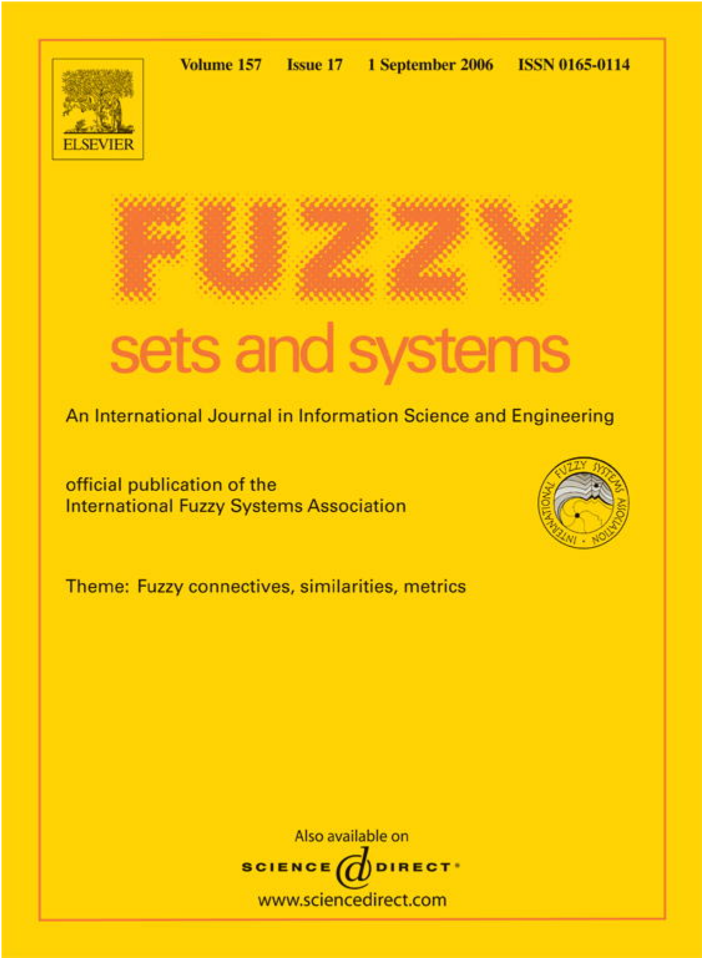
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Contrapositive symmetrisation of fuzzy implications—Revisited

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Abstract

Contrapositive symmetry (CPS) is a tautology in classical logic. In fuzzy logic, not all fuzzy implications have CPS with respect to a given strong negation. Given a fuzzy implication J, towards imparting contrapositive symmetry to J with respect to a strong negation N two techniques, viz., upper and lower contrapositivisation, have been proposed by Bandler and Kohout [Semantics of implication operators and fuzzy relational products, Internat. J. Man Machine Stud. 12 (1980) 89–116]. In this work we investigate the N-compatibility of these contrapositivisation techniques, i.e., conditions under which the natural negation of the contrapositivised implication is equal to the strong negation employed. This property is equivalent to the neutrality of the contrapositivised implication. We have shown that while upper contrapositivisation has N-compatibility when the natural negation of the fuzzy implication J (given by $N_J(x) = J(x, 0)$) is less than the strong negation N considered, the lower contrapositivisation is N-compatible in the other case. Also we have proposed a new contrapositivisation technique, viz., M-contrapositivisation, which is N-compatible independent of the ordering on the negations. Some interesting properties of M-contrapositivisation are also discussed. Since all S-implications have contrapositive symmetry, we investigate whether these contrapositivisations can be written as S-implications for suitable fuzzy disjunctions. Also some sufficient conditions for these fuzzy disjunctions to become t-conorms are given. In line with Fodor's [Contrapositive symmetry of fuzzy implications, Fuzzy Sets and Systems 69 (1995) 141–156] investigation, it is shown that the lower contrapositivisation of an R-implication J_T can also be seen as the residuation of a suitable binary operator.

Keywords: Residuated implications; S-implications; Contrapositive symmetry; Contrapositivisation; N-compatibility

1. Introduction

In the framework of classical two-valued logic, contrapositivity of a binary implication operator is a tautology, i.e., $\alpha \Rightarrow \beta \equiv \neg\beta \Rightarrow \neg\alpha$. In fuzzy logic, contrapositive symmetry of a fuzzy implication J with respect to strong negation N—CPS(N)—plays an important role in the applications of fuzzy implications, viz., approximate reasoning, deductive systems, decision support systems, formal methods of proof, etc. (see also [7,10]). Usually, the contrapositive symmetry of a fuzzy implication J is studied with respect to its natural negation, denoted by N_J and defined as $N_J(x) = J(x, 0)$ for all $x \in [0, 1]$, i.e., CPS(N_J). However, not all fuzzy implications have CPS(N_J), either because the natural negation N_J is not strong or N_J is strong but still J does not have CPS(N_J).

For example, consider the fuzzy implication $J_{GG}(x, y) = \min\{1, (1 - x)/(1 - y)\}$. The natural negation of J_{GG} is $N_{J_{GG}}(x) = 1 - x$ which is a strong negation but J_{GG} does not have CPS($1 - x$). Similarly the natural negation of the

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fuzzy implication $J(x, y) = [1 - x + x \cdot y^2]^{1/2}$ is $N_J(x) = J(x, 0) = [1 - x]^{1/2}$ which is not a strong negation and hence J does not have CPS(N_J).

