 1], the natural generalizations of the above definitions, viz., (S,N)- and R-implications, are not equivalent. This variety has

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Table 1
Truth table for the classical implication

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

led to some intensive research on fuzzy implications for close to three decades. Quite understandably then, the most established and well-studied classes of fuzzy implications are the above (S,N)- and R-implications (cf. [9,10,12,17]). For a broader analysis of fuzzy implications and their applications we refer the readers to Mas et al. [22] and the latest monograph [5].

Yet another popular way of obtaining fuzzy implications is as a generalization of the following implication defined in quantum logic:

$$p \to q \equiv \neg p \vee (p \wedge q).$$

Needless to state, when the truth values are restricted to {0, 1} its truth table coincides with that of the material and intuitionistic-logic implications.

However, QL-implications have not received as much attention as (S,N)- and R-implications within fuzzy logic. Perhaps, one of the reasons can be attributed to the fact that not all members of this family satisfy one of the main properties expected of a fuzzy implication, viz., left antitonicity (I1) (see Definition 3.1 and Remark 4.3). Moreover, in the earlier works, some conditions imposed on the fuzzy logic operations employed in the definition of QL-implications restricted both the class of operations from which QL-implications could be obtained and the properties these implications satisfied (see Remark 4.22 for details).

Interest on QL-implications has seen some rise in the recent past and some works have appeared on them. These can be broadly classified as follows:

(i) Studies that focus on QL-implications and their basic algebraic properties as in Trillas et al. [29], Mas et al. [21], Shi et al. [26], and Jayaram and Baczyński [14].
(ii) Works that investigate QL-implications as part of determining which families of implications satisfy a property under consideration, viz., Fodor [11], Trillas and Alsina [27], Trillas et al. [28], Shi et al. [25], and Jayaram [13].

Once again, most of these studies have been done after restricting the underlying T, S, N operations to certain families and hence are less general in their obtained results. For example, in Trillas et al. [28] their investigations have been done only in the context of continuous T and S and strong N, while Mas et al. [21] consider QL-implications where the underlying N is strong, but they do consider the non-continuous t-conorm $S_{\mathbf{nM}}$. However, recently Shi et al. [26] have investigated QL-implications, where T and S are not always assumed to be continuous, though N is always a strong fuzzy negation and their subsequent analysis is predominantly for the class of continuous operations.

In this work, we study the family of QL-implications in fuzzy logic, without any restrictions on the underlying operations. We propose necessary and/or sufficient conditions on the underlying operations under which QL-implications satisfy some of the most desirable algebraic properties. Following this, a partial characterization of the intersections that exist between the family of QL-implications and the families of (S,N)- and R-implications is given. Most importantly, it is shown that QL-implications contain the set of all R-implications obtained from left-continuous t-norms that are also (S,N)-implications. Finally, we also investigate the overlaps that exist between QL-implications and the recently proposed f- and g-implications (see [32]).

2. Preliminaries

We assume that the reader is familiar with the classical results concerning basic fuzzy logic connectives, but to make this work more self-contained, we introduce basic notations used in the text and we briefly mention some of the concepts and results employed in the rest of the work. By Φ we denote the family of all increasing bijections $\varphi: [0, 1] \to [0, 1]$. We say that functions $f, g: [0, 1]^n \to [0, 1]$, where $n \in \mathbb{N}$, are Φ-conjugate (cf. [18, p. 156]), if there exists $\varphi \in \Phi$ such that $g = f_\varphi$, where

$$f_\varphi(x_1, \ldots, x_n) := \varphi^{-1}(f(\varphi(x_1), \ldots, \varphi(x_n))), \quad x_1, \ldots, x_n \in [0, 1].$$

Equivalently, g is said to be the Φ-conjugate of f.

2.1. Fuzzy negations, t-norms and t-conorms

Definition 2.1 (*see Fodor and Roubens [10, p. 3], Klement et al. [16, Definition 11.3]*). A decreasing function $N\colon [0, 1] \to [0, 1]$ is called a *fuzzy negation*, if $N(0) = 1$, $N(1) = 0$. A fuzzy negation N is called

(i) *strict*, if it is strictly decreasing and continuous;
(ii) *strong*, if it is an involution, i.e., $N(N(x)) = x$ for all $x \in [0, 1]$;
(iii) *non-vanishing*, if $N(x) = 0 \Longleftrightarrow x = 1$.

Example 2.2. The classical negation $N_{\mathbf{C}}(x) = 1-x$ is a strong negation, while $N_{\mathbf{K}}(x) = 1-x^2$ is only strict, whereas the Gödel negations, $N_{\mathbf{D1}}$ and $N_{\mathbf{D2}}$—which are the least and greatest fuzzy negations—are non-strong negations:

$$N_{\mathbf{D1}}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x > 0, \end{cases} \qquad N_{\mathbf{D2}}(x) = \begin{cases} 1 & \text{if } x < 1, \\ 0 & \text{if } x = 1. \end{cases}$$

For more examples of fuzzy negations see [10,17].

Definition 2.3 (*Schweizer and Sklar [24], Klement et al. [16]*).

(i) An associative, commutative and increasing operation $T\colon [0, 1]^2 \to [0, 1]$ is called a *triangular norm* (t-norm, for short), if it has the neutral element equal to 1.
(ii) An associative, commutative and increasing operation $S\colon [0, 1]^2 \to [0, 1]$ is called a *triangular conorm* (t-conorm, for short), if it has the neutral element equal to 0.

If F is an associative binary operation on $[a, b]$ with the neutral element e, then the power notation $x_F^{[n]}$, where $n \in \mathbb{N}_0$, is defined by

$$x_F^{[n]} := \begin{cases} e & \text{if } n = 0, \\ x & \text{if } n = 1, \\ F(x, x_F^{[n-1]}) & \text{if } n > 1. \end{cases}$$

Definition 2.4 (*Klement et al. [16, Definitions 1.23, 2.9 and 2.13]*). A t-norm T (t-conorm S, respectively) is said to be

(i) continuous, if it is continuous in both the arguments;
(ii) left-continuous, if it is left-continuous in each component;
(iii) right-continuous, if it is right-continuous in each component;
(iv) idempotent, if $T(x, x) = x$ ($S(x, x) = x$, respectively) for all $x \in [0, 1]$;
(v) Archimedean, if for every $x, y \in (0, 1)$ there is $n \in \mathbb{N}$ such that $x_T^{[n]} < y$ ($x_S^{[n]} > y$, respectively);
(vi) strict, if T (S, respectively) is continuous and strictly monotone, i.e., $T(x, y) < T(x, z)$ whenever $x > 0$ ($S(x, y) < S(x, z)$ whenever $x < 1$, respectively) and $y < z$;
(vii) nilpotent, if T (S, respectively) is continuous and if each $x \in (0, 1)$ is a nilpotent element, i.e., if for each $x \in (0, 1)$ there exists $n \in \mathbb{N}$ such that $x_T^{[n]} = 0$ ($x_S^{[n]} = 1$, respectively);
(viii) positive, if $T(x, y) = 0$ ($S(x, y) = 1$, respectively) implies that either $x = 0$ or $y = 0$ ($x = 1$ or $y = 1$, respectively).

Remark 2.5.

(i) For a continuous t-norm T the Archimedean property is given by the simpler condition, that $T(x, x) < x$, for all $x \in (0, 1)$ (see [12, Proposition 5.1.2]).
(ii) If a t-norm T is continuous and Archimedean, then T is nilpotent if and only if there exists some nilpotent element of T, which is equivalent to the existence of some zero divisor of T, i.e., there exist $x, y \in (0, 1)$ such that $T(x, y) = 0$ (see [16, Theorem 2.18]).
(iii) If a t-norm T is strict or nilpotent, then it is Archimedean. Conversely, every continuous and Archimedean t-norm is either strict or nilpotent (see [16, p. 33]).
(iv) By the duality between t-norms and t-conorms, similar properties as above hold for t-conorms with the appropriate changes in either the inequality or the neutral element (cf. [16, Remark 2.20, 10, Chapter 1]).

Definition 2.6 (*cf. Klement et al. [16, Definition 1.25]*). A t-norm T is said to satisfy 1-*Lipschitz condition* or 1-Lipschitzianity, if

$$|T(x_1, y_1) - T(x_2, y_2)| \leqslant |x_1 - x_2| + |y_1 - y_2|, \quad x_1, x_2, y_1, y_2 \in [0, 1]. \tag{3}$$

Remark 2.7.

(i) Any 1-Lipschitz t-norm is also continuous but the converse, in general, is not true (see [16, Example 1.26]).
(ii) One well-known family of t-norms that satisfies Lipschitzianity is the family of Frank t-norms $T_{\mathbf{F}}^{\lambda}$, where $\lambda \in [0, \infty]$, defined as follows:

$$T_{\mathbf{F}}^{\lambda}(x, y) = \begin{cases} T_{\mathbf{M}}(x, y) & \text{if } \lambda = 0, \\ T_{\mathbf{P}}(x, y) & \text{if } \lambda = 1, \\ T_{\mathbf{LK}}(x, y) & \text{if } \lambda = \infty, \\ \log_{\lambda}\left(1 + \dfrac{(\lambda^x - 1)\cdot(\lambda^y - 1)}{\lambda - 1}\right) & \text{otherwise,} \end{cases} \quad x, y \in [0, 1].$$

In fact, this family was obtained while characterizing the following so-called Frank functional equation $T(x, y) + S(x, y) = x + y$, for all $x, y \in [0, 1]$. We will refer to the above family in the sequel.

Example 2.8 (*see Klement et al. [16]*). Tables 2 and 3 list the basic t-norms and t-conorms with the properties they satisfy. Note that $T_{\mathbf{M}}$, $T_{\mathbf{P}}$ are positive t-norms, while $T_{\mathbf{LK}}$, $T_{\mathbf{D}}$ and $T_{\mathbf{nM}}$ are not. Similarly, $S_{\mathbf{M}}$, $S_{\mathbf{P}}$ are positive t-conorms, while $S_{\mathbf{LK}}$, $S_{\mathbf{D}}$ and $S_{\mathbf{nM}}$ are not.

2.2. Negations from t-conorms and t-norms

One can associate a fuzzy negation to any t-norm or t-conorm as given in the definition below.

Definition 2.9 *(see Nguyen and Walker [23, Definition 5.5.2], Klement et al. [16, p. 232] or Baczyński and Jayaram [4]).*

(i) Let T be a t-norm. A function $N_T\colon [0, 1] \to [0, 1]$ defined as

$$N_T(x) := \sup\{t \in [0, 1] \mid T(x, t) = 0\}, \quad x \in [0, 1], \tag{4}$$

is called the *natural negation of* T.
(ii) Let S be a t-conorm. A function $N_S\colon [0, 1] \to [0, 1]$ defined as

$$N_S(x) := \inf\{t \in [0, 1] \mid S(x, t) = 1\}, \quad x \in [0, 1], \tag{5}$$

is called the *natural negation of* S.

Table 2
Examples of basic t-norms and their properties

Name	Formula	Properties
Minimum	$T_{\mathbf{M}}(x,y)=\min(x,y)$	Continuous, idempotent
Product	$T_{\mathbf{P}}(x,y)=xy$	Strict
Łukasiewicz	$T_{\mathbf{LK}}(x,y)=\max(x+y-1,0)$	Nilpotent
Drastic product	$T_{\mathbf{D}}(x,y)=\begin{cases}0 & \text{if } x,y\in[0,1)\\ \min(x,y) & \text{otherwise}\end{cases}$	Archimedean, non-continuous
Nilpotent minimum	$T_{\mathbf{nM}}(x,y)=\begin{cases}0 & \text{if } x+y\leqslant 1\\ \min(x,y) & \text{otherwise}\end{cases}$	Non-Archimedean, left-continuous

Table 3
Examples of basic t-conorms and their properties

Name	Formula	Properties
Maximum	$S_{\mathbf{M}}(x,y)=\max(x,y)$	Continuous, idempotent
Probabilistic sum	$S_{\mathbf{P}}(x,y)=x+y-xy$	Strict
Łukasiewicz	$S_{\mathbf{LK}}(x,y)=\min(x+y,1)$	Nilpotent
Drastic sum	$S_{\mathbf{D}}(x,y)=\begin{cases}1 & \text{if } x,y\in(0,1]\\ \max(x,y) & \text{otherwise}\end{cases}$	Archimedean, non-continuous
Nilpotent maximum	$S_{\mathbf{nM}}(x,y)=\begin{cases}1 & \text{if } x+y\geqslant 1\\ \max(x,y) & \text{otherwise}\end{cases}$	Non-Archimedean, right-continuous

Table 4
Examples of natural negations from basic t-norms and t-conorms

t-norm T	N_T	t-conorm S	N_S
Positive	$N_{\mathbf{D1}}$	Positive	$N_{\mathbf{D2}}$
$T_{\mathbf{LK}}$	$N_{\mathbf{C}}$	$S_{\mathbf{LK}}$	$N_{\mathbf{C}}$
$T_{\mathbf{D}}$	$N_{\mathbf{D2}}$	$S_{\mathbf{D}}$	$N_{\mathbf{D1}}$
$T_{\mathbf{nM}}$	$N_{\mathbf{C}}$	$S_{\mathbf{nM}}$	$N_{\mathbf{C}}$

Remark 2.10.

(i) It is easy to prove that both N_T and N_S are fuzzy negations. In the literature N_T is also called the contour line C_0 of T, while N_S is called the contour line D_1 of S (see [19,20]).

(ii) Since for any t-norm T and any t-conorm S we have $T(x,0)=0$ and $S(x,1)=1$ for all $x\in[0,1]$, the appropriate sets in (4) and (5) are non-empty.

(iii) Notice that if $S(x,y)=1$ for some $x,y\in[0,1]$, then $y\geqslant N_S(x)$ and if $T(x,y)=0$ for some $x,y\in[0,1]$, then $y\leqslant N_T(x)$. Moreover, if any $z<N_T(x)$, then $T(x,z)=0$ and if any $z>N_S(x)$, then $S(x,z)=1$.

Example 2.11. Table 4 gives the natural negations of the basic t-norms and t-conorms.

The next result will be useful in the sequel.

Proposition 2.12 (*Baczyński and Jayaram [4, Proposition 2.11], cf. Maes and De Baets [19]*). *If a t-conorm S is right-continuous, then*

(i) *for every $x,y\in[0,1]$ the following equivalence holds*:

$$S(x,y)=1 \iff N_S(x)\leqslant y;$$

(ii) *the infimum in* (5) *is the minimum*, *i.e.*,

$$N_S(x) = \min\{t \in [0,1] \mid S(x,t) = 1\}, \quad x \in [0,1],$$

where the right side exists for all $x \in [0,1]$;

(iii) N_S *is right-continuous*.

It is well-known that if an N_T obtained from a left-continuous t-norm T is continuous, then it is strong. In the proof of this result, the equality $N_T \circ N_T \circ N_T = N_T$ plays an important role (see [10, p. 28]). However, the above equality is not true for all t-norms, as shown in the following example.