Towards imparting contrapositive symmetry to such fuzzy implications J with respect to a strong negation N the following two contrapositivisation techniques—upper and lower contrapositivisation—have been proposed in [2]:

$$x \overset{U:N}{\Longrightarrow} y = \max\{J(x, y), J(N(y), N(x))\},$$

$$x \overset{L:N}{\Longrightarrow} y = \min\{J(x, y), J(N(y), N(x))\},$$

for any $x, y \in [0, 1]$. These techniques transform a fuzzy implication J that does not have CPS(N_J) into a contrapositivised implication $J^* \equiv \overset{*:N}{\Longrightarrow}$ that has CPS(N) with respect to a strong negation N. Of particular interest is when N is taken as N_J, provided N_J is strong.

In this work we investigate the conditions under which J^* has contrapositive symmetry with respect to its natural negation N_{J^*}, i.e., when will J^* have CPS(N_{J^*})? In other words, we investigate the conditions under which $N_{J^*} \equiv N$.

1.1. Motivation for this work

In [7] Fodor has discussed the contrapositive symmetry of fuzzy implications for the three main families, viz., S-, R- and QL-implications. Fodor has considered the upper contrapositivisation of the R-implication J_T, with T a strict t-norm, and cites the lack of neutrality of the lower contrapositivisation of J_T as one of the reasons for not considering it.

In this work we investigate the conditions under which both the upper and lower contrapositivised fuzzy implications have the neutrality property when applied to a general fuzzy implication, by showing that it is equivalent to investigating the conditions under which the natural negations obtained from these, i.e., $x \overset{U:N}{\Longrightarrow} 0$, $x \overset{L:N}{\Longrightarrow} 0$, are equal to the strong negation N employed therein. When a contrapositivisation $\overset{*:N}{\Longrightarrow}$ has the property that its natural negation is equal to the strong negation N employed therein, we say $\overset{*:N}{\Longrightarrow}$ is N-compatible.

1.2. Outline of the paper

After detailing the necessary preliminaries in Section 2, in Section 3 we investigate the conditions under which these two techniques, viz., $\overset{U:N}{\Longrightarrow}$, $\overset{L:N}{\Longrightarrow}$, are N-compatible.

We show that an ordering between N and the natural negation N_J of the original implication J considered is necessary for $\overset{U:N}{\Longrightarrow}$, $\overset{L:N}{\Longrightarrow}$ to be N-compatible. Also, in Section 4, we propose a new contrapositivisation technique $\overset{M:N}{\Longrightarrow}$ such that the transformed implication not only has contrapositive symmetry with respect to N, but also is N-compatible independent of any ordering between N and N_J.

It is well-known that S-implications have contrapositive symmetry with respect to the strong negation used in its definition. Since all the three contrapositivisation techniques, viz, $\overset{U:N}{\Longrightarrow}$, $\overset{L:N}{\Longrightarrow}$, $\overset{M:N}{\Longrightarrow}$, in general, can be applied to any fuzzy implication, we investigate, in Section 5, whether they can be written as S-implications for suitable fuzzy disjunctions $\circ, \diamond, \triangle$, respectively. Subsequently, we propose some sufficient conditions for these fuzzy disjunctions to become t-conorms.

In [7] Fodor has shown the upper contrapositivisation of an R-implication J_T obtained from a (strict) t-norm T as the residuation of an appropriate conjunction, $*_T$, and discussed conditions under which it becomes a t-norm. In Section 6, an analogous study is done in the case of lower contrapositivisation. Finally, some concluding remarks are given.

2. Preliminaries

To make this work self-contained, we briefly mention some of the concepts and results employed in the rest of the work.

2.1. Negations

Definition 1 (*Fodor and Roubens [8, Definition 1.1, p. 3]*). A negation N is a function from [0, 1] to [0, 1] such that

- $N(0) = 1$; $N(1) = 0$;
- N is non-increasing.

Definition 2. A negation N is said to be

- non-vanishing if $N(x) \neq 0$ for any $x \in [0, 1)$, i.e., $N(x) = 0$ iff $x = 1$;
- non-filling if $N(x) \neq 1$ for any $x \in (0, 1]$, i.e., $N(x) = 1$ iff $x = 0$.

A negation N that is not non-filling (non-vanishing) will be called filling (vanishing).

Definition 3 (*Fodor and Roubens [8, Definition 1.2, p. 3]*). A negation N is called strict if in addition N is strictly decreasing and continuous.

Note that if a negation N is strict it is both non-vanishing and non-filling, but the converse is not true as shown in Fig. 1, which gives the graphs of some continuous filling and vanishing negations.

Definition 4 (*Fodor and Roubens [8, Definition 1.2, p. 3]*). A strong negation N is a strict negation N that is also involutive, i.e., $N(N(x)) = x, \ \forall x \in [0, 1]$.

2.2. t-norms and t-conorms

Definition 5 (*Klement et al. [11, Definition 1.1, p. 4]*). A t-norm T is a function from $[0, 1]^2$ to [0, 1] such that for all $x, y, z \in [0, 1]$,

$$T(x, y) = T(y, x), \tag{T1}$$

$$T(x, T(y, z)) = T(T(x, y), z), \tag{T2}$$

$$T(x, y) \leqslant T(x, z) \quad \text{whenever } y \leqslant z, \tag{T3}$$

$$T(x, 1) = x. \tag{T4}$$

Definition 6 (*Klement et al. [11, Definition 1.13 p. 11]*). A t-conorm S is a function from $[0, 1]^2$ to [0, 1] such that for all $x, y, z \in [0, 1]$,

$$S(x, y) = S(y, x), \tag{S1}$$

$$S(x, S(y, z)) = S(S(x, y), z), \tag{S2}$$

$$S(x, y) \leqslant S(x, z) \quad \text{whenever } y \leqslant z, \tag{S3}$$

$$S(x, 0) = x. \tag{S4}$$

Definition 7 (*Klement et al. [11, Definition 2.9, p. 26, Definition 2.13, p. 28]*). A t-norm T is said to be

- continuous if it is continuous in both the arguments;
- Archimedean if for each $(x, y) \in (0, 1)^2$ there is an $n \in \mathbb{N}$ with $x_T^{(n)} < y$, where $x_T^{(n)} = T(\underbrace{x, \ldots, x}_{n \text{ times}})$;
- Strict if T is continuous and strictly monotone, i.e., $T(x, y) < T(x, z)$ whenever $x > 0$ and $y < z$;
- Nilpotent if T is continuous and if each $x \in (0, 1)$ is such that $x_T^{(n)} = 0$ for some $n \in \mathbb{N}$.