Example 2.13. Consider the non-left-continuous t-norm given in [16, Example 1.24(i)] as follows:

$$T_{\mathbf{B}}(x,y) = \begin{cases} 0 & \text{if } (x,y) \in (0,0.5)^2, \\ \min(x,y) & \text{otherwise}, \end{cases} \qquad x, y \in [0,1],$$

whose natural negation is the following:

$$N_{\mathbf{TB}}(x) = \begin{cases} 1 & x = 0, \\ 0.5 & x \in (0,0.5), \\ 0 & x \in [0.5,1]. \end{cases}$$

It can be quite easily verified that $N_{\mathbf{TB}} \circ N_{\mathbf{TB}} \circ N_{\mathbf{TB}} \neq N_{\mathbf{TB}}$. However, as we show below, the result still remains valid for any t-norm T.

Theorem 2.14. *Let* T *be any t-norm*.

(i) *If* N_T *is continuous*, *then it is strong*.

(ii) *If* N_T *is discontinuous*, *then it is not strictly decreasing*.

Proof.

(i) Firstly, we show that N_T is strict. Assume to the contrary that N_T is not strict, i.e., it is constant on some interval $[x, y]$, where without loss of the generality, we assume that $0 < x < y < 1$. Therefore there exists $p \in [0,1]$ such that

$$N_T(x) = N_T(y) = p.$$

If $p = 0$, then $N_T(x) = 0$, which implies that $T(\varepsilon, x) = T(x, \varepsilon) > 0$ for an arbitrary small $\varepsilon > 0$. Therefore, $N_T(\varepsilon) < x$. Since N_T is continuous, as $\varepsilon \to 0$, we have that $N_T(0) \leqslant x$. However, $N_T(0) = 1$, a contradiction. If $p = 1$, then $T(1-\varepsilon, x) = T(x, 1-\varepsilon) = 0$ for an arbitrary small $\varepsilon > 0$. Thus, $N_T(1-\varepsilon) \geqslant x$. Since N_T is continuous, as $\varepsilon \to 0$, we have that $N_T(1) \geqslant x$. However, $N_T(1) = 0$, a contradiction.
Hence, we consider now the situation, when $p \in (0,1)$. Since $N_T(z) = p$ for any $z \in (x, y)$, by the definition of N_T we have

$$T(z, p-\varepsilon) = T(p-\varepsilon, z) = 0,$$
$$T(z, p+\varepsilon) = T(p+\varepsilon, z) > 0,$$

for any arbitrary small $\varepsilon > 0$. Thus

$$N_T(p+\varepsilon) < z \leqslant N_T(p-\varepsilon).$$

Since N_T is continuous, as $\varepsilon \to 0$, we have that $N_T(p) = z$. Now this happens for every $z \in (x, y)$, which once again contradicts the fact that N_T is a function itself, or the fact that N_T is continuous. Hence N_T is strict.

To show that N_T is strong, we show that $N_T(N_T(N_T(x))) = N_T(x)$ for all $x \in [0,1]$. The above is clear for any $x \in \{0,1\}$. Since N_T is strict, for any $x \in (0,1)$ and an arbitrary small $\varepsilon > 0$, we have the following inequalities:

$$\begin{aligned} & x - \varepsilon < x < x + \varepsilon \\ & \Longrightarrow N_T(x-\varepsilon) > N_T(x) > N_T(x+\varepsilon) \\ & \Longrightarrow N_T(N_T(x-\varepsilon)) < N_T(N_T(x)) < N_T(N_T(x+\varepsilon)). \end{aligned}$$

By the definition of N_T, we have

$$T(x, N_T(x+\varepsilon)) = T(N_T(x+\varepsilon), x) = 0,$$

thus $x \leqslant N_T(N_T(x+\varepsilon))$, which implies that $N_T(x) \geqslant N_T(N_T(N_T(x+\varepsilon)))$. Once again, by the continuity of N_T, as $\varepsilon \to 0$, we have

$$N_T(x) \geqslant N_T(N_T(N_T(x))).$$

Recall, from Remark 2.10(iii), that if $z < N_T(x)$, then $T(z,x) = 0$. Now, since $N_T(N_T(x-\varepsilon)) < N_T(N_T(x))$, we also have, $T(N_T(N_T(x-\varepsilon)), N_T(x)) = 0$. Once again, by the definition of N_T, we have $N_T(x) \leqslant N_T(N_T(N_T(x-\varepsilon)))$, and as $\varepsilon \to 0$, we get

$$N_T(x) \leqslant N_T(N_T(N_T(x))).$$

From the above inequalities, we have $N_T(x) = N_T(N_T(N_T(x)))$ for any $x \in [0,1]$. Now, by the continuity of N_T, one can easily see that N_T is involutive.

(ii) Let N_T be discontinuous at some $p \in [0,1]$. By the monotonicity of N_T there exist constants $x, y \in [0,1]$ such that

$$x = \begin{cases} \lim\limits_{t \to p^+} N_T(t) & \text{if } p < 1, \\ 0 & \text{if } p = 1, \end{cases} \qquad y = \begin{cases} \lim\limits_{t \to p^-} N_T(t) & \text{if } p > 0, \\ 1 & \text{if } p = 0. \end{cases}$$

From the discontinuity of decreasing negation N_T at p we have $x < y$. Now, we consider the following two cases:

(a) Let $N_T(p) = x$. In particular this implies that $p > 0$. Let us fix arbitrarily $z \in (x,y)$. It is obvious that $N_T(p) < z$, so by the definition of N_T we get $T(p,z) > 0$, which implies $N_T(z) \leqslant p$. On the other side, $N_T(p-\varepsilon) \geqslant y$ for an arbitrary small $\varepsilon > 0$, thus $T(p-\varepsilon, z) = 0$, therefore $N_T(z) \geqslant p - \varepsilon$. Taking the limit $\varepsilon \to 0$ we get $N_T(z) \geqslant p$. Since z was arbitrarily fixed, the above implies that $N_T(z) = p$ for every $z \in (x,y)$ and hence N_T is constant on this interval (x,y).

(b) Let $N_T(p) = z'$, where $x < z' \leqslant y$. In particular this implies that $p < 1$. We now claim that N_T is a constant on the interval (x, z'). Let us fix arbitrarily $z \in (x, z')$. In this case we have that $T(p,z) = 0$, so $N_T(z) \geqslant p$. Once again, we claim that $N_T(z) = p$. Instead, if $N_T(z) > p$, then $T(p+\varepsilon, z) = 0$ for some $\varepsilon > 0$. Thus, by the definition of N_T, we have that $N_T(p+\varepsilon) \geqslant z$, and from the decreasing nature of N_T and the definition of x we obtain

$$x \geqslant N_T(p+\varepsilon) \geqslant z,$$

a contradiction to the fact that $x < z$. Therefore $N_T(z) = p$ for every $z \in (x, z')$ and hence N_T is a constant on this interval (x, z'). □

Corollary 2.15. *For a t-norm T the following statements are equivalent*:

(i) N_T *is strictly decreasing*.
(ii) N_T *is continuous*.
(iii) N_T *is strict*.
(iv) N_T *is strong*.

Similarly, one can prove the following results.

Theorem 2.16. *Let S be any t-conorm.*

(i) *If N_S is continuous, then it is strong.*
(ii) *If N_S is discontinuous, then it is not strictly decreasing.*

Corollary 2.17. *For a t-conorm S the following statements are equivalent:*

(i) *N_S is strictly decreasing.*
(ii) *N_S is continuous.*
(iii) *N_S is strict.*
(iv) *N_S is strong.*

2.3. The law of excluded middle

Now we analyze the law of excluded middle, which in the classical case has the following form: $p \vee \neg p = \top$.

Definition 2.18. Let S be a t-conorm and N a fuzzy negation. We say that the pair (S, N) satisfies the *law of excluded middle*, if

$$S(N(x), x) = 1, \quad x \in [0, 1]. \tag{LEM}$$

Now the following result is easy to see.

Lemma 2.19. *Let S be a t-conorm and N a fuzzy negation. If the pair (S,N) satisfies* (LEM), *then*

(i) $N \geqslant N_S$;
(ii) $N_S \circ N(x) \leqslant x$, *for all* $x \in [0, 1]$.

Example 2.20.

(i) Any t-conorm satisfies (LEM) with the greatest fuzzy negation $N_{\mathbf{D2}}$. Indeed, for any t-conorm S and $x \in [0, 1]$ we have

$$S(N_{\mathbf{D2}}(x), x) = \begin{cases} S(1, x) & \text{if } x < 1 \\ S(0, x) & \text{if } x = 1 \end{cases} = \begin{cases} 1 & \text{if } x < 1 \\ x & \text{if } x = 1 \end{cases} = 1.$$

From the previous result and Table 4 it follows that if S is a positive t-conorm, then it satisfies (LEM) only with the greatest fuzzy negation $N_{\mathbf{D2}}$.

(ii) However, no t-conorm satisfies (LEM) with the least fuzzy negation $N_{\mathbf{D1}}$. Indeed, for any t-conorm S and $x \in (0, 1)$ we have $S(N_{\mathbf{D1}}(x), x) = S(0, x) = x \neq 1$.

Example 2.21. The fact that the conditions in Lemma 2.19 are only necessary and not sufficient follow from the following example. Consider the non-right-continuous nilpotent maximum t-conorm

$$S_{\mathbf{nM}^*}(x, y) = \begin{cases} 1 & \text{if } x + y > 1, \\ \max(x, y) & \text{otherwise,} \end{cases} \quad x, y \in [0, 1].$$

Then its natural negation is the classical negation, i.e., $N_{S_{\mathbf{nM}^*}}(x) = N_{\mathbf{C}}(x) = 1 - x$ and $N_{\mathbf{C}} \circ N_{\mathbf{C}}(x) = x$ for all $x \in [0, 1]$. However, the pair $(S_{\mathbf{nM}^*}, N_{\mathbf{C}})$ does not satisfy (LEM). Indeed, for $x = 0.5$ we get

$$S_{\mathbf{nM}^*}(N_{\mathbf{C}}(0.5), 0.5) = S_{\mathbf{nM}^*}(0.5, 0.5) = 0.5.$$

Interestingly, for the right-continuous t-conorms the condition (i) from Lemma 2.19 is both necessary and sufficient.

Proposition 2.22 (*Baczyński and Jayaram [4, Proposition 2.16]*). *For a right-continuous t-conorm S and a fuzzy negation N the following statements are equivalent*:

(i) *The pair* (S, N) *satisfies* (LEM).
(ii) $N \geqslant N_S$.

In the class of continuous functions we get the following important fact.

Proposition 2.23 (*Baczyński and Jayaram [4, Proposition 2.17]*). *For a continuous t-conorm S and a continuous fuzzy negation N the following statements are equivalent*:

(i) *The pair* (S, N) *satisfies* (LEM).
(ii) *S is a nilpotent t-conorm, i.e., S is* Φ*-conjugate with the Łukasiewicz t-conorm* $S_{\mathbf{LK}}$, *i.e., there exists* $\varphi \in \Phi$, *which is uniquely determined, such that S has the representation*

$$S(x, y) = \varphi^{-1}(\min(\varphi(x) + \varphi(y), 1)), \quad x, y \in [0, 1],$$

and

$$N(x) \geqslant N_S(x) = \varphi^{-1}(1 - \varphi(x)), \quad x \in [0, 1].$$

2.4. De Morgan triples

Finally, in this subsection we present some results regarding De Morgan triples.

Definition 2.24 (*Klement et al. [16, p. 232]*). A triple (T, S, N), where *T* is a t-norm, *S* is a t-conorm and *N* is a strict negation, is called a *De Morgan triple*, if

$$T(x, y) = N^{-1}(S(N(x), N(y))), \quad S(x, y) = N^{-1}(T(N(x), N(y))),$$

for all $x, y \in [0, 1]$.

Theorem 2.25 (*Klement et al. [16, p. 232]*). *For a t-norm T, t-conorm S and a strict fuzzy negation N the following statements are equivalent*:

(i) (T, S, N) *is a De Morgan triple*.
(ii) *N is a strong negation and S is the N-dual of T, i.e.*, $S(x, y) = N(T(N(x), N(y)))$, *for all* $x, y \in [0, 1]$.

Using the above theorem it can be shown that the following relation exists between N_T and N_S.

Theorem 2.26 (*Baczyński and Jayaram [4, Proposition 2.21]*). *Let T be a left-continuous t-norm and S be a t-conorm. If* (T, N_T, S) *is a De Morgan triple, then*

(i) $N_S = N_T$ *is a strong negation*,
(ii) *S is right-continuous*.

3. Fuzzy implications

In the literature, especially at the beginnings, we can find several different definitions of fuzzy implications. In this article we will use the following one, which is equivalent to the definition introduced by Fodor and Roubens [10, Definition 1.15] (see also [15, p. 50]).

Definition 3.1. A function $I: [0, 1]^2 \to [0, 1]$ is called a *fuzzy implication*, if it satisfies, for all $x, y, z \in [0, 1]$, the following conditions:

$$\text{if } x \leqslant y \quad \text{then } I(x, z) \geqslant I(y, z), \tag{I1}$$

Table 5
Examples of basic fuzzy implications

Name	Formula
Łukasiewicz	$I_{\mathbf{LK}}(x, y) = \min(1, 1 - x + y)$
Gödel	$I_{\mathbf{GD}}(x, y) = \begin{cases} 1 & \text{if } x \leqslant y \\ y & \text{if } x > y \end{cases}$
Reichenbach	$I_{\mathbf{RC}}(x, y) = 1 - x + xy$
Kleene–Dienes	$I_{\mathbf{KD}}(x, y) = \max(1 - x, y)$
Goguen	$I_{\mathbf{GG}}(x, y) = \begin{cases} 1 & \text{if } x \leqslant y \\ \frac{y}{x} & \text{if } x > y \end{cases}$
Weber	$I_{\mathbf{WB}}(x, y) = \begin{cases} 1 & \text{if } x < 1 \\ y & \text{if } x = 1 \end{cases}$
Dubois–Prade	$I_{\mathbf{DP}}(x, y) = \begin{cases} y & \text{if } x = 1 \\ 1 - x & \text{if } y = 0 \\ 1 & \text{if } x < 1 \text{ and } y > 0 \end{cases}$
Fodor	$I_{\mathbf{FD}}(x, y) = \begin{cases} 1 & \text{if } x \leqslant y \\ \max(1 - x, y) & \text{if } x > y \end{cases}$

$$\text{if } y \leqslant z \quad \text{then } I(x, y) \leqslant I(x, z), \tag{I2}$$

$$I(0, 0) = 1, \tag{I3}$$

$$I(1, 1) = 1, \tag{I4}$$

$$I(1, 0) = 0. \tag{I5}$$

The set of all fuzzy implications will be denoted by $\mathcal{FI}$.

Remark 3.2. Directly from Definition 3.1 we see that each fuzzy implication *I* satisfies the following *left* and *right boundary conditions*, respectively:

$$I(0, y) = 1, \quad y \in [0, 1], \tag{LB}$$

$$I(x, 1) = 1, \quad x \in [0, 1]. \tag{RB}$$

Therefore, *I* satisfies also the *normality condition*:

$$I(0, 1) = 1. \tag{NC}$$

Example 3.3. Table 5 lists a few basic fuzzy implications.