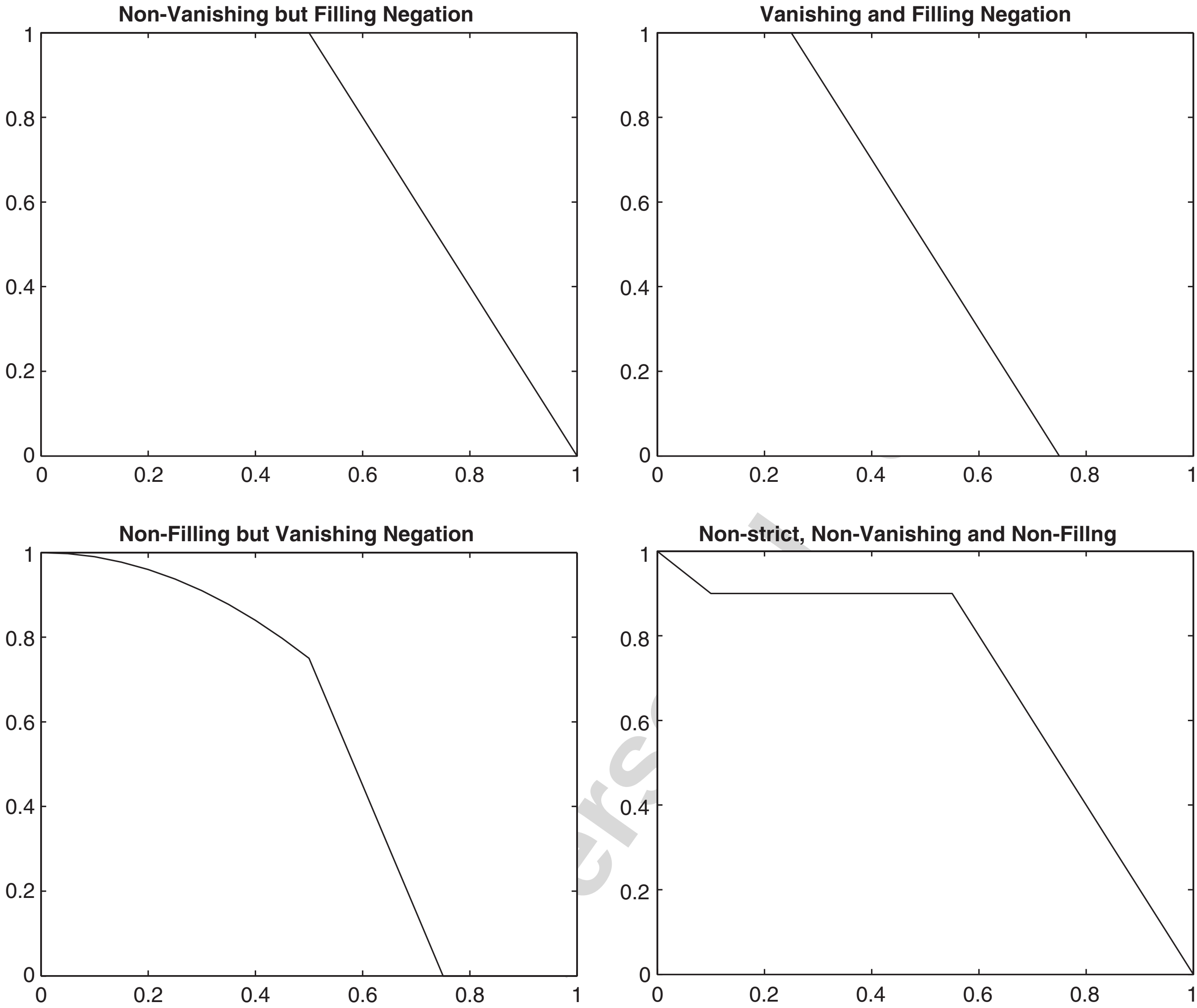

Fig. 1. Graphs of some continuous filling and vanishing negations.

2.3. Fuzzy implications

Definition 8 (*Fodor and Roubens [8, Definition 1.15, p. 22]*). A function $J : [0,1]^2 \to [0,1]$ is called a fuzzy implication if it has the following properties, for all $x, y, z \in [0,1]$,

$$J(x,z) \geqslant J(y,z) \quad \text{if } x \leqslant y, \tag{J1}$$

$$J(x,y) \geqslant J(x,z) \quad \text{if } y \geqslant z, \tag{J2}$$

$$J(0,z) = 1, \tag{J3}$$

$$J(x,1) = 1, \tag{J4}$$

$$J(1,0) = 0. \tag{J5}$$

Definition 9 (*cf. Trillas and Valverde [14]*). A fuzzy implication J is said to have

- contrapositive symmetry with respect to a strong negation N, CPS(N), if

$$J(x,y) = J(N(y),N(x)), \quad \forall x, y \in [0,1]; \tag{CP}$$

Table 1
Some fuzzy implications with the properties they satisfy

Name	Fuzzy implication J	Properties satisfied
Lukasiewicz	$J_{\mathrm{L}}(x, y) = \min(1, 1 - x + y)$	(OP), (NP), (EP), CPS$(1 - x)$
Goguen	$J_{\mathrm{G}}(x, y) = \begin{cases} 1 & \text{if } x \leqslant y \\ \frac{y}{x} & \text{if } x > y \end{cases}$	(OP), (NP) and (EP)
Kleene–Dienes	$J_{\mathrm{KD}}(x, y) = \max(1 - x, y)$	(NP), (EP), CPS$(1 - x)$
Reciprocal Goguen	$J_{\mathrm{GG}}(x, y) = \min\left\{1, \frac{1 - x}{1 - y}\right\}$	Only (OP)
Baczynski	$J_{\mathrm{B}}(x, y)$ from Example 1	(OP), CPS$(1 - x)$

- the ordering property if for all $x, y \in [0, 1]$,

$$x \leqslant y \quad \Leftrightarrow \quad J(x, y) = 1; \tag{OP}$$

- the neutrality property or is said to be neutral if

$$J(1, y) = y, \quad \forall y \in [0, 1]; \tag{NP}$$

- the exchange property if

$$J(x, J(y, z)) \equiv J(y, J(x, z)), \quad \forall x, y, z \in [0, 1]. \tag{EP}$$

Table 1 lists a few fuzzy implications along with the properties they satisfy among (CP)–(EP) (see [12]).

Definition 10. Let J be any fuzzy implication. By the natural negation of J, denoted by N_J, we mean $N_J(x) = J(x, 0)$, $\forall x \in [0, 1]$.

Clearly, $N_J(0) = 1$ and $N_J(1) = 0$. Also by the non-increasingness of J in its first place N_J is a non-increasing function.

Example 1. Let us consider the fuzzy implication J_{B} given in [1, Example 17, p. 274]:

$$J_{\mathrm{B}}(x, y) = \min\left\{\max\left[\frac{1}{2}, \min(1 - x + y, 1)\right], 2 - 2x + 2y\right\}. \tag{1}$$

The following are easy to see:

- $x \leqslant y \Leftrightarrow J_{\mathrm{B}}(x, y) = 1$ and thus J_{B} has the ordering property (OP).
- The natural negation of J_{B} is given by

$$J_{\mathrm{B}}(x, 0) = N_{J_{\mathrm{B}}}(x) = \begin{cases} 1 - x & \text{if } \frac{1}{2} \geqslant x \geqslant 0, \\ \frac{1}{2} & \text{if } \frac{3}{4} \geqslant x \geqslant \frac{1}{2}, \\ 2(1 - x) & \text{if } 1 \geqslant x \geqslant \frac{3}{4}. \end{cases}$$

- $N_{J_{\mathrm{B}}}$ though continuous is not strict (hence not strong) and thus J_{B} does not have CPS$(N_{J_{\mathrm{B}}})$ as per Definition 9. Note that $N_{J_{\mathrm{B}}}$ is both a non-filling and a non-vanishing negation.
- J_{B} does not have the neutrality property (NP). For example, $J_{\mathrm{B}}\left(1, \frac{1}{4}\right) = \frac{1}{2}$.

Lemma 1 (*cf. Bustince et al. [3, Lemma 1, p. 214]*). *Let J be a fuzzy implication and N a strong negation. Then*

(i) *if J has* CPS(N) *and is neutral* (NP) *then $N_J(x) = N(x)$;*
(ii) *if $N(x) = N_J(x)$ and J has the exchange property* (EP) *then J has* CPS(N) *and is neutral* (NP);
(iii) *if J has* CPS(N), *the exchange* (EP) *and ordering* (OP) *properties then $N_J(x) = N(x)$.*

Proof.

(i) Let J have CPS(N) and be Neutral. Then

$$\begin{aligned} N_J(x) &= J(x,0) \\ &= J(N(0), N(x)) \quad [\because J \text{ has CPS}(N)] \\ &= J(1, N(x)) = N(x) \quad [\because J \text{ is neutral}]. \end{aligned}$$

(ii) Let $N(x) = N_J(x)$ and J have the exchange property. Then

$$\begin{aligned} J(N(y), N(x)) &= J(N(y), J(x,0)) \quad [\text{by definition of } N_J] \\ &= J(x, J(N(y),0)) \quad [J \text{ has (EP)}] \\ &= J(x, N(N(y))) = J(x,y) \quad [N \text{ is strong}] \end{aligned}$$

and $J(1,y) = J(N(y),0) = N(N(y)) = y$.

(iii) By Lemma 1.3 in [8], if J satisfies (OP) and (EP) then J satisfies (NP). Now using (i) we have the result. □

Example 2. Consider the reciprocal Goguen implication J_{GG}. Then, as can be seen from Table 1, $N_{J_{GG}}(x) = J_{GG}(x,0) = 1-x$, a strong negation, but J_{GG} does not have either CPS($1-x$) or (NP). Thus the reverse implication of (i) in Lemma 1 does not always hold.

Example 3. To see that the reverse implication of (ii) in Lemma 1 does not always hold, consider the upper contrapositivisation J_G^* of Goguen's implication J_G(see [7]):

$$J_G(x,y) = \min\left\{1, \frac{y}{x}\right\}, \quad J_G^*(x,y) = \min\left\{1, \max\left[\frac{y}{x}, \frac{1-x}{1-y}\right]\right\}.$$

J_G^* has CPS($1-x$), (NP) (see Proposition 17 below) and $N_{J_{G^*}}(x) = J_G^*(x,0) = 1-x$, but not (EP).

Example 4. Again from Example 2 and Table 1, we see that $N_{J_{GG}}(x) = J_{GG}(x,0) = 1-x$, a strong negation, but J_{GG} does not have either (OP) or (EP). Thus the reverse implication of (iii) in Lemma 1 is not always true.