Additional properties of fuzzy implications were postulated in many works (see, for example [31,10,12]). The most important of them are presented below.

Definition 3.4. A fuzzy implication *I* is said to satisfy

(i) the *left neutrality property*, if

$$I(1, y) = y, \quad y \in [0, 1]. \tag{NP}$$

(ii) the *exchange principle*, if

$$I(x, I(y, z)) = I(y, I(x, z)), \quad x, y, z \in [0, 1]. \tag{EP}$$

(iii) the *identity principle*, if

$$I(x, x) = 1, \quad x \in [0, 1]. \tag{IP}$$

(iv) the *ordering property*, if

$$I(x,y)=1 \iff x \leqslant y, \quad x,y \in [0,1]. \tag{OP}$$

(v) the *law of contraposition* with respect to a fuzzy negation N, CP(N), if

$$I(x,y)=I(N(y),N(x)), \quad x,y \in [0,1]. \tag{CP}$$

Just as in the case of t-norms or t-conorms, a fuzzy negation can be obtained from fuzzy implications too as follows.

Definition 3.5. Let $I\colon [0,1]^2 \to [0,1]$ be any function. If the function $N_I\colon [0,1] \to [0,1]$ given by

$$N_I(x) := I(x,0), \quad x \in [0,1],$$

is a fuzzy negation, then it is called the *natural negation* of I.

It should be noted that for any $I \in \mathcal{FI}$ we have (I3) and (I5), so N_I is a fuzzy negation in this case. In the following results we discuss some relationships that exist between the above properties of fuzzy implications. They will be useful in the sequel.

Proposition 3.6 (*cf. Fodor and Roubens [10, Corollary 1.1]*). *If a function* $I\colon [0,1]^2 \to [0,1]$ *satisfies* (EP) *and* (OP), *then* N_I *is either a strong negation or a discontinuous negation.*

Lemma 3.7 (*Baczyński and Jayaram [3, Lemma 2.2], cf. Bustince et al. [7, Lemma 1]*). *Let I be any* $[0,1]^2 \to [0,1]$ *function and N be a fuzzy negation. If I satisfies* (NP) *and CP* (*N*), *then* $N = N_I$ *is a strong negation.*

Lemma 3.8 (*Baczyński and Jayaram [3, Corollary 2.3]*). *Let* $I \in \mathcal{FI}$ *satisfy* (NP). *If* N_I *is not a strong negation, then I does not satisfy the contrapositive symmetry* (CP) *with any fuzzy negation.*

Lemma 3.9 (*Baczyński and Jayaram [3, Lemma 2.4], cf. Bustince et al. [7, Lemma 1]*). *Let I be any* $[0,1]^2 \to [0,1]$ *function and* N_I *be a strong negation.*

(i) *If I satisfies CP*(N_I), *then I satisfies* (NP).
(ii) *If I satisfies* (EP), *then I satisfies* (I3), (NP) *and CP*(N_I).

Corollary 3.10 (*Baczyński and Jayaram [3, Corollary 2.5]*). *Let* $I \in \mathcal{FI}$ *satisfy* (NP) *and* (EP). *Then I satisfies CP*(N) *with some fuzzy negation N if and only if* $N = N_I$ *is a strong negation.*

4. QL-operations and QL-implications

In this section, we define a QL-operation as a generalization of the quantum logic implication. However, it should be noted that not every such operation defined is a fuzzy implication and hence we find some suitable necessary or sufficient conditions for this to happen. Then we give many examples of QL-operations that are fuzzy implications. Following this, we investigate this family of fuzzy implications with respect to the desirable properties as proposed in Definition 3.4.

4.1. Definitions, examples and basic properties

Definition 4.1. A function $I\colon [0,1]^2 \to [0,1]$ is called a QL-*operation*, if there exist a t-norm T, a t-conorm S and a fuzzy negation N such that

$$I(x,y)=S(N(x),T(x,y)), \quad x,y \in [0,1].$$

If I is a QL-operation generated from the triple (T,S,N), then we will often denote it by $I_{T,S,N}$.

Table 6
Examples of basic QL-operations

T	S	N	QL-operation $I_{T,S,N}$	$I_{T,S,N} \in \mathcal{FI}$
$T_{\mathbf{M}}$	$S_{\mathbf{M}}$	$N_{\mathbf{C}}$	$I_{\mathbf{ZD}}$	×
$T_{\mathbf{M}}$	$S_{\mathbf{P}}$	$N_{\mathbf{C}}$	$I(x,y) = \begin{cases} 1-x+x^2 & \text{if } x \leqslant y \\ 1-x+xy & \text{otherwise} \end{cases}$	×
$T_{\mathbf{M}}$	$S_{\mathbf{LK}}$	$N_{\mathbf{C}}$	$I_{\mathbf{LK}}$	✓
$T_{\mathbf{M}}$	$S_{\mathbf{D}}$	$N_{\mathbf{C}}$	$I_{\mathbf{DP}}$	✓
$T_{\mathbf{M}}$	$S_{\mathbf{nM}}$	$N_{\mathbf{C}}$	$I_{\mathbf{FD}}$	✓
$T_{\mathbf{P}}$	$S_{\mathbf{M}}$	$N_{\mathbf{C}}$	$I(x,y) = \max(1-x, xy)$	×
$T_{\mathbf{P}}$	$S_{\mathbf{P}}$	$N_{\mathbf{C}}$	$I(x,y) = 1-x+x^2y$	×
$T_{\mathbf{P}}$	$S_{\mathbf{LK}}$	$N_{\mathbf{C}}$	$I_{\mathbf{RC}}$	✓
$T_{\mathbf{P}}$	$S_{\mathbf{D}}$	$N_{\mathbf{C}}$	$I_{\mathbf{DP}}$	✓
$T_{\mathbf{P}}$	$S_{\mathbf{nM}}$	$N_{\mathbf{C}}$	$I(x,y) = \begin{cases} 1 & \text{if } y = 1 \\ \max(1-x, xy) & \text{otherwise} \end{cases}$	×
$T_{\mathbf{LK}}$	$S_{\mathbf{M}}$	$N_{\mathbf{C}}$	$I(x,y) = \max(1-x, x+y-1)$	×
$T_{\mathbf{LK}}$	$S_{\mathbf{P}}$	$N_{\mathbf{C}}$	$I(x,y) = \begin{cases} 1-x & \text{if } y \leqslant 1-x \\ 1+x^2+xy-2x & \text{otherwise} \end{cases}$	×
$T_{\mathbf{LK}}$	$S_{\mathbf{LK}}$	$N_{\mathbf{C}}$	$I_{\mathbf{KD}}$	✓
$T_{\mathbf{LK}}$	$S_{\mathbf{D}}$	$N_{\mathbf{C}}$	$I(x,y) = \begin{cases} y & \text{if } x = 1 \\ 1-x & \text{if } y \leqslant 1-x \\ 1 & \text{otherwise} \end{cases}$	×
$T_{\mathbf{LK}}$	$S_{\mathbf{nM}}$	$N_{\mathbf{C}}$	$I(x,y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 1 \\ 1-x & \text{if } y \leqslant 2-2x \\ y & \text{otherwise} \end{cases}$	×
$T_{\mathbf{D}}$	Any S	$N_{\mathbf{C}}$	$I(x,y) = \begin{cases} S(N(x), x) & \text{if } y = 1 \\ y & \text{if } x = 1 \\ 1-x & \text{otherwise} \end{cases}$	×
$T_{\mathbf{nM}}$	$S_{\mathbf{nM}}$	$N_{\mathbf{C}}$	$I(x,y) = \begin{cases} 1 & \text{if } x \leqslant y \text{ and } y > 1-x \\ y & \text{if } x > y \text{ and } y > 1-x \\ 1-x & \text{otherwise} \end{cases}$	×
Any T	Any S	$N_{\mathbf{D1}}$	$I(x,y) = \begin{cases} T(x,y) & \text{if } x > 0 \\ 1 & \text{if } x = 0 \end{cases}$	×
Any T	Any S	$N_{\mathbf{D2}}$	$I_{\mathbf{WB}}$	✓

Firstly, we investigate some properties of QL-operations. We will see that not all QL-operations are fuzzy implications in the sense of Definition 3.1. The following fact can be proven by an easy verification.

Proposition 4.2 (*cf. Shi et al. [26, Proposition 3.1]*). *If $I_{T,S,N}$ is a QL-operation, then*

(i) $I_{T,S,N}$ *satisfies* (I2)–(I5), (NC), (LB) *and* (NP);
(ii) $N_{I_{T,S,N}} = N$.

From the above proposition, it follows that a QL-operation is generated by a unique negation.

Remark 4.3. A QL-operation does not always satisfy (I1). For example, consider the following function:

$$I_{\mathbf{ZD}}(x,y) = \max(1-x, \min(x,y)), \quad x, y, \in [0,1],$$

also called in the literature as the Zadeh implication. As can be seen in Table 6, it is the QL-operation obtained from the triple $(T_{\mathbf{M}}, S_{\mathbf{M}}, N_{\mathbf{C}})$, but it does not satisfy (I1). However, the QL-operation obtained from the triple $(T_{\mathbf{LK}}, S_{\mathbf{LK}}, N_{\mathbf{C}})$ satisfies (I1). In fact, it is the Kleene–Dienes implication $I_{\mathbf{KD}}$, which is a fuzzy implication.

Example 4.4. Table 6 lists QL-operations obtained from the basic t-norms, t-conorms and negations. In the last column we indicate whether the QL-operation is also a fuzzy implication.

Therefore the first main problem is the characterization of those QL-operations which satisfy (I1). Unfortunately, only partial results are known in the literature (cf. [26,29]). Following the terminology used by Trillas et al. [29] and Mas et al. [21], only if the QL-operation is a fuzzy implication we use the term QL-*implication*, and often use the equivalent expression $I_{T,S,N} \in \mathcal{FI}$.

Lemma 4.5. *If a QL-operation $I_{T,S,N} \in \mathcal{FI}$, then the pair (S, N) satisfies* (LEM).

Proof. If $I_{T,S,N}$ is a fuzzy implication, then by Remark 3.2 it satisfies (RB). Thus $I_{T,S,N}(x, 1) = 1$ if and only if $S(N(x), T(x, 1)) = 1$, i.e., $S(N(x), x) = 1$, for every $x \in [0, 1]$. □

Remark 4.6.

(i) From Example 2.20(ii) we know that there does not exist any t-conorm S such that the pair $(S, N_{\mathbf{D1}})$ satisfies (LEM). Therefore, by Lemma 4.5, we see that no QL-operation obtained from the triple (T, S, N), where $N = N_{\mathbf{D1}}$ is the least fuzzy negation, can be a fuzzy implication.

(ii) The fact that the condition in Lemma 4.5 is only necessary and not sufficient can be seen from the QL-operation I obtained from the triple $(T_{\mathbf{P}}, S_{\mathbf{nM}}, N_{\mathbf{C}})$, which is given in Table 6. Although the pair $(S_{\mathbf{nM}}, N_{\mathbf{C}})$ satisfies (LEM), it can be verified, by letting $x_1 = 0.8$, $x_2 = 0.9$ and $y = 0.3$, that $x_1 < x_2$ but $I(0.8, 0.3) = 0.24 < 0.27 = I(0.9, 0.3)$, so this I does not satisfy (I1).

(iii) From Lemma 2.19, it is easy to see that if a negation N in the triple (T, S, N) is less than the natural negation of S, i.e., if $N(x) < N_S(x)$ for some $x \in [0, 1]$, then the pair (S, N) does not satisfy (LEM) and hence the QL-operation $I_{T,S,N}$ is not a fuzzy implication.

(iv) Let S be any t-conorm and $N = N_{\mathbf{D2}}$, the greatest fuzzy negation. From Example 2.20(i) we see that the pair $(S, N_{\mathbf{D2}})$ satisfies (LEM). Now, for any t-norm T we have that the QL-operation obtained from the triple $(T, S, N_{\mathbf{D2}})$ is a fuzzy implication and is, in fact, the Weber implication $I_{\mathbf{WB}}$.

In fact, we have the following result.

Proposition 4.7. *A QL-operation $I_{T,S,N}$, where S is a positive t-conorm, is a fuzzy implication if and only if $N = N_{\mathbf{D2}}$. Moreover, $I_{T,S,N} = I_{\mathbf{WB}}$ in this case.*

Before considering special examples of QL-implications, we show some relationship between the Φ-conjugates of QL-implications.

Theorem 4.8. *If $I_{T,S,N}$ is a QL-implication (QL-operation, respectively), then the Φ-conjugate of $I_{T,S,N}$ is also a QL-implication (QL-operation, respectively) generated from the Φ-conjugate t-norm of T, the Φ-conjugate t-conorm of S and the Φ-conjugate fuzzy negation of N, i.e., if $\varphi \in \Phi$, then*

$$(I_{T,S,N})_\varphi = I_{T_\varphi, S_\varphi, N_\varphi}.$$

Proof. Let $\varphi \in \Phi$ and let $I_{T,S,N}$ be a QL-implication based on the suitable operations. We now know that the operations T_φ, S_φ and N_φ are a t-norm, t-conorm and a fuzzy negation, respectively. It is obvious that if $I_{T,S,N}$ is a fuzzy implication, then $(I_{T,S,N})_\varphi$ is also a fuzzy implication. Now, we have

$$\begin{aligned}
(I_{T,S,N})_\varphi(x, y) &= \varphi^{-1}(I_{T,S,N}(\varphi(x), \varphi(y))) \\
&= \varphi^{-1}(S(N(\varphi(x)), T(\varphi(x), \varphi(y)))) \\
&= \varphi^{-1}(S(\varphi \circ \varphi^{-1}(N(\varphi(x))), \varphi \circ \varphi^{-1}(T(\varphi(x), \varphi(y))))) \\
&= \varphi^{-1}(S(\varphi(N_\varphi(x)), \varphi(T_\varphi(x, y)))) = S_\varphi(N_\varphi(x), T_\varphi(x, y)) \\
&= I_{T_\varphi, S_\varphi, N_\varphi}(x, y),
\end{aligned}$$

for every $x, y \in [0, 1]$. □

Example 4.9. Let S be a t-conorm and N a fuzzy negation such that the pair (S, N) satisfies (LEM).

(i) If T is the minimum t-norm $T_{\mathbf{M}}$, then it can be easily seen that the QL-operation obtained from the triple $(T_{\mathbf{M}}, S, N)$ is always a fuzzy implication given by

$$I_{T_{\mathbf{M}},S,N}(x,y) = \begin{cases} 1 & \text{if } x \leqslant y, \\ S(N(x), y) & \text{if } x > y, \end{cases} \quad x, y \in [0,1]. \tag{6}$$

(ii) If T is the drastic t-norm $T_{\mathbf{D}}$, then the QL-operation obtained from the triple $(T_{\mathbf{D}}, S, N)$ is given by

$$I_{T_{\mathbf{D}},S,N}(x,y) = \begin{cases} 1 & \text{if } y = 1, \\ y & \text{if } x = 1, \\ N(x) & \text{otherwise}, \end{cases} \quad x, y \in [0,1].$$

This function is not always a fuzzy implication, even if S and N satisfy (LEM). Observe that it is a fuzzy implication if and only if $N(x) \geqslant y$ for all $x, y \in [0, 1)$, which means that $N = N_{\mathbf{D2}}$. In this case, of course, the QL-operation reduces, once again, to the Weber implication $I_{\mathbf{WB}}$.