From parts (i) and (ii) of Lemma 1 we have the following corollary:

Corollary 11. *Let J be a fuzzy implication that has* CPS(N), *where N is a strong negation. J is neutral if and only if $N_J \equiv N$.*

Lemma 2. *Let J be a fuzzy implication and N a strong negation. Let us define a binary operation S_J on* $[0,1]$ *as follows*:

$$S_J(x,y) = J(N(x), y), \quad \text{for all } x, y \in [0,1]. \tag{2}$$

Then for all $x, y \in [0,1]$, we have

(i) $S_J(x,1) = S_J(1,x) = 1$;
(ii) *S_J is non-decreasing in both the variables.*

In addition, if J has CPS(N), *then*

(iii) *S_J is commutative*;
(iv) *$S_J(x,0) = S_J(0,x) = x$ if and only if J is neutral* (NP);
(v) *S_J is associative if and only if J has the exchange property* (EP).

Proof.

(i) $S_J(x,1) = J(N(x),1) = 1$, by the boundary condition on J. Also, $S_J(1,x) = J(N(1),x) = J(0,x) = 1$ again by the boundary condition of J.

(ii) That S_J is non-decreasing in both the variables is a direct consequence of the non-increasingness (non-decreasingness) of J in the first (second) variable.

Let J have CPS(N).

(iii) Then $S_J(x, y) = J(N(x), y) = J(N(y), N(N(x))) = J(N(y), x) = S_J(y, x)$.

(iv) If J is neutral, then $S_J(x, 0) = S_J(0, x) = J(N(0), x) = J(1, x) = x$. On the other hand, $J(1, x) = S_J(N(1), x) = S_J(0, x) = x$, for all $x \in [0, 1]$.

(v) Let J have the exchange principle. Then

$$\begin{aligned} S_J(x, S_J(y, z)) &= J(N(x), J(N(y), z)) \\ &= J(N(x), J(N(z), y)) \quad [\because J \text{ has CPS}(N)\ ] \\ &= J(N(z), J(N(x), y)) \quad [\because J \text{ has exchange principle}\ ] \\ &= J(N[J(N(x), y)], z) \\ &= J(N[S_J(x, y)], z) \\ &= S_J(S_J(x, y), z). \end{aligned}$$

On the other hand, if S_J is associative, then

$$\begin{aligned} J(x, J(y, z)) &= S_J(N(x), S_J(N(y), z)) \quad [\text{by definition}\ ] \\ &= S_J(S_J(N(x), N(y)), z)) \quad [\because S_J \text{ is associative}\ ] \\ &= S_J(S_J(N(y), N(x)), z)) \quad [\because S_J \text{ is commutative}\ ] \\ &= S_J(N(y), S_J(N(x), z)) \\ &= J(y, J(x, z)). \qquad \square \end{aligned}$$

The following are the two important classes of fuzzy implications well-established in the literature:

Definition 12 (*[8, Definition 1.16, p. 24]*). An S-implication $J_{S,N}$ is obtained from a t-conorm S and a strong negation N as follows:

$$J_{S,N}(a, b) = S(N(a), b), \quad \forall a, b \in [0, 1]. \tag{3}$$

Definition 13 (*Fodor and Roubens [8, Definition 1.16, p. 24]*). An R-implication J_T is obtained from a t-norm T as its residuation as follows:

$$J_T(a, b) = \mathrm{Sup}\{x \in [0, 1] : T(a, x) \leqslant b\}, \quad \forall a, b \in [0, 1]. \tag{4}$$

Remark 14.

- Though an S-implication can be defined for any negation N, not necessarily strong, herein we employ the above restricted definition from [8]. Likewise, though an R-implication can be defined for any t-norm T it has some important properties, like the residuation principle, only if T is a left-continuous t-norm. J_T is also called the residuum of T.
- All S-implications $J_{S,N}$ possess CPS(N_J), with respect to their natural negation N_J which is also the strong negation N used in its definition (see Corollary 11 above), (NP) and (EP).
- All R-implications J_T possess properties (NP) and (EP)(see [14]).
- If the R-implication J_T is obtained from a nilpotent t-norm T, its natural negation $N_{J_T}(x) = J_T(x, 0)$ is a strong negation (see [7, Corollary 2, p. 145]). Also, from [10, Corollary 2, p. 162], and by the characterisation of nilpotent t-norms [11, Corollary 5.7, p. 125–126], we have that an R-implication J_T obtained from a nilpotent t-norm T has CPS($J_T(x, 0)$).
- On the other hand, if the R-implication J_T is obtained from a strict t-norm T then, by definition, $N_{J_T}(x) = 0, \forall x \in (0, 1]$ and $N_{J_T}(x) = 1$ if $x = 0$, which is neither strict nor continuous and hence is not strong. Thus, J_T does not have contrapositive symmetry with respect to its natural negation, in the sense of Definition 9.

The following theorems characterise S- and R-implications:

Theorem 1 (*Fodor and Roubens [8, Theorem 1.13, p. 24]*). *A fuzzy implication J is an S-implication for an appropriate t-conorm S and a strong negation N if and only if J has* CPS(N), *the exchange property* (EP) *and is neutral* (NP).

Theorem 2 (*Fodor and Roubens [8, Theorem 1.14, p. 25]*). *A function J from* $[0, 1]^2$ *to* $[0, 1]$ *is an R-implication based on a left-continuous t-norm T if and only if J is non-decreasing in the second variable* (J2), *has the ordering property* (OP), *exchange property* (EP) *and $J(x, .)$ is right-continuous for any* $x \in [0, 1]$.