Now, let us consider QL-implications obtained from triples (T, S, N), where S is some continuous t-conorm. Firstly, if S is a continuous but positive t-conorm from Proposition 4.7 we know that for the QL-operation obtained from the triple (T, S, N) to be a fuzzy implication N has to be the greatest negation $N_{\mathbf{D2}}$ and that $I_{T,S,N} = I_{\mathbf{WB}}$ in this situation. Hence we consider now only non-positive continuous t-conorms.

Let S be a continuous t-conorm and N a continuous fuzzy negation such that the pair (S, N) satisfies (LEM). Then, from Proposition 2.23, there exists a unique $\varphi \in \Phi$ such that

$$S(x,y) = (S_{\mathbf{LK}})_\varphi(x,y) = \varphi^{-1}(\min(\varphi(x)+\varphi(y), 1)),$$
$$N(x) \geqslant (N_{\mathbf{C}})_\varphi(x) = \varphi^{-1}(1-\varphi(x)),$$

for all $x, y \in [0, 1]$. Note that in this case S is a nilpotent t-conorm, i.e., it is non-positive and continuous. Let us consider the extreme case when $N(x) = (N_{\mathbf{C}})_\varphi(x) = \varphi^{-1}(1-\varphi(x))$ (with the same increasing bijection φ), in which case we have that N is a strong negation. Now, if we consider the QL-operation obtained from the triple $(T, (S_{\mathbf{LK}})_\varphi, (N_{\mathbf{C}})_\varphi)$, then since $T(x, y) \leqslant x$ for any t-norm T and $x \in [0, 1]$, we obtain the following function, denoted by $I_{\varphi,T}$ for ease of notation (see also [29,21]):

$$I_{\varphi,T}(x,y) = (S_{\mathbf{LK}})_\varphi((N_{\mathbf{C}})_\varphi(x), T(x,y)) = \varphi^{-1}(1-\varphi(x)+\varphi(T(x,y))), \quad x, y \in [0,1]. \tag{7}$$

The following result has been obtained by Mas et al. [21].

Theorem 4.10. *For a QL-operation $I_{\varphi,T}$ given by (7), where T is any t-norm and $\varphi \in \Phi$, the following statements are equivalent*:

(i) $I_{\varphi,T} \in \mathcal{FI}$.
(ii) $T_{\varphi^{-1}}$ *satisfies the* 1*-Lipschitz condition* (3).

Remark 4.11. Since the class of t-norms satisfying the Lipschitz condition is contained in the class of continuous t-norms, we have that $T_{\varphi^{-1}}$, and hence T itself, is a continuous t-norm.

The case when T is an Archimedean or an idempotent t-norm has been investigated by Fodor [11]. In fact, it is shown there that an equivalence relation exists between the t-norms T employed below and the resulting QL-implications.

Example 4.12. All QL-operations $I_{\varphi,T}$ obtained using the following t-norms satisfy (I1) and hence are fuzzy implications (cf. Table 6).

(i) If the t-norm T in (7) is Φ-conjugate with the Łukasiewicz t-norm $T_{\mathbf{LK}}$ with the same $\varphi \in \Phi$, then $I_{\varphi,(T_{\mathbf{LK}})_\varphi}$ is Φ-conjugate with the Kleene–Dienes implication $I_{\mathbf{KD}}$, i.e.,

$$I_{\varphi,(T_{\mathbf{LK}})_\varphi}(x,y) = (I_{\mathbf{KD}})_\varphi(x,y) = \max(N_\varphi(x), y), \quad x, y \in [0,1].$$

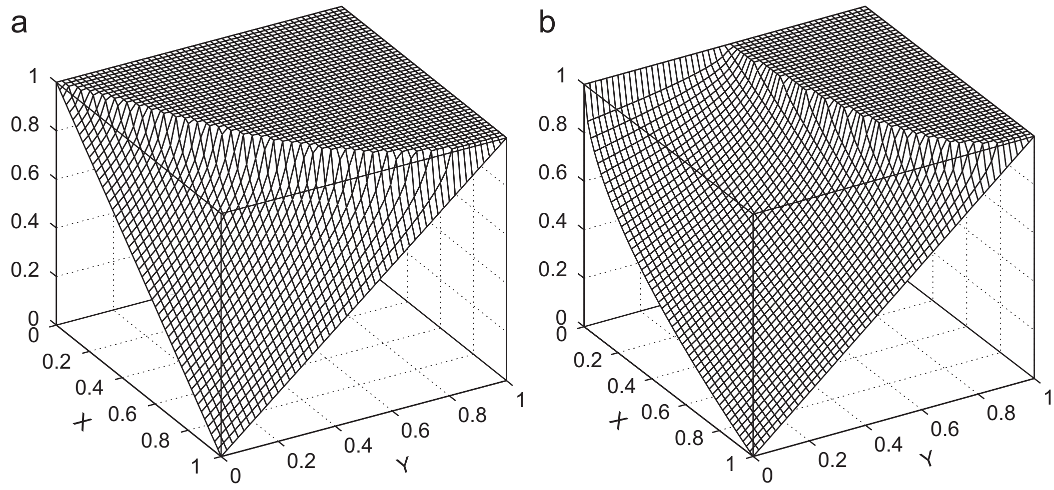

Fig. 1. Plots of the QL-implications $I_{\mathbf{PC}}$ and $I_{\mathbf{PR}}$ from Example 4.13: (a) $I_{\mathbf{PC}}$ from the triple $(T_{\mathbf{P}}, S^2_{\mathbf{SS}}, N_{\mathbf{C}})$; (b) $I_{\mathbf{PR}}$ from the triple $(T_{\mathbf{P}}, S^2_{\mathbf{SS}}, N_{\mathbf{R}})$.

(ii) If the t-norm T in (7) is Φ-conjugate with the product t-norm $T_{\mathbf{P}}$ with the same $\varphi \in \Phi$, then $I_{\varphi,(T_{\mathbf{P}})_\varphi}$ is Φ-conjugate with the Reichenbach implication $I_{\mathbf{RC}}$, i.e.,

$$I_{\varphi,(T_{\mathbf{P}})_\varphi}(x, y) = (I_{\mathbf{RC}})_\varphi(x, y) = \varphi^{-1}(1 - \varphi(x) + \varphi(x)\varphi(y)), \quad x, y \in [0, 1].$$

(iii) Firstly, note that $(T_{\mathbf{M}})_\varphi = T_{\mathbf{M}}$ for any $\varphi \in \Phi$ (see [16, Proposition 2.31]). Now, if the t-norm T in (7) is the minimum t-norm $T_{\mathbf{M}}$, then $I_{\varphi,T_{\mathbf{M}}}$ is Φ-conjugate with the Łukasiewicz implication $I_{\mathbf{LK}}$, i.e.,

$$I_{\varphi,T_{\mathbf{M}}}(x, y) = (I_{\mathbf{LK}})_\varphi(x, y) = \min(\varphi^{-1}(1 - \varphi(x) + \varphi(y), 1)), \quad x, y \in [0, 1].$$

In the following we show yet other examples of QL-implications generated from continuous functions.

Example 4.13. Let S be the Schweizer–Sklar t-conorm $S^\lambda_{\mathbf{SS}}$ for $\lambda = 2$ given by

$$S^2_{\mathbf{SS}}(x, y) = 1 - (\max((1 - x)^2 + (1 - y)^2 - 1, 0))^{1/2}, \quad x, y \in [0, 1],$$

and T be the product t-norm $T_{\mathbf{P}}$. It can be easily verified that the pairs $(S^2_{\mathbf{SS}}, N_{\mathbf{C}})$ and $(S^2_{\mathbf{SS}}, N_{\mathbf{R}})$, where $N_{\mathbf{R}}(x) = 1 - \sqrt{x}$, satisfy (LEM).

(i) The QL-operation obtained from the triple $(T_{\mathbf{P}}, S^2_{\mathbf{SS}}, N_{\mathbf{C}})$ is given by

$$I_{\mathbf{PC}}(x, y) = 1 - (\max(x(x + xy^2 - 2y), 0))^{1/2}, \quad x, y \in [0, 1].$$

(ii) The QL-operation obtained from the triple $(T_{\mathbf{P}}, S^2_{\mathbf{SS}}, N_{\mathbf{R}})$ is given by

$$I_{\mathbf{PR}}(x, y) = 1 - (\max(x(1 + xy^2 - 2y), 0))^{1/2}, \quad x, y \in [0, 1].$$

It can be easily checked that both $I_{\mathbf{PC}}$ and $I_{\mathbf{PR}}$ satisfy (I1) and hence are QL-implications, whose plots are given in Fig. 1.

In the rest of this section we give examples of QL-implications obtained from triples (T, S, N), where S is a non-continuous t-conorm.

Example 4.14. Let S be the drastic t-conorm $S_{\mathbf{D}}$ and N any non-vanishing negation. Then the pair $(S_{\mathbf{D}}, N)$ satisfies (LEM). If the t-norm T is positive, then, as can be verified, the QL-operation obtained from the triple $(T, S_{\mathbf{D}}, N)$ is a fuzzy implication given by

$$I_{T,S_{\mathbf{D}},N}(x, y) = \begin{cases} y & \text{if } x = 1, \\ N(x) & \text{if } y = 0, \\ 1 & \text{otherwise}, \end{cases} \qquad x, y \in [0, 1].$$

Fig. 2(a) gives the plot of the QL-implication obtained from the triple $(T, S_{\mathbf{D}}, N_{\mathbf{C}})$, where T is any positive t-norm, which is in fact the Dubois–Prade implication $I_{\mathbf{DP}}$ (see [8]).

Example 4.15. Let N be a strong negation. Consider the following t-conorm:

$$S^N_{\mathbf{nM}}(x, y) = \begin{cases} 1 & \text{if } x \geqslant N(y), \\ \max(x, y) & \text{if } x < N(y), \end{cases} \qquad x, y \in [0, 1], \tag{8}$$

which is only right-continuous. If N^* is any negation such that $N^* \geqslant N$, then $S^N_{\mathbf{nM}}(N^*(x), x) = 1$.

(i) The QL-operation from the triple $(T_{\mathbf{M}}, S^N_{\mathbf{nM}}, N^*)$ is

$$I_{T_{\mathbf{M}},S^N_{\mathbf{nM}},N^*}(x, y) = \begin{cases} 1 & \text{if } x \leqslant y, \\ \max(N^*(x), y) & \text{if } x > y, \end{cases} \qquad x, y \in [0, 1]. \tag{9}$$

In the case $N = N^* = N_{\mathbf{C}}$ the QL-operation in (9) is a QL-implication, indeed, it is the Fodor implication $I_{\mathbf{FD}}$. Fig. 2(b) gives the plot of the QL-implication obtained from the triple $(T_{\mathbf{M}}, S^{N_{\mathbf{C}}}_{\mathbf{nM}}, N_{\mathbf{K}})$, where $N_{\mathbf{K}}(x) = 1 - x^2$.

(ii) Let us consider the following N-dual t-norm of $S^N_{\mathbf{nM}}$ given by

$$T^N_{\mathbf{nM}}(x, y) = \begin{cases} 0 & \text{if } x \leqslant N(y), \\ \min(x, y) & \text{if } x > N(y), \end{cases} \qquad x, y \in [0, 1].$$

The QL-operation obtained from the triple $(T^N_{\mathbf{nM}}, S^N_{\mathbf{nM}}, N^*)$ is given by

$$I_{T^N_{\mathbf{nM}},S^N_{\mathbf{nM}},N^*}(x, y) = \begin{cases} N^*(x) & \text{if } x \leqslant N(y), \\ 1 & \text{if } N^*(x) \geqslant N(y), \\ \max(N^*(x), y) & \text{if } N^*(x) < N(y), \end{cases} \qquad x, y \in [0, 1].$$

Figs. 2(c) and (d) give plots of the QL-implications obtained from the triple $(T^N_{\mathbf{nM}}, S^N_{\mathbf{nM}}, N^*)$, when $N = N^* = N_{\mathbf{C}}$ and $N = N_{\mathbf{C}}$, $N^* = N_{\mathbf{K}}$, respectively.

In fact, the following result was proven by Mas et al. [21, Corollary 2].

Proposition 4.16. *Let N be a strong negation with the fixed point $e \in (0, 1)$, T a continuous t-norm and $S^N_{\mathbf{nM}}$ the t-conorm obtained from N as given in (8). Let $I_{T,S^N_{\mathbf{nM}},N}$ be the QL-operation obtained from the triple $(T, S^N_{\mathbf{nM}}, N)$. Then the following statements are equivalent*:

(i) $I_{T,S^N_{\mathbf{nM}},N} \in \mathcal{FI}$.
(ii) $T(x, x) = x$ *for all* $x \in [e, 1]$.

Moreover, the corresponding QL-implication is then given by

$$I_{T,S^N_{\mathbf{nM}},N}(x, y) = \begin{cases} 1 & \text{if } x, e \leqslant y \quad \text{or} \quad (x \leqslant y < e \text{ and } T(x, y) = x), \\ y & \text{if } N(x) \leqslant y < x, \\ N(x) & \text{otherwise}, \end{cases} \qquad x, y \in [0, 1].$$

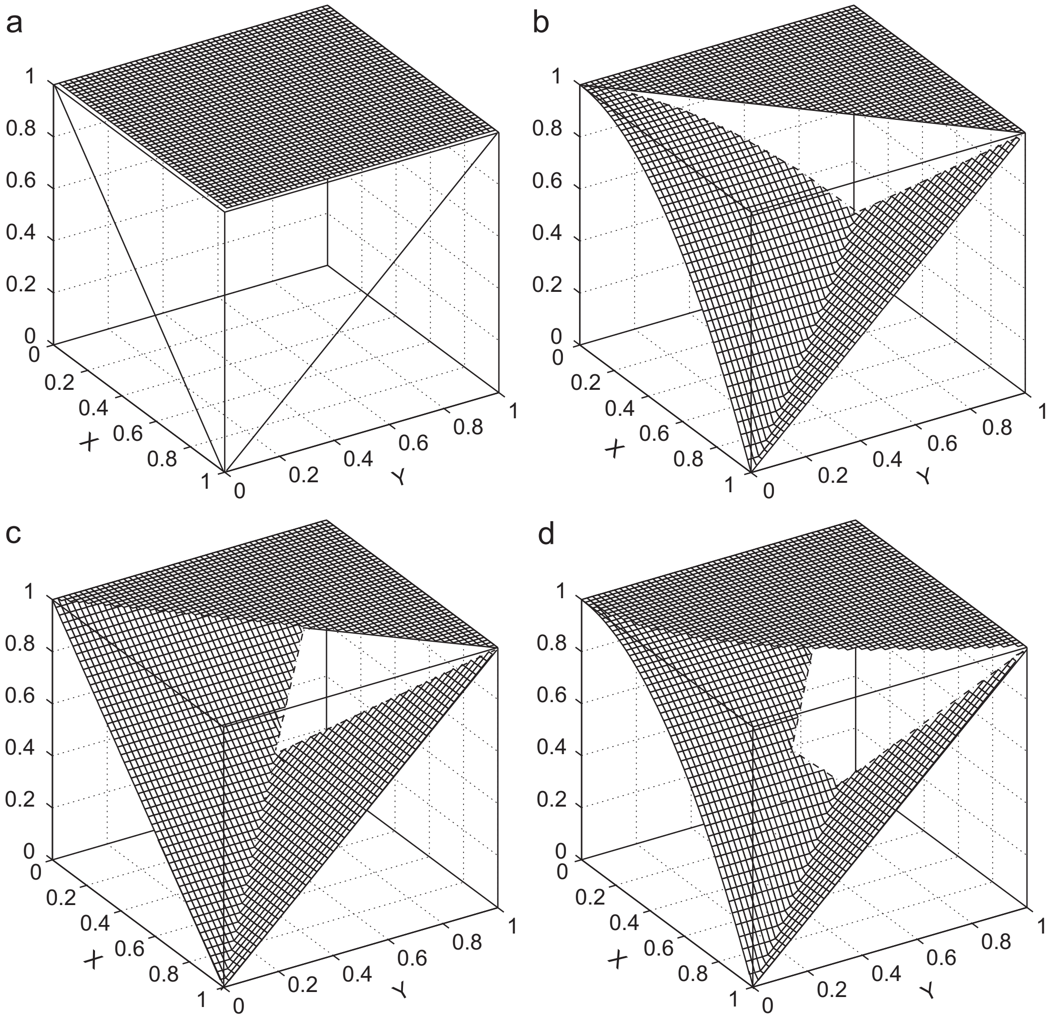

Fig. 2. Plots of QL-implications from non-continuous t-conorms. (a) The QL-implication $I_{\mathbf{DP}}$ from Example 4.14 with $N = N_{\mathbf{C}}$. (b) The QL-implication $I_{T_{\mathbf{M}},S^N_{\mathbf{nM}},N^*}$ from Example 4.15(i) with $N^* = N_{\mathbf{K}}$. (c) The QL-implication $I_{T^N_{\mathbf{nM}},S^N_{\mathbf{nM}},N^*}$ from Example 4.15(ii) with $N = N^* = N_{\mathbf{C}}$. (d) The QL-implication $I_{T^N_{\mathbf{nM}},S^N_{\mathbf{nM}},N^*}$ from Example 4.15(ii) with $N = N_{\mathbf{C}}$ and $N^* = N_{\mathbf{K}}$.

4.2. QL-implications and the exchange principle

Not all QL-implications satisfy (EP). However, in Theorem 6.2 we show that if a QL-operation $I_{T,S,N}$ is obtained from a triple (T, S, N), where N is a continuous negation, then $I_{T,S,N}$ satisfies (EP) if and only $I_{T,S,N}$ is an (S,N)-implication (see Section 5.1). We deal with this topic in more detail in Section 6.

4.3. QL-implications and the identity principle

The following proposition is immediate from Remark 4.6(iv), Examples 4.9 and 4.14.

Proposition 4.17. *A QL-implication $I_{T,S,N}$ satisfies* (IP) *if*

(i) *$N = N_{\mathbf{D2}}$, S is any t-conorm and T any t-norm, or*
(ii) *$T = T_{\mathbf{M}}$, S is any t-conorm and N any negation such that the pair (S, N) satisfies* (LEM), *or*
(iii) *$S = S_{\mathbf{D}}$, N is any non-vanishing negation and T is a positive t-norm.*

Remark 4.18.

(i) If S is a positive t-conorm, then we know, from Proposition 4.7, that the obtained QL-implication is the Weber implication $I_{\mathbf{WB}}$, which satisfies (IP).
(ii) However, from Example 4.12 and Proposition 4.17(iii), we see that in the case when S is not positive, we can obtain QL-implications that satisfy the identity principle (IP) for many fuzzy negations N. In fact, we have the following easy to obtain result.

Proposition 4.19. *If a QL-implication $I_{T,S,N}$ satisfies* (IP), *then $T(x,x) \geqslant N_S \circ N(x)$ for all $x \in [0,1]$.*

Proof. If $I_{T,S,N}$ satisfies (IP), then for any $x \in [0,1]$ we have $I_{T,S,N}(x,x) = S(N(x), T(x,x)) = 1$. From Remark 2.10(iii), we have that $T(x,x) \geqslant N_S \circ N(x)$, for all $x \in [0,1]$. □

In the case, when the t-conorm considered in Proposition 4.19 is right-continuous, then the above condition is also sufficient.

Theorem 4.20. *For a QL-implication $I_{T,S,N}$ with a right-continuous t-conorm S the following statements are equivalent*:

(i) *$I_{T,S,N}$ satisfies* (IP).
(ii) *$T(x,x) \geqslant N_S \circ N(x)$ for all $x \in [0,1]$.*

Proof. (i) $\Longrightarrow$ (ii) It is obvious from Proposition 4.19.
(ii)$\Longrightarrow$ (i) By the right-continuity of S, from Proposition 2.12(i) we have that, $S(N(x), N_S \circ N(x)) = 1$ for all $x \in [0,1]$. Now by the monotonicity of the t-conorm S we have that

$$I_{T,S,N}(x,x) = S(N(x), T(x,x)) \geqslant S(N(x), N_S \circ N(x)) = 1, \quad x \in [0,1],$$

i.e., $I_{T,S,N}$ satisfies (IP). □

Example 4.21. Let us consider the Łukasiewicz t-conorm $S_{\mathbf{LK}}$ and the strict negation $N_{\mathbf{K}}(x) = 1 - x^2$. The pair $(S_{\mathbf{LK}}, N_{\mathbf{K}})$ satisfies (LEM) and also $S_{\mathbf{LK}}$ is continuous, and hence is right-continuous. Let $T = T_{\mathbf{P}}$ be the product t-norm. Then the QL-operation obtained from the triple $(T_{\mathbf{P}}, S_{\mathbf{LK}}, N_{\mathbf{K}})$ is

$$I_{\mathbf{KP}}(x,y) = \min(1, 1 - x^2 + xy), \quad x, y \in [0,1].$$

Firstly, note that $I_{\mathbf{KP}}$ satisfies (I1) and hence is a fuzzy implication. Since $N_{S_{\mathbf{LK}}}(x) = 1 - x$, note also that, $N_{S_{\mathbf{LK}}} \circ N_{\mathbf{K}}(x) = 1 - N_{\mathbf{K}}(x) = 1 - (1 - x^2) = x^2$ and hence $T_{\mathbf{P}}(x,x) = N_{S_{\mathbf{LK}}} \circ N_{\mathbf{K}}(x)$ for all $x \in [0,1]$. It is easy to observe that $I_{\mathbf{KP}}$ satisfies (IP).

Remark 4.22. In one of the earliest works on QL-implications, Trillas and Valverde [30] (see also their recent work [28]) required the negation N in Definition 4.1 to be strong. Moreover, the t-norm T and t-conorm S are continuous, and are expected to form a De Morgan triple with the negation N. In fact, in Theorem 3.2 of the same work, under these restrictions, condition (ii) of Theorem 4.20 has been obtained. From their proof, it is clear that the considered T and S are both continuous and Archimedean and hence either they are strict or nilpotent, in which case they show that the aforementioned condition is not satisfied and hence the claim that "QL-implications never satisfy (IP)". Whereas, from the QL-implications $I_{\mathbf{WB}}$ and $I_{\mathbf{KP}}$ (see Example 4.21) we see that $I_{T,S,N}$ can satisfy (IP).

4.4. QL-implications and the ordering property

From Proposition 4.7 and Remark 4.18(i) it is clear that if S is a positive t-conorm, then the QL-implication obtained from the triple (T, S, N) does not satisfy (OP). The following result gives a necessary condition for a QL-implication to satisfy (OP).

Proposition 4.23. *If a QL-implication $I_{T,S,N}$ obtained from a non-positive t-conorm S satisfies* (OP), *then the negation N is strictly decreasing.*

Proof. To see this, if possible, let there exist $x, y \in [0, 1]$ such that $x < y$ but $N(x) = N(y)$. By (OP) we have

$$\begin{aligned} I_{T,S,N}(x, y) = 1 &\Longrightarrow S(N(x), T(x, y)) = 1 \\ &\Longrightarrow S(N(y), T(y, x)) = 1 \\ &\Longrightarrow I_{T,S,N}(y, x) = 1 \\ &\Longrightarrow y \leqslant x, \end{aligned}$$

a contradiction. □

Note that, from Remark 4.6(iii), we require that $N \geqslant N_S$, which implies that the natural negation N_S of the t-conorm S should be non-filling, i.e., $N(x) = 1 \iff x = 0$. From Definition 2.2, we see that this can happen only if every $x \in (0, 1)$ has $y \in (0, 1)$ such that $S(x, y) = 1$. Noting that a fuzzy implication that satisfies (OP) also satisfies (IP), using also Theorem 4.20, we summarize the above discussion in the following result.

Theorem 4.24. *If a QL-implication $I_{T,S,N}$ satisfies* (OP), *then*

(i) $T(x, x) \geqslant N_S \circ N(x)$ *for all* $x \in [0, 1]$;
(ii) *N is a strictly decreasing negation*;
(iii) *S is a non-positive t-conorm such that for every $x \in (0, 1)$ there exists $y \in (0, 1)$ such that $S(x, y) = 1$.*

Remark 4.25.

(i) In fact, the QL-implication $I_{\mathbf{KP}}$ obtained from the triple $(T_{\mathbf{P}}, S_{\mathbf{LK}}, N_{\mathbf{K}})$ (see Example 4.21) not only satisfies (IP), but also—as it can be easily verified—(OP).
(ii) The fact that the above conditions are not sufficient can be seen from Example 4.14 where the drastic sum t-conorm $S_{\mathbf{D}}$ satisfies condition (iii) of Theorem 4.24. Since $N_{S_{\mathbf{D}}} = N_{\mathbf{D1}}$, if N is any strictly decreasing negation we have that N is non-vanishing and $N > N_{\mathbf{D1}}$ for all $x \in (0, 1)$. Notice also that any t-norm T satisfies condition (i) since $N_{\mathbf{D1}} \circ N(x) = 0$ for all $x \in (0, 1)$. However, as can be seen from Example 4.14, the QL-implication $I_{T,S_{\mathbf{D}},N}$ obtained from such a triple $(T, S_{\mathbf{D}}, N)$ does not satisfy (OP).
(iii) We only emphasize that point (iii) of Theorem 4.24 is different from the pair (S, N) satisfying (LEM), in that, for some $x \in (0, 1)$ the N may be such that $N(x) = 1$, but the y in Theorem 4.24(iii) has to be in $(0, 1)$.

In the case, when the t-norm $T = T_{\mathbf{M}}$, we have the following stronger result.

Theorem 4.26. *Let S be a t-conorm and N a fuzzy negation such that the pair (S, N) satisfies conditions in Theorem* 4.24. *Further, for a t-norm T, let $I_{T,S,N}$ be a QL-implication which satisfies* (OP). *Then the following statements are equivalent*:

(i) *T is the minimum t-norm $T_{\mathbf{M}}$.*
(ii) $N_S \circ N = \mathrm{id}_{[0,1]}$.

Proof. Let the pair (S, N) satisfy conditions in Theorem 4.24 and, for a t-norm T, let $I_{T,S,N} \in \mathcal{FI}$ satisfy (OP).

(i) $\Longrightarrow$ (ii) If $T = T_{\mathbf{M}}$, then the QL-implication obtained from the triple $(T_{\mathbf{M}}, S, N)$ is the $I_{T_{\mathbf{M}},S,N}$ given in Example 4.9. From (6) we see that $x \leqslant y \Longrightarrow I_{T,S,N}(x, y) = 1$. The reverse implication is violated only if there exists $y < x$ such that $S(N(x), y) = 1$. From Remark 2.10(iii), we know that for this to happen $y \geqslant N_S \circ N(x)$. However, from Lemma 2.19(ii), we see that $y \in [N_S \circ N(x), x)$. Now it is obvious that the reverse implication holds only if $x = N_S \circ N(x)$.

(ii) $\Longrightarrow$ (i) Now, let $N_S \circ N(x) = x$ for all $x \in [0, 1]$. Since $I_{T,S,N}$ satisfies (OP), from Theorem 4.24(i), we have $x = N_S \circ N(x) \leqslant T(x, x) \leqslant x$ which implies that $T(x, x) = x$ for all $x \in [0, 1]$, i.e., T is idempotent, or equivalently, $T = T_{\mathbf{M}}$. □

Remark 4.27.

(i) From Example 4.14, we see that with the positive t-norm $T = T_{\mathbf{M}}$, if N is both non-vanishing and $N_S \circ N \neq \mathrm{id}_{[0,1]}$, then $I_{T,S,N}$ does not satisfy (OP).

(ii) Let S be a nilpotent t-conorm, i.e., Φ-conjugate with the Łukasiewicz t-conorm $S_{\mathbf{LK}}$. We know that the QL-implication obtained from the triple $(T, (S_{\mathbf{LK}})_\varphi, (N_{\mathbf{C}})_\varphi)$, where T is any t-norm, is $I_{\varphi,T}$ given by (7). Since $N_S = N = (N_{\mathbf{C}})_\varphi$ is a strong negation, from Theorems 4.26 and 4.20, we obtain the following result (cf. [21]).

Corollary 4.28. *For a QL-operation $I_{\varphi,T}$ given by (7), where T is any t-norm and $\varphi \in \Phi$, the following statements are equivalent*:

(i) *$I_{\varphi,T}$ satisfies (IP).*
(ii) *$I_{\varphi,T}$ satisfies (OP).*
(iii) *$T = T_{\mathbf{M}}$.*

Remark 4.29. The QL-implication $I_{\mathbf{KP}}$ in Example 4.21 shows that in the case $N \neq (N_{\mathbf{C}})_\varphi$ in Corollary 4.28, there do exist t-norms T other than $T_{\mathbf{M}}$ such that the QL-implication obtained from the triple $(T, (S_{\mathbf{LK}})_\varphi, N)$ satisfies (OP).

4.5. QL-implications and the law of contraposition

Since every QL-operation satisfies (NP) (see Proposition 4.2), it is immediate from Lemma 3.7, that if $I_{T,S,N}$ satisfies CP(N), then $N = N_I$ is strong. If S is a positive t-conorm, from Proposition 4.7 we see that a QL-operation $I_{T,S,N}$ is a fuzzy implication if and only if $N = N_{\mathbf{D2}}$, which is a non-strong negation. In fact, the QL-implication in this case is the Weber implication $I_{\mathbf{WB}}$, which does not satisfy (CP) with any negation N. Once again, if the QL-implication is obtained from the triple (T, S, N) where N is not strong, then by Lemma 3.8 we see that it does not satisfy (CP) with any fuzzy negation N. Of course, if N is strong and $I_{T,S,N}$ satisfies (EP), then by Lemma 3.9(ii), we have that $I_{T,S,N}$ satisfies CP(N).