3. Upper and lower contrapositivisations

Bandler and Kohout, in [2], have proposed two techniques, viz., upper and lower contrapositivisation, towards imparting contrapositive symmetry to a fuzzy implication J with respect to a strong negation N, whose definitions we give below.

Definition 15. Let J be any fuzzy implication and N a strong negation. The upper and lower contrapositivisations of J with respect to N, denoted herein as $\overset{U:N}{\Longrightarrow}$ and $\overset{L:N}{\Longrightarrow}$, respectively, are defined as follows:

$$x \overset{U:N}{\Longrightarrow} y = \max\{J(x, y), J(N(y), N(x))\}, \tag{5}$$

$$x \overset{L:N}{\Longrightarrow} y = \min\{J(x, y), J(N(y), N(x))\} \tag{6}$$

for any $x, y \in [0, 1]$.

As can be seen, $\overset{U:N}{\Longrightarrow}$ and $\overset{L:N}{\Longrightarrow}$ are both fuzzy implications, as per Definition 8, and always have the contrapositive symmetry with respect to the strong negation N employed in their definitions.

In [7], Fodor has dealt with the contrapositive symmetry of residuated implications J_T obtained from strict t-norms T. When N is a strong negation, Fodor in [7] discusses the upper contrapositivisation of J_T, denoted by $\rightarrow_T$, as given in (5) and cites the lack of neutrality of the lower contrapositivisation of J_T as one of the reasons for not considering it. We show in Example 5 below that upper contrapositivisation also suffers from the same malady. Also from Corollary 11 we have that the natural negation of $\overset{U:N}{\Longrightarrow}$ is not equal to the strong N employed.

Example 5. Consider the fuzzy implication $J(x, y) = [1 - x + x \cdot y^2]^{1/2}$ with natural negation $N_J(x) = J(x, 0) = [1 - x]^{1/2}$ which is not a strong negation. To see this let $x = 0.75$. Then $N_J(0.75) = \sqrt{1 - x} = \sqrt{0.25} = 0.5$, $N_J(N_J(0.75)) = \sqrt{N_J(0.75} = \sqrt{0.5} = 0.707 \neq 0.75 = x$. Thus J does not have CPS($\sqrt{1 - x}$). Note also that J is neutral, i.e., $J(1, y) = [y^2]^{1/2} = y$, for all $y \in [0, 1]$.

Let us consider the strong negation $N(x) = 1 - x$. The upper contrapositivisation of J with respect to N is given by

$$J^*(x, y) = x \overset{U:N}{\Longrightarrow} y = \max\{J(x, y), J(N(y), N(x))\} = J^*(N(y), N(x)).$$

Let $y = 0.5$, then $J^*(1, 0.5) = \max\{J(1, 0.5), J(0.5, 0)\} = \max\{0.5, N_J(0.5)\} = \sqrt{1 - 0.5} = 0.707 \neq 0.5 = y$. Thus the upper contrapositivised J is not neutral and hence from Corollary 11 its natural negation too is not equal to $1 - x$.

Definition 16. Let J be a fuzzy implication and N a strong negation. A contrapositivisation technique $\overset{*:N}{\Longrightarrow}$ is said to be *N-compatible* if the contrapositivisation of J with respect to N, denoted as $J^*(x, y) \equiv x \overset{*:N}{\Longrightarrow} y$, is such that the natural negation of J^*, $J^*(., 0) = N_{J^*}(.) = N(.)$, the strong negation employed. Or, equivalently J^* is neutral.

In Sections 3.1 and 3.2, we investigate the conditions under which $\overset{U:N}{\Longrightarrow}$, $\overset{L:N}{\Longrightarrow}$ applied to any general fuzzy implication are N-compatible, and show that an ordering on N and the natural negation N_J of the original implication J considered is both necessary and sufficient for $\overset{U:N}{\Longrightarrow}$, $\overset{L:N}{\Longrightarrow}$ to be N-compatible.

3.1. The upper contrapositivisation and N-compatibility

Proposition 17. *Let J be a neutral fuzzy implication with natural negation $J(x, 0) = N_J(x)$ and N a strong negation. The upper contrapositivisation of J with respect to N is N-compatible if and only if $N(x) \geqslant N_J(x)$, for all $x \in [0, 1]$.*

Proof. Let $x \in [0, 1]$. By definition of $\overset{U:N}{\Longrightarrow}$ we have

$$
\begin{aligned}
\overset{U:N}{\Longrightarrow} \text{ is N-compatible} \quad & \text{iff } N(x) = x \overset{U:N}{\Longrightarrow} 0,\\
& \text{iff } N(x) = \max\{J(x, 0), J(1, N(x))\},\\
& \text{iff } N(x) = \max(N_J(x), N(x)),\\
& \text{iff } N(x) \geqslant N_J(x), \text{ for all } x \in [0, 1]. \qquad \square
\end{aligned}
$$

Proposition 18. *If J is a neutral fuzzy implication and N is a strong negation such that the natural negation $J(x, 0) = N_J(x) \geqslant N(x)$, $\forall x \in [0, 1]$, then the upper contrapositivisation of J with respect to N is such that*

(i) $x \overset{U:N}{\Longrightarrow} 0 = N_J(x), \ \forall x \in [0, 1]$;
(ii) $1 \overset{U:N}{\Longrightarrow} y = N_J(N(y)), \ \forall y \in [0, 1]$.

3.2. The lower contrapositivisation and N-compatibility

We present below the counterparts of Propositions 17 and 18 for lower contrapositivisation.

Proposition 19. *Let J be a neutral fuzzy implication with natural negation $J(x, 0) = N_J(x)$ and N a strong negation. The lower contrapositivisation of J with respect to N is N-compatible if and only if $N(x) \leqslant N_J(x)$, for all $x \in [0, 1]$.*

Proof. Let $x \in [0, 1]$. By definition of $\overset{L:N}{\Longrightarrow}$ we have

$$
\begin{aligned}
\overset{L:N}{\Longrightarrow} \text{ is N-compatible} \quad & \text{iff } N(x) = x \overset{L:N}{\Longrightarrow} 0,\\
& \text{iff } N(x) = \min\{J(x, 0), J(1, N(x))\},\\
& \text{iff } N(x) = \min(N_J(x), N(x)),\\
& \text{iff } N(x) \leqslant N_J(x), \text{ for all } x \in [0, 1]. \qquad \square
\end{aligned}
$$

Proposition 20. *If J is a neutral fuzzy implication and N a strong negation such that the natural negation $J(x, 0) = N_J(x) \leqslant N(x)$, $\forall x \in [0, 1]$, then the lower contrapositivisation of J with respect to N is such that*

(i) $x \overset{L:N}{\Longrightarrow} 0 = N_J(x), \ \forall x \in [0, 1]$;
(ii) $1 \overset{L:N}{\Longrightarrow} y = N_J(N(y)), \ \forall y \in [0, 1]$.

Corollary 21. *Let J be a neutral fuzzy implication such that N_J is strong. Then both the upper and lower contrapositivisation of J with respect to N_J, $\overset{U:N_J}{\Longrightarrow}$, $\overset{L:N_J}{\Longrightarrow}$, are N_J-compatible. If J has* CPS(N_J), *then $J(x, y) = x \overset{U:N_J}{\Longrightarrow} y = x \overset{L:N_J}{\Longrightarrow} y$, for all $x, y \in [0, 1]$.*

3.3. Some new classes of fuzzy implications and contrapositivisation

Let the upper contrapositivisation of J with respect to a strong N be N-compatible. Then from Proposition 17 we know $N \geqslant N_J$. Since N is strong $N(x) = 1 \Leftrightarrow x = 0$ and we have that for all $x \in (0, 1]$, $1 > N(x) \geqslant N_J(x)$ and N_J is a non-filling negation. In other words, if the natural negation of the fuzzy implication J is a *filling* negation we cannot find any strong N with which the upper contrapositivisation of J becomes N-compatible. Similarly, if the natural negation of the fuzzy implication J is a *vanishing* negation we cannot find any strong N with which the lower contrapositivisation of J becomes N-compatible.

As noted in Remark 14, when T is strict, the natural negation, $N_{J_T}(x)$, of an R-implication J_T obtained from T is such that $N_{J_T}(x) = 0$ for all x in $(0, 1]$ and $N_{J_T}(0) = 1$ and hence N_{J_T} is a non-filling but a *vanishing* negation. Since any strong negation $N \geqslant N_{J_T}$ only the upper contrapositivisation is N-compatible, as cited by Fodor [7].

In Section 3.3.1, we present the recently proposed classes of fuzzy implications by Yager in [15] and another new class of fuzzy implications inspired by it and discuss their contrapositive symmetry. These classes of implications, denoted herein as J_f and J_h, do not fall under the established categorisations of R-, S- or QL-implications and thus are of interest in their own right. An application of the contrapositivisation techniques discussed above with respect to J_f and J_h highlights the general nature of these techniques and also motivates an alternate contrapositivisation technique as presented in Section 4.

3.3.1. J_f and J_h fuzzy implications and contrapositivisation

Definition 22 (*Yager [15, p. 196]*). An f-generator is a function $f : [0, 1] \to [0, \infty]$ that is a strictly decreasing and continuous function with $f(1) = 0$. Also we denote its pseudo-inverse by $f^{(-1)}$, given by

$$f^{(-1)}(x) = \begin{cases} f^{-1}(x) & \text{if } x \in [0, f(0)], \\ 0 & \text{if } x \in [f(0), \infty]. \end{cases} \tag{7}$$

Definition 23 (*Yager [15, p. 197]*). A function from $[0, 1]^2$ to $[0, 1]$ defined by an f-generator as $J_f(a, b) = f^{(-1)}(a \cdot f(b))$, with the understanding that $0 \times \infty = 0$, is called an f-generated implication.

It can easily be shown, as in [15, p. 197], that J_f is a fuzzy implication.

Table 2 gives a few examples from the above class J_f (see [15, p. 198–200]).

Definition 24. An h-generator is a function $h : [0, 1] \to [0, 1]$, that is strictly decreasing and continuous, such that $h(0) = 1$. Let $h^{(-1)}$ be its pseudo-inverse given by

$$h^{(-1)}(x) = \begin{cases} h^{-1}(x) & \text{if } x \in [h(1), 1], \\ 1 & \text{if } x \in [0, h(1)]. \end{cases} \tag{8}$$

Lemma 3. *Let J_h from $[0, 1]^2$ to $[0, 1]$ be defined as*

$$J_h(x, y) =_{\text{def}} h^{(-1)}(x \cdot h(y)), \quad \forall x, y \in [0, 1]. \tag{9}$$

J_h is a fuzzy implication and called the h-generated implication.