Let S be a nilpotent t-conorm. Then it is non-positive, continuous and is Φ-conjugate with the Łukasiewicz t-conorm $S_{\mathbf{LK}}$. Consider, once again, the QL-implication $I_{\varphi,T}$ given by (7) and obtained from the triple $(T, (S_{\mathbf{LK}})_\varphi, (N_{\mathbf{C}})_\varphi)$. Since $(N_{\mathbf{C}})_\varphi$ is strong, we know that $I_{\varphi,T}$ satisfies (CP) only with $(N_{\mathbf{C}})_\varphi$. Now we have the following result firstly obtained by Fodor [11].

Theorem 4.30. *For a QL-operation $I_{\varphi,T}$ given by (7), where T is any t-norm and $\varphi \in \Phi$, the following statements are equivalent*:

(i) *$I_{\varphi,T}$ satisfies CP($(N_{\mathbf{C}})_\varphi$).*
(ii) *T belongs to the family of Frank t-norms $T_{\mathbf{F}}^{\lambda}$.*

We also have the following result which is stronger than the original result of Fodor [11, Theorem 5].

Theorem 4.31. *For the QL-implication $I_{T,S_{\mathbf{nM}}^N,N}$ the following statements are equivalent*:

(i) *$I_{T,S_{\mathbf{nM}}^N,N}$ satisfies (CP) with some fuzzy negation N^*.*
(ii) *$N^* = N$ is strong and $T = T_{\mathbf{M}}$.*

Proof. (i)$\Longrightarrow$ (ii) Let $I_{T,S_{\mathbf{nM}}^N,N}$ satisfy (CP) with some fuzzy negation N^*. Since any QL-implication satisfies (NP), from Lemma 3.7 we see that $N^* = N$ and is a strong negation. The rest of the proof is very much along the lines given in [11, Theorem 5].

(ii) $\Longrightarrow$ (i) In the case $T = T_{\mathbf{M}}$ we have that the QL-implication obtained from the triple $(T_{\mathbf{M}}, S_{\mathbf{nM}}^N, N)$ is as given in (9) with $N^* = N$. By a straightforward verification we see that $I_{T_{\mathbf{M}},S_{\mathbf{nM}}^N,N}$ indeed satisfies CP(N). □

5. (S,N)-implications and R-implications

In this section we introduce the two most established families of fuzzy implications, viz., (S,N)- and R-implications by giving their definitions, some examples, the conditions under which they satisfy the desirable algebraic properties and the relevant characterization/representation results. Following this, we give some results pertaining to the intersections that exist between these families.

Table 7
Examples of basic (S,N)-implications

S	N	(S,N)-implication $I_{S,N}$
$S_{\mathbf{M}}$	$N_{\mathbf{C}}$	$I_{\mathbf{KD}}$
$S_{\mathbf{P}}$	$N_{\mathbf{C}}$	$I_{\mathbf{RC}}$
$S_{\mathbf{LK}}$	$N_{\mathbf{C}}$	$I_{\mathbf{LK}}$
$S_{\mathbf{D}}$	$N_{\mathbf{C}}$	$I_{\mathbf{DP}}$
$S_{\mathbf{nM}}$	$N_{\mathbf{C}}$	$I_{\mathbf{FD}}$
Any S	$N_{\mathbf{D1}}$	$I_{\mathbf{D}}(x, y) = \begin{cases} 1 & \text{if } x = 0 \\ y & \text{if } x > 0 \end{cases}$
Any S	$N_{\mathbf{D2}}$	$I_{\mathbf{WB}}$

5.1. (S,N)-implications

Definition 5.1 *(cf. Trillas and Valverde [31], Fodor and Roubens [10], Alsina and Trillas [1], Baczyński and Jayaram [3])*. A function $I : [0, 1]^2 \to [0, 1]$ is called an (S,N)-*implication* if there exist a t-conorm S and a fuzzy negation N such that

$$I(x, y) = S(N(x), y), \quad x, y \in [0, 1].$$

If N is a strong negation, then I is called a *strong implication* or *S-implication*. Moreover, if I is an (S,N)-implication generated from S and N, then we will often denote it by $I_{S,N}$.

Example 5.2. Table 7 lists a few well-known (S,N)-implications along with their t-conorms and negations from which they have been obtained.

Remark 5.3 (*see Trillas and Valverde [31], Baczyński and Jayaram [3]*).

(i) All (S,N)-implications are fuzzy implications which satisfy (NP) and (EP).
(ii) If I is an (S,N)-implication obtained from a fuzzy negation N, then $N = N_I$.
(iii) Because of Corollary 3.10 we get that an (S,N)-implication I satisfies CP(N) with some fuzzy negation N if and only if $N = N_I$ is a strong negation, i.e., I is an S-implication.

Not all (S,N)-implications satisfy the identity principle (IP) or the ordering property (OP), see, for example $I_{\mathbf{RC}}$ and $I_{\mathbf{KD}}$. An equivalence condition under which (S,N)-implications satisfy them are given by the following results.

Lemma 5.4 (*Baczyński and Jayaram [4, Lemma 4.5]*). *For a t-conorm S and a fuzzy negation N the following statements are equivalent*:

(i) *The (S,N)-implication $I_{S,N}$ satisfies* (IP).
(ii) *The pair (S, N) satisfies* (LEM).

Theorem 5.5 (*Baczyński and Jayaram [4, Theorem 4.7]*). *For a t-conorm S and a fuzzy negation N the following statements are equivalent*:

(i) *The (S,N)-implication $I_{S,N}$ satisfies* (OP).
(ii) *$N = N_S$ is a strong negation and the pair (S, N_S) satisfies* (LEM).

The following characterization of (S,N)-implications is from [2], which is an extension of a result in [30].

Table 8
Examples of basic R-implications

t-norm T	R-implication I_T
$T_{\mathbf{M}}$	$I_{\mathbf{GD}}$
$T_{\mathbf{P}}$	$I_{\mathbf{GG}}$
$T_{\mathbf{LK}}$	$I_{\mathbf{LK}}$
$T_{\mathbf{D}}$	$I_{\mathbf{WB}}$
$T_{\mathbf{nM}}$	$I_{\mathbf{FD}}$

Theorem 5.6. *For a function $I: [0, 1]^2 \to [0, 1]$ the following statements are equivalent*:

(i) *I is an (S,N)-implication generated from some t-conorm S and some continuous (strict, strong) fuzzy negation N.*
(ii) *I satisfies (I2), (EP) and the function N_I is a continuous (strict, strong) fuzzy negation.*

Remark 5.7.

(i) The representations of (S,N)-implications in the above theorem is unique.
(ii) In Theorem 5.6 we can substitute the axiom (I2) by (I1).

5.2. R-implications

Definition 5.8 (*cf. Trillas and Valverde [31], Fodor and Roubens [10], Gottwald [12]*). A function $I: [0, 1]^2 \to [0, 1]$ is called an R-*implication*, if there exists a t-norm T such that

$$I(x, y) = \sup\{t \in [0, 1] \mid T(x, t) \leqslant y\}, \quad x, y \in [0, 1].$$

If I is an R-implication generated from a t-norm T, then we will often denote it by I_T.

Example 5.9. Table 8 lists a few well-known R-implications along with their t-norms from which they have been obtained.

Theorem 5.10 (*cf. Fodor and Roubens [10], Gottwald [12]*). *If T is any t-norm (not necessarily left-continuous), then $I_T \in \mathcal{FI}$. Moreover, I_T satisfies (NP) and (IP).*

Theorem 5.11 (*Fodor and Roubens [10, Theorem 1.14]*). *For a function $I: [0, 1]^2 \to [0, 1]$ the following statements are equivalent*:

(i) *I is an R-implication based on some left-continuous t-norm T.*
(ii) *I satisfies (I2), (OP), (EP) and $I(x, \cdot)$ is right-continuous for any fixed $x \in [0, 1]$.*

Remark 5.12. It can be immediately noted that $N_T(\cdot) = I_T(\cdot, 0)$, where I_T is obtained from a t-norm T. From Theorem 5.11 and Corollary 3.6, we see that for a left-continuous t-norm T, the fuzzy negation N_T is either strong or discontinuous.

5.3. Intersections between (S,N)- and R-implications

The following results are from Baczyński and Jayaram [4].

Theorem 5.13 (*Baczyński and Jayaram [4, Theorem 6.2]*). *For a left-continuous t-norm T, a t-conorm S and a fuzzy negation N the following statements are equivalent*:

(i) *The R-implication I_T is also an (S,N)-implication $I_{S,N}$, i.e., $I_T = I_{S,N}$.*
(ii) *$N = N_T$ is a strong negation and (T, N, S) form a De Morgan triple.*

Theorem 5.14 (*Baczyński and Jayaram [4, Theorem 6.5]*). *For a right-continuous t-conorm S and a t-norm T the following statements are equivalent*:

(i) *The (S,N)-implication I_{S,N_S} is also the R-implication I_T.*
(ii) (S, N_S, T) *form a De Morgan triple*.

Let us denote by

- $\mathbb{I}_{\mathbb{S},\mathbb{N}}$—the family of all (S,N)-implications;
- $\mathbb{I}_{\mathbb{S}}$—the family of all S-implications, i.e., (S,N)-implications where *N* is a strong negation;
- $\mathbb{I}_{\mathbb{S}^*,\mathbb{N}_{\mathbb{S}}^*}$—the family of all (S,N)-implications obtained from right-continuous t-conorms and their natural negations which are strong;
- $\mathbb{I}_{\mathbb{T}}$—the family of all R-implications;
- $\mathbb{I}_{\mathbb{T}_{\mathbb{LC}}}$—the family of all R-implications obtained from left-continuous t-norms;
- $\mathbb{I}_{\mathbb{T}^*}$—the family of all R-implications obtained from left-continuous t-norms having strong induced negations;
- $\mathbb{I}_{\mathbb{N}_{\mathbb{T}}(\mathbb{T}),\mathbb{N}_{\mathbb{T}}}$—the family of all (S,N)-implications obtained from the N_T-dual of the left-continuous t-norm *T* whose natural negation N_T is strong.

As a consequence of the presented facts, the following equalities hold:

$$\mathbb{I}_{\mathbb{S}} \cap \mathbb{I}_{\mathbb{T}_{\mathbb{LC}}} = \mathbb{I}_{\mathbb{S},\mathbb{N}} \cap \mathbb{I}_{\mathbb{T}_{\mathbb{LC}}} = \mathbb{I}_{\mathbb{N}_{\mathbb{T}}(\mathbb{T}),\mathbb{N}_{\mathbb{T}}} = \mathbb{I}_{\mathbb{T}^*} = \mathbb{I}_{\mathbb{S}^*,\mathbb{N}_{\mathbb{S}}^*}.$$

Note, that as yet, it is only known that $I_{\mathbf{WB}} \in \mathbb{I}_{\mathbb{S},\mathbb{N}} \cap \mathbb{I}_{\mathbb{T}}$. A complete characterization of the intersection of these classes is still an open problem.

6. Intersections between (S,N)- and QL-implications

Let us denote by

- $\mathbb{I}_{\mathbb{QL}}$—the family of all QL-implications.

Firstly, note that if the t-conorm *S* is positive, the following result is obvious from Proposition 4.7.

Theorem 6.1. *The QL-implication $I_{T,S,N}$, where S is a positive t-conorm, is an* (S,N)-*implication*. *In fact*, $I_{T,S,N} = I_{\mathbf{WB}}$.

Hence

$$\mathbb{I}_{\mathbb{S},\mathbb{N}} \cap \mathbb{I}_{\mathbb{QL}} \neq \emptyset.$$

The following result shows that if a QL-operation $I_{T,S,N}$ is obtained from a triple (T, S, N), where *N* is a continuous negation, then it being an (S,N)-implication is equivalent to $I_{T,S,N}$ satisfies (EP). We give a slightly more general result than was shown by Mas et al. [21].

Theorem 6.2 (*cf. Mas et al. [21, Proposition 8]*). *For a QL-implication $I_{T,S,N}$, with a continuous negation N, the following statements are equivalent*:

(i) $I_{T,S,N}$ *satisfies* (EP).
(ii) $I_{T,S,N}$ *is an (S,N)-implication generated from N.*

Proof. (i) $\Longrightarrow$(ii) Let $I_{T,S,N}$ be a QL-implication with *N* a continuous negation. Firstly, observe that $I_{T,S,N}$ satisfies (I2) and $N_{I_{T,S,N}} = N$ is a continuous negation. If $I_{T,S,N}$ satisfies (EP), then by virtue of Theorem 5.6 and Remark 5.7(ii), the function $I_{T,S,N}$ is an (S,N)-implication generated from *N*.

(ii)$\Longrightarrow$ (i) The reverse implication is obvious and follows from Remark 5.3(i). □

Remark 6.3.

(i) When N is a strict negation in Theorem 6.2, then $I_{T,S,N}$ is an (S,N)-implication generated from N and a t-conorm S^* given by $S^*(x, y) = S(x, T(N^{-1}(x), y))$.
(ii) Theorem 6.2 also gives a sufficient condition for a QL-operation obtained from the triple (T, S, N) with a continuous negation N to be a fuzzy implication.
(iii) However, the QL-implications $I_{\mathbf{WB}}$ and $I_{T,S_{\mathbf{D}},N}$ (see Example 4.14), with any N that is discontinuous but non-vanishing, show that the continuity of N is not necessary for QL-operation to satisfy (EP).
(iv) It is interesting to note that both $I_{\mathbf{WB}}$ and $I_{T,S_{\mathbf{D}},N}$, under the conditions of Example 4.14, are also (S,N)-implications. While $I_{\mathbf{WB}}$ is an (S,N)-implication obtained from any t-conorm S and $N = N_{\mathbf{D2}}$, i.e., $I_{\mathbf{WB}} = I_{S,N_{\mathbf{D2}}}$, the QL-implication $I_{T,S_{\mathbf{D}},N}$ is the (S,N)-implication $I_{S_{\mathbf{D}},N}$.

In the case, when T is the minimum t-norm $T_{\mathbf{M}}$, we have the following stronger results.

Proposition 6.4. *Let S be a t-conorm and N a fuzzy negation such that the pair (S, N) satisfies* (LEM). *Then the QL-operation $I_{T_{\mathbf{M}},S,N}$ is also an (S,N)-implication obtained from the same t-conorm S and negation N, i.e., $I_{T_{\mathbf{M}},S,N} = I_{S,N}$. In other words, $I_{S,N}$ can be represented as a QL-implication obtained from the triple $(T_{\mathbf{M}}, S, N)$.*

Proof. Consider the QL-operation generated from the minimum t-norm $T_{\mathbf{M}}$, a t-conorm S and a fuzzy negation N such that the pair (S, N) satisfies (LEM). Then $I_{T_{\mathbf{M}},S,N}$ is a fuzzy implication given by (6). On the one hand, if $x \leqslant y$, then

$$I_{T,S,N}(x, y) = S(N(x), T_{\mathbf{M}}(x, y)) = S(N(x), x) = 1$$

and

$$I_{S,N}(x, y) = S(N(x), y) \geqslant S(N(x), x) = 1.$$

On the other hand, if $x > y$, then $I_{T,S,N}(x, y) = S(N(x), y) = I_{S,N}(x, y)$. □

Remark 6.5. In the above Proposition 6.4 the condition that the pair (S, N) satisfies (LEM) is essential. Otherwise, the QL-operation may not be a fuzzy implication, as can be seen for $I_{\mathbf{ZD}}$ in Remark 4.3.

From [16, Proposition 2.31] we know that $(T_{\mathbf{M}})_{\varphi} = T_{\mathbf{M}}$ for all $\varphi \in \Phi$, from whence we have the following result.

Corollary 6.6. *The Φ-conjugate of the QL-implication $I_{T_{\mathbf{M}},S,N}$ is also the (S,N)-implication generated from the Φ-conjugate t-conorm of S and the Φ-conjugate fuzzy negation of N, i.e., if $\varphi \in \Phi$, then*

$$(I_{T_{\mathbf{M}},S,N})_{\varphi}(x, y) = I_{S_{\varphi},N_{\varphi}}(x, y), \quad x, y \in [0, 1].$$

As an interesting consequence of the above fact we obtain the following characterizations of some special classes of QL-implications.

Theorem 6.7. *For a function $I: [0, 1]^2 \to [0, 1]$ the following statements are equivalent*:

(i) *I is a QL-implication obtained from the triple $(T_{\mathbf{M}}, S, N)$ with a continuous negation N.*
(ii) *I satisfies* (I2), (IP), (EP) *and N_I is a continuous negation.*

Proof. (i) $\Longrightarrow$ (ii) From Proposition 6.4 we see that I is an (S,N)-implication obtained from the t-conorm S and the continuous negation N. Hence it satisfies (I2) and (EP) too. Since I is a QL-implication we know that the pair (S, N) satisfies (LEM) and hence, by Lemma 5.4 I satisfies (IP).

(ii)$\Longrightarrow$ (i) Since I satisfies (I2), (EP) and N_I is a continuous negation, I is an (S,N)-implication obtained from some t-conorm S and $N = N_I$. Consider the QL-operation J obtained from the triple $(T_{\mathbf{M}}, S, N)$ with $N = N_I$ and the above S. Since I satisfies (IP), once again by Lemma 5.4, we know that the pair (S, N) satisfies (LEM). Now with $T = T_{\mathbf{M}}$ we know from Proposition 6.4, that J is the (S,N)-implication obtained from the above t-conorm S and $N = N_I$, i.e., $J = I$. □

Theorem 6.8. *For a function* $I: [0,1]^2 \to [0,1]$ *the following statements are equivalent*:

(i) *I is a QL-implication obtained from the triple* $(T_{\mathbf{M}}, S, N_S)$ *where* N_S *is a strong negation and the pair* (S, N_S) *satisfies* (LEM).
(ii) *I satisfies* (I2), (OP), (EP) *and* N_I *is a strong negation*.

Proof. (i)$\Longrightarrow$ (ii) From Proposition 6.4 we see that $I = I_{S,N_S}$, the (S,N)-implication obtained from the t-conorm *S* and the strong negation N_S. Hence it satisfies (I2) and (EP). Moreover, $N_I = N_{I_{S,N_S}} = N_S$ is a strong negation. Finally, from Theorem 5.5 we see that *I* satisfies (OP).

(ii)$\Longrightarrow$ (i) Since *I* satisfies (I2), (EP) and N_I is a strong negation it is an S-implication obtained from the t-conorm $S(x, y) = I(N_I(x), y)$ and $N = N_I$. Once again, from Theorem 5.5 we see that $N = N_I = N_S$ and the pair (S, N_S) satisfies (LEM). Consider the QL-operation *J* obtained from the triple $(T_{\mathbf{M}}, S, N_S)$. From Proposition 6.4, we get that *J* is the (S,N)-implication obtained from the above t-conorm *S* and $N = N_S$, i.e., $J = I$. □

Corollary 6.9. *For a function* $I: [0,1]^2 \to [0,1]$ *the following statements are equivalent*:

(i) *I is a QL-implication obtained from the triple* $(T_{\mathbf{M}}, S, N_S)$ *where S is a right-continuous t-conorm and* N_S *is a strong negation*.
(ii) *I satisfies* (I2), (OP), (EP), *I is right-continuous in the second variable and* N_I *is a strong negation*.

Proof. (i) $\Longrightarrow$ (ii) Since *S* is right-continuous from Proposition 2.12(iv) we have that the pair (S, N_S) satisfies (LEM). Then, from Theorem 6.8, we see that *I* satisfies (I2), (OP), (EP) and N_I is a strong negation. Obviously, *I* is right-continuous in the second variable.

(ii)$\Longrightarrow$ (i) Since I is right-continuous in the second variable and $S(x, y) = I(N_I(x), y)$ we have that *S* is also right-continuous in the second variable. But *S* is also a t-conorm in this case, so *S* is right-continuous. Once again, from Theorem 6.8 the rest of the proof is obvious. □

Theorem 6.10. *Let* $I_{T,S,N}$ *be a QL-implication, where S is a non-positive t-conorm with strong induced negation* N_S. *Consider the following statements*:

(i) $I_{T,S,N}$ *is an (S,N)-implication obtained from the same S and N, i.e.,* $I_{T,S,N} = I_{S,N}$.
(ii) $N = N_S$.
(iii) $T = T_{\mathbf{M}}$.

Then the following relationships exist among the above statements: (i) *and* (ii) $\Longrightarrow$ (iii); (ii) *and* (iii) $\Longrightarrow$ (i).

Proof. Firstly, note that if $I_{T,S,N} \in \mathcal{FI}$, then by virtue of Lemma 4.5 the pair (S, N) satisfies (LEM).

(i) and (ii) $\Longrightarrow$ (iii) We know that for any t-norm $T(x, x) \leqslant x$ for all $x \in [0, 1]$. Moreover, for any $x \in (0, 1)$, we have $N_S(x) \neq 1$. Let us assume that $I_{T,S,N_S} = I_{S,N_S}$ for some non-positive t-conorm *S* with strong natural negation N_S and some t-norm *T*. Then $S(N_S(x), T(x, x)) = S(N_S(x), x) = 1$, for all $x \in [0, 1]$, since the pair (S, N_S) satisfies (LEM). From Remark 2.10(iii), we obtain $T(x, x) \geqslant N_S \circ N_S(x) = x$, from whence we obtain $T(x, x) = x$, for all $x \in [0, 1]$, i.e., $T = T_{\mathbf{M}}$.

(ii) and (iii) $\Longrightarrow$ (i) Since the pair (S, N) satisfies (LEM), this follows from Proposition 6.4. □

Remark 6.11. Let us consider a t-conorm *S* whose natural negation N_S is discontinuous. Note that, in this remark, by points (i)–(iii) we refer to the items described in Theorem 6.10.

From Proposition 6.4, we always have that (iii) $\Longrightarrow$ (i). Let us define a lenient version of (i) as follows:

(i′) $I_{T,S,N}$ is an (S,N)-implication obtained from a (possibly different) t-conorm S' and a negation N', i.e., $I_{T,S,N} = I_{S',N'}$.

Then, from Table 9, the following observations can be made:

(a) From the first entry, we notice that $N = N_S$ is not strong and $I_{T,S,N} = I_{S,N}$, but $T \neq T_{\mathbf{M}}$, i.e., (i) and (ii)$\not\Longrightarrow$ (iii), when *N* is not strong. Note that the t-conorm $S_{\mathbf{P}}$ can be replaced by any positive t-conorm.

Table 9
Some QL-implications that are also (S,N)-implications

S	T	N	N_S	$I_{T,S,N}$
$S_{\mathbf{P}}$	Any T	$N_{\mathbf{D2}}$	$N_{\mathbf{D2}}$	$I_{\mathbf{WB}}$
$S_{\mathbf{B}}$	$T_{\mathbf{M}}$	$N_{\mathbf{D2}}$	$N_{S_{\mathbf{B}}}$	$I_{\mathbf{WB}}$
$S_{\mathbf{D}}$	$T_{\mathbf{M}}$	$N_{\mathbf{C}}$	$N_{\mathbf{D1}}$	$I_{\mathbf{DP}}$
$S_{\mathbf{LK}}$	$T_{\mathbf{LK}}$	$N_{\mathbf{C}}$	$N_{\mathbf{C}}$	$I_{\mathbf{KD}}$
$S_{\mathbf{LK}}$	$T_{\mathbf{P}}$	$N_{\mathbf{C}}$	$N_{\mathbf{C}}$	$I_{\mathbf{RC}}$

(b) From the second and third entries, it is clear that even if $I_{T,S,N} = I_{S,N}$ and $T = T_{\mathbf{M}}$ we can have $N_S \neq N$, i.e., (i) and (iii)$\not\Longrightarrow$ (ii), when N is not strong. Note that the t-conorm $S_{\mathbf{B}}$ and its natural negation $N_{S_{\mathbf{B}}}$ are given as follows:

$$S_{\mathbf{B}}(x, y) = \begin{cases} 1 & \text{if } (x, y) \in (0.5, 1)^2, \\ \max(x, y) & \text{otherwise,} \end{cases} \qquad x, y \in [0, 1],$$

$$N_{S_{\mathbf{B}}}(x) = \begin{cases} 1 & \text{if } x \in [0, 0.5), \\ 0.5 & \text{if } x \in [0.5, 1), \\ 0 & \text{if } x = 1, \end{cases} \qquad x \in [0, 1].$$

(c) From the fourth and fifth entries, we see that $I_{T,S,N} = I_{S',N'}$ and $N = N' = N_{\mathbf{C}}$, a strong negation, but $T \neq T_{\mathbf{M}}$, i.e., (i$'$) and (ii)$\not\Longrightarrow$ (iii).

Let us denote by

- $\mathbb{I}_{\mathbb{S},\mathbb{N}_{\mathbb{S}}}$—the family of all (S,N)-implications obtained from t-conorms and their natural negations;
- $\mathbb{I}_{\mathbb{S},\hat{\mathbb{N}}}$—the family of all (S,N)-implications, where N is greater than or equal to the natural negation obtained from S, i.e., $N \geqslant N_S$.

Summarizing the above discussion, we get

$$\mathbb{I}_{\mathbb{S}^*,\mathbb{N}^*_{\mathbb{S}}} \subsetneq \mathbb{I}_{\mathbb{S},\mathbb{N}_{\mathbb{S}}} \subsetneq \mathbb{I}_{\mathbb{S},\hat{\mathbb{N}}} \subsetneq \mathbb{I}_{\mathbb{QL}}.$$

The following examples illustrate the above chain of inclusions.

- $I_{\mathbf{LK}}, I_{\mathbf{FD}} \in \mathbb{I}_{\mathbb{S}^*,\mathbb{N}^*_{\mathbb{S}}} \subsetneq \mathbb{I}_{\mathbb{QL}}$.
- $I_{\mathbf{WB}} \in \mathbb{I}_{\mathbb{S},\mathbb{N}_{\mathbb{S}}} \setminus \mathbb{I}_{\mathbb{S}^*,\mathbb{N}^*_{\mathbb{S}}} \subsetneq \mathbb{I}_{\mathbb{QL}}$.
- $I_{\mathbf{DP}} \in \mathbb{I}_{\mathbb{S},\hat{\mathbb{N}}} \setminus \mathbb{I}_{\mathbb{S},\mathbb{N}_{\mathbb{S}}} \subsetneq \mathbb{I}_{\mathbb{QL}}$.
- $I_{\mathbf{KD}}, I_{\mathbf{RC}} \in \mathbb{I}_{\mathbb{QL}} \setminus \mathbb{I}_{\mathbb{S},\hat{\mathbb{N}}}$.
- The QL-implications $I_{\mathbf{PC}}, I_{\mathbf{PR}}$ from Example 4.13 and $I_{\mathbf{KP}}$ from Example 4.21 do not satisfy the exchange principle (EP) and hence are not (S,N)-implications, i.e.,

$$I_{\mathbf{PC}}, I_{\mathbf{PR}}, I_{\mathbf{KP}} \in \mathbb{I}_{\mathbb{QL}} \setminus \mathbb{I}_{\mathbb{S},\mathbb{N}}.$$

- Similarly, the fuzzy implication $I_{\mathbf{D}}$ (see Table 7) is an (S,N)-implication obtained from the least negation $N_{\mathbf{D1}}$ and hence, by Remark 4.6(i), it is not a QL-implication, i.e.,

$$I_{\mathbf{D}} \in \mathbb{I}_{\mathbb{S},\mathbb{N}} \setminus \mathbb{I}_{\mathbb{QL}}.$$

7. Intersections between R- and QL-implications

Firstly, if S is a positive t-conorm or if $N = N_{\mathbf{D2}}$, then the QL-implication $I_{T,S,N}$ is the R-implication $I_{\mathbf{WB}}$ obtained from the non-left-continuous t-norm $T_{\mathbf{D}}$. Hence

$$\mathbb{I}_{\mathbb{QL}} \cap \mathbb{I}_{\mathbb{T}} \neq \emptyset.$$

A complete characterization of the above intersection is as yet unknown. However, as we show below, the exact intersection of the family of QL-implications $\mathbb{I}_{\mathbb{QL}}$ with the family of R-implications obtained from left-continuous t-norms $\mathbb{I}_{\mathbb{T}_{\mathbb{LC}}}$ can be precisely determined.

Proposition 7.1. *If a QL-implication $I_{T,S,N}$ is an R-implication obtained from a left-continuous t-norm T^*, then*

(i) *$N = N_{T^*}$ is strong*;
(ii) *$I_{T,S,N}$ is also an S-implication obtained from a t-conorm S^*, such that S^* is the N-dual of T^*, and $N = N_{S^*}$, i.e., $I_{T,S,N} = I_{S^*,N_{S^*}}$.*

Proof. Let a QL-implication obtained from the triple (T, S, N) also be an R-implication obtained from a left-continuous t-norm T^*, i.e., let $I_{T,S,N} = I_{T^*}$.

(i) From Theorem 5.11, we see that I_{T^*} satisfies both (EP) and (OP). Now, from Propositions 4.23 and 3.6, we get that

$$N = N_{I_{T,S,N}} = N_{I_{T^*}} = N_{T^*},$$

is either strong or discontinuous but strictly decreasing. However, we know from Corollary 2.15, that the natural negation N_{T^*} of a (left-continuous) t-norm T^*, if discontinuous, is not strictly decreasing. Hence $N = N_{T^*}$ is strong.

(ii) Since $I_{T,S,N}$ satisfies (EP), Theorem 6.2 implies that $I_{T,S,N}$ is also an S-implication $I_{S^*,N}$ for some t-conorm S^*, i.e., $I_{T,S,N} = I_{S^*,N} = I_{T^*}$. Now, from Theorem 5.13, we see that (T^*, S^*, N) forms a De Morgan triple, i.e., S^* is the N-dual of T^* and that $N = N_{S^*}$. $\square$

Theorem 7.2. *For a function $I: [0, 1]^2 \to [0, 1]$ the following statements are equivalent*:

(i) *I is both a QL-implication obtained from the triple (T, S, N) and an R-implication obtained from some left-continuous t-norm T^*.*
(ii) *I can be represented as a QL-implication obtained from $(T_{\mathbf{M}}, S^*, N_{S^*})$, where S^* is a right-continuous t-conorm with a strong natural negation N_{S^*}.*
(iii) *I is both an (S,N)- and an R-implication obtained from a left-continuous t-norm.*

Proof. (i) $\Longrightarrow$ (ii) Let $I = I_{T,S,N} = I_{T^*}$. From Proposition 7.1, we have that $N = N_{I_{T^*}} = N_{T^*}$ is strong and $I = I_{S^*,N}$, where S^* is the right-continuous t-conorm that is the N-dual of T^*. Moreover, $N = N_{T^*} = N_{S^*}$. Now, since S^* is right-continuous, by Proposition 2.22, we see that the pair (S^*, N_{S^*}) indeed satisfies (LEM). Further, by Proposition 6.4, we see that $I_{S^*,N_{S^*}}$ can also be represented as a QL-implication obtained from the triple $(T_{\mathbf{M}}, S^*, N_{S^*})$, i.e., $I = I_{T^*} = I_{S^*,N_{S^*}} = I_{T_{\mathbf{M}},S^*,N_{S^*}}$.

(ii) $\Longrightarrow$ (iii) Firstly, from Proposition 6.4, we see that such a QL-implication is also an (S,N)-implication. In fact, we have $I_{T_{\mathbf{M}},S^*,N_{S^*}} = I_{S^*,N_{S^*}}$. Since S^* is a right-continuous t-conorm with a strong natural negation N_{S^*}, we see that (S^*, N_{S^*}, T^*) form a De Morgan triple, where T^* is the left-continuous t-norm which is N_{S^*}-dual of S^*. Now, from Theorem 5.14, we see that $I_{S^*,N_{S^*}}$ is also the R-implication obtained from T^*, i.e., $I_{S^*,N_{S^*}} = I_{T^*}$.

(iii) $\Longrightarrow$ (i) If I is both an (S,N)- and an R-implication obtained from a left-continuous t-norm, then we know, from Theorem 5.13, that $(T, S, N_T = N_S)$ form a De Morgan triple and $I = I_{S,N_S} = I_T$. Once again, invoking Proposition 6.4, we see that I_{S,N_S} can also be represented as a QL-implication obtained from the triple $(T_{\mathbf{M}}, S, N_S)$, i.e., $I = I_T = I_{S,N_S} = I_{T_{\mathbf{M}},S,N_S}$. $\square$

From Theorem 7.2, we see that

$$\begin{aligned}\mathbb{I}_{\mathbb{QL}} \cap \mathbb{I}_{\mathbb{T}_{\mathbb{LC}}} &= \mathbb{I}_{\mathbb{T}_{\mathbb{LC}}} \cap \mathbb{I}_{\mathbb{S},\mathbb{N}} \\ &= \mathbb{I}_{\mathbb{QL}} \cap \mathbb{I}_{\mathbb{T}_{\mathbb{LC}}} \cap \mathbb{I}_{\mathbb{S},\mathbb{N}} \\ &= \mathbb{I}_{\mathbb{S}^*,\mathbb{N}_{\mathbb{S}}^*} = \mathbb{I}_{\mathbb{T}^*}.\end{aligned}$$

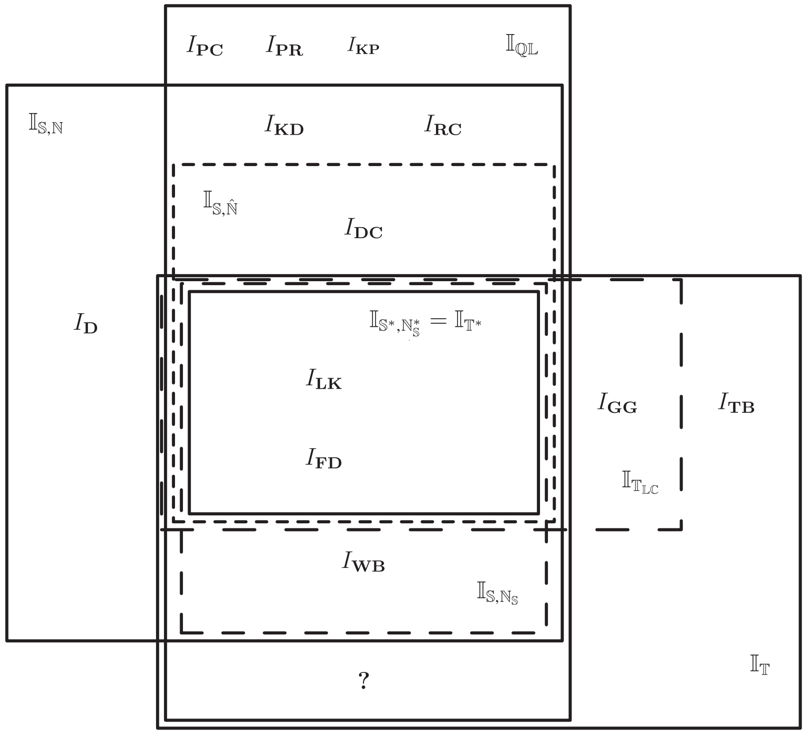

Fig. 3. Intersections between families of (S,N)-, R- and QL-implications.

Example 7.3.

(i) Consider the R-implication obtained from the t-norm $T_{\mathbf{B}}$ given in Example 2.13, which is given as follows:

$$I_{\mathbf{TB}}(x,y) = \begin{cases} 1 & \text{if } x \leqslant y, \\ 0.5 & \text{if } x > y \quad \text{and} \quad x \in [0, 0.5), \\ y & \text{otherwise}, \end{cases} \qquad x, y \in [0,1].$$

It is clear that $I_{\mathbf{TB}}$ satisfies (OP) but its natural negation $N_{I_{\mathbf{TB}}} = N_{\mathbf{TB}}$ given in Example 2.13 is not strictly decreasing and hence, by Proposition 4.23, we have that $I_{\mathbf{TB}}$ is not a QL-implication. For the same reason the Goguen implication $I_{\mathbf{GG}}$ cannot be obtained as a QL-implication for any triple (T, S, N).

(ii) Consider the QL-implications $I_{\mathbf{PC}}$, $I_{\mathbf{PR}}$ from Example 4.13. Since they do not satisfy (IP), from Theorem 5.10 we see that they cannot be represented as R-implications of any t-norm. Let us now consider the QL-implication $I_{\mathbf{KP}}$ in Example 4.21. Since a QL-operation is generated by a unique negation (see Proposition 4.2(ii)), if $I_{\mathbf{KP}}$ is also an R-implication I_T obtained from some t-norm T, then $N_{I_{\mathbf{KP}}} = N_{\mathbf{K}} = N_{I_T} = N_T$. However, $N_{\mathbf{K}}$ is a strict negation that is not involutive, hence by Corollary 2.15, we see that $I_{\mathbf{KP}}$ cannot be an R-implication.

To summarize, we have the following facts:

- $I_{\mathbf{PC}}, I_{\mathbf{PR}}, I_{\mathbf{KP}} \in \mathbb{I}_{\mathbb{QL}} \setminus \mathbb{I}_{\mathbb{T}}$,
- $I_{\mathbf{WB}} \in \mathbb{I}_{\mathbb{QL}} \setminus \mathbb{I}_{\mathbb{T}_{\mathbb{LC}}}$,
- $I_{\mathbf{TB}} \in \mathbb{I}_{\mathbb{T}} \setminus \mathbb{I}_{\mathbb{QL}}$,
- $I_{\mathbf{GG}} \in \mathbb{I}_{\mathbb{T}_{\mathbb{LC}}} \setminus \mathbb{I}_{\mathbb{QL}}$.

The results presented in this section are also diagrammatically represented in Fig. 3.

8. Intersections between Yager's and QL-implications

Recently, Yager [32] introduced two new classes of fuzzy implications from the additive generators of t-norms and t-conorms called *f*- and *g*-implications, respectively. We only give the relevant results here. For more details, we refer the readers to Yager [32] and Baczyński and Jayaram [2].

Definition 8.1 (*Yager [32, p. 197]*). Let $f: [0, 1] \to [0, \infty]$ be a strictly decreasing and continuous function with $f(1) = 0$. The function $I: [0, 1]^2 \to [0, 1]$ defined by

$$I(x, y) = f^{-1}(x \cdot f(y)), \quad x, y \in [0, 1], \tag{10}$$

with the understanding $0 \cdot \infty = 0$, is called an *f-generated implication*. The function *f* itself is called an *f*-generator of the *I* generated as in (10). In such a case, to emphasize the apparent relation we will write I_f instead of *I*.

Definition 8.2 (*Yager [32, p. 202]*). Let $g: [0, 1] \to [0, \infty]$ be a strictly increasing and continuous function with $g(0) = 0$. The function $I: [0, 1]^2 \to [0, 1]$ defined by

$$I(x, y) = g^{(-1)}\left(\frac{1}{x} \cdot g(y)\right), \quad x, y \in [0, 1], \tag{11}$$

with the understanding $1/0 = \infty$ and $\infty \cdot 0 = \infty$, is called a *g-generated implication*, where the function $g^{(-1)}$ in (11) is the pseudo-inverse of *g* given by

$$g^{(-1)}(x) = \begin{cases} g^{-1}(x) & \text{if } x \in [0, g(1)], \\ 1 & \text{if } x \in [g(1), \infty]. \end{cases}$$

The function *g* itself is called a *g*-generator of the *I* generated as in (11). Once again, we will write I_g instead of *I*.

Proposition 8.3 (*Baczyński and Jayaram [2, Propositions 2 and 4]*).

(i) *If an f-generator is such that* $f(0) = \infty$, *then the natural negation of* I_f *is the Gödel negation* $N_{\mathbf{D1}}$, *which is non-continuous*.
(ii) *If g is a g-generator*, *then the natural negation of* I_g *is the Gödel negation* $N_{\mathbf{D1}}$, *which is not continuous*.

Let us denote by

- $\mathbb{I}_{\mathbb{F},\infty}$—the family of all *f*-generated implications such that $f(0) = \infty$;
- $\mathbb{I}_{\mathbb{G}}$—the family of all *g*-generated implications.

Unfortunately, there does not exist any complete characterization of the family of QL-implications. Interestingly, some results can still be proven regarding the intersections between QL-implications and $\mathbb{I}_{\mathbb{F},\infty}$, $\mathbb{I}_{\mathbb{G}}$. From Remark 4.6(i), we see that if the natural negation N_I of a fuzzy implication *I* is the Gödel negation $N_{\mathbf{D1}}$, then *I* is not a QL-implication. Now, from Proposition 8.3(i) and (ii), we have the following result.

Theorem 8.4. *If I is either*

(i) *a g-implication obtained from a g-generator*, *or*
(ii) *an f-implication obtained from an f-generator with* $f(0) = \infty$,

then I is not a QL-implication.

Summarizing the above results we have

$$\mathbb{I}_{\mathbb{F},\infty} \cap \mathbb{I}_{\mathbb{QL}} = \emptyset,$$

$$\mathbb{I}_{\mathbb{G}} \cap \mathbb{I}_{\mathbb{QL}} = \emptyset.$$

9. Concluding remarks

In this paper, we have systematically studied the class of QL-implications without imposing any conditions on the underlying operations. Firstly, we have shown that not all QL-operations are fuzzy implications and have given some necessary/sufficient conditions to this end. However, the following question, as yet, remains unsolved:

Problem 9.1. Characterize triples (T, S, N) such that $I_{T,S,N}$ satisfies (I1).

Following this, we have discussed the conditions under which this family satisfies some desirable algebraic properties. However, only some necessary conditions are known for a QL-implication to satisfy (OP) and hence we have the following:

Problem 9.2. What extra sufficient condition(s) should we impose, other than the ones in Theorem 4.24, so that the QL-implication obtained from the triple (T, S, N) satisfies (OP)?

Although Theorem 6.2 gives an equivalence condition for a QL-implication $I_{T,S,N}$, with a continuous negation *N*, to satisfy (EP), it is obvious that its utility is quite limited. Nevertheless, all the examples, so far, seem to point that a QL-implication that satisfies (EP) also turns out to be an (S,N)-implication. The answers to the following posers will be of immense help in resolving the exact intersection between families of (S,N)- and QL-implications.

Problem 9.3.

(i) Is Theorem 6.2 true even when *N* is not continuous, i.e., is any QL-implication $I_{T,S,N}$ that satisfies (EP) also an (S,N)-implication?
(ii) If not, give a counter-example and hence obtain an alternate necessary and sufficient condition for a QL-implication $I_{T,S,N}$ to satisfy (EP).

Based on the obtained results and existing characterization results, the intersections between QL-implications and the two most established families of fuzzy implications, viz., (S,N)- and R-implications have been determined. It is shown that QL-implications contain the set of all R-implications obtained from left-continuous t-norms that are also (S,N)-implications. However, it is not yet clear if there is a QL-implication which can be represented as an R-implication of some non-left-continuous t-norm, but which is not an (S,N)-implication. Note that Theorem 7.2 assumes the left-continuity of the t-norm T^* (see Fig. 3). The overlaps between QL-implications and the recently proposed *f*- and *g*-implications have also been studied.

In this context, denoting the family of all *f*-generated implications with $f(0) < \infty$ by $\mathbb{I}_{\mathbb{F},\aleph}$, the following questions remain to be solved.

Problem 9.4.

(i) Characterize the non-empty intersection $\mathbb{I}_{\mathbb{S},\mathbb{N}} \cap \mathbb{I}_{\mathbb{QL}}$.
(ii) Is the Weber implication $I_{\mathbf{WB}}$ the only QL-implication that is also an R-implication obtained from a non-left continuous t-norm? If not, give other examples from the above intersection and hence, characterize the non-empty intersection $\mathbb{I}_{\mathbb{QL}} \cap \mathbb{I}_{\mathbb{T}}$.
(iii) Prove or disprove by giving an example: $(\mathbb{I}_{\mathbb{QL}} \cap \mathbb{I}_{\mathbb{T}}) \setminus \mathbb{I}_{\mathbb{S},\mathbb{N}} = \emptyset$.
(iv) Is the intersection $\mathbb{I}_{\mathbb{F},\aleph} \cap \mathbb{I}_{\mathbb{QL}}$ non-empty? If yes, then characterize the intersection $\mathbb{I}_{\mathbb{F},\aleph} \cap \mathbb{I}_{\mathbb{QL}}$.

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