Proof. That J_h is a fuzzy implication can be seen from the following:

- $J_h(1, 0) = h^{(-1)}(1 \cdot h(0)) = h^{(-1)}(1 \cdot 1) = 0$.
- $J_h(0, 1) = h^{(-1)}(0 \cdot h(1)) = h^{(-1)}(0) = 1 = J_h(0, 0)$.
- $J_h(1, 1) = h^{(-1)}(1 \cdot h(1)) = h^{(-1)}(h(1)) = 1$, since $h^{(-1)} \circ h$ is the identity on the range of h.
- $a \leqslant a' \Rightarrow a \cdot h(b) \leqslant a' \cdot h(b) \Rightarrow h^{(-1)}(a \cdot h(b)) \geqslant h^{(-1)}(a' \cdot h(b)) \Rightarrow J_h(a, b) \geqslant J_h(a', b)$. Thus J_h is non-increasing in the first variable.
- $b \leqslant b' \Rightarrow a \cdot h(b) \geqslant a \cdot h(b') \Rightarrow h^{(-1)}(a \cdot h(b)) \leqslant h^{(-1)}(a \cdot h(b')) \Rightarrow J_h(a, b) \leqslant J_h(a, b')$. Thus J_h is non-decreasing in the second variable.
- Since $0 \leqslant a \cdot h(1) \leqslant h(1)$, $\forall a \in [0, 1]$, we have $J_h(a, 1) = h^{(-1)}(a \cdot h(1)) = 1$, by definition of $h^{(-1)}$.
- $J_h(0, b) = h^{(-1)}(0 \cdot h(b)) = h^{(-1)}(0) = 1$, $\forall b \in [0, 1]$. $\square$

Remark 25. If f is an f-generator such that $f(0) = \infty$, then $h(x) = \exp\{-f(1 - x)\}$ is a decreasing bijection on the unit interval $[0, 1]$ and thus can act as an h-generator. Table 3 gives a few examples from the above class J_h.

Table 2
Examples of some J_f implications with their f-generators

Name	$f(x)$	$f(0)$	$J_f(a,b)$
Yager	$-\log x$	∞	b^a
Frank	$-\ln\left\{\frac{s^x-1}{s-1}\right\};\ s>0, s\neq 1$	∞	$\log_s\{1+(s-1)^{1-a}(s^b-1)^a\}$
Trigonometric	$\cos\left(\frac{\pi}{2}x\right)$	1	$\cos^{-1}\left[a\cos\left(\frac{\pi}{2}b\right)\right]$
Yager's class	$(1-x)^\lambda;\ \lambda>0$	1	$1-a^{1/\lambda}(1-b)$

Table 3
Examples of some J_h implications with their h-generators

Name	$h(x)$	$h(1)$	$J_h(a,b)$
Schweizer–Sklar	$1-x^p;\ p\neq 0$	0	$[1-a+ab^p]^{1/p}$
Yager's	$(1-x)^\lambda;\ \lambda>0$	0	$1-a^{1/\lambda}(1-b)$
–	$1-\frac{x^n}{n};\ n\geqslant 1$	$1-\frac{1}{n}$	$\min\{[n-na+ab^n]^{1/n},1\}$

3.3.2. J_f and J_h implications, CPS(N_J) *and contrapositivisation*

In the following, we consider the natural negations of the J_f and J_h implications and discuss their contrapositive symmetry with respect to their natural negations.

3.3.2.1. J_f and its natural negation The natural negation of J_f, given by $N_J(x) = J_f(x,0) = f^{(-1)}(x\cdot f(0))$, is a negation by the decreasing nature of f. To discuss the properties of N_J, we consider the following two cases:

Case I: $f(0) < \infty$.

If $f(0)<\infty$ then $N_J(x) = J_f(x,0) = f^{(-1)}(x\cdot f(0))$, $\forall x\in(0,1)$. Since f and thus $f^{(-1)}(\equiv f^{-1})$ are strictly decreasing continuous functions, we have that N_J is a strict negation. N_J is not strong always (see Example 6 below).

Example 6. Consider the f-generated implication $J_{f_Y}(x,y) = 1-x^{1/\lambda}(1-y)$ obtained from Yager's class of f-generators $f(x)=(1-x)^\lambda$ with $f(0)=1<\infty$ (see Table 2). Now, if $\lambda=0.5$, i.e. $1/\lambda=2$, then $N_{J_{f_Y}}(x) = J_{f_Y}(x,0) = 1-x^2$ is a strict negation. That it is not strong can be seen by letting $x=0.5$ in which case $N_{J_{f_Y}}(N_{J_{f_Y}}(x)) = 1-[1-x^2]^2 = 1-(1-0.25)^2 = 0.4375 \neq 0.5 = x$. On the other hand, if $\lambda=1$, then $N_{J_{f_Y}}(x) = J_{f_Y}(x,0) = 1-x$, which is a strong negation.

Case II: $f(0)=\infty$.

In the case when $f(0)=\infty$ it is easy to see that N_J is not even strict, since $\forall x\in(0,1]$, we have $J_f(x,0) = N_{J_f}(x) = f^{-1}(x\cdot f(0)) = f^{-1}(x\cdot\infty) = f^{-1}(\infty) = 0$, i.e.,

$$N_{J_f}(x) = \begin{cases} 0 & \text{if } x\in(0,1], \\ 1 & \text{if } x=0. \end{cases}$$

Quite obviously, it is not strong either.

Thus, as per Definition 9, J_f does not have contrapositive symmetry with respect to its natural negation.

Now, in the case $f(0)<\infty$, we have that the natural negation of J_f is at least strict and so both upper and lower contrapositivisation techniques are N-compatible, with respect to strong negations N, depending on whether $N\geqslant N_{J_f}$ or $N\leqslant N_{J_f}$, respectively. On the other hand, when $f(0)=\infty$, N_{J_f} is a non-filling but a *vanishing* negation and thus only the upper contrapositivisation technique is N-compatible with respect to strong negations $N\geqslant N_{J_f}$.

Table 4
Examples of classes of implications J whose natural negations N_J are either non-vanishing (NV) or non-filling (NF) or both

$J_f/J_h : J(a,b)$	$f(0)/h(1)$	N_J	Type
$\log_s\{1+(s-1)^{1-a}(s^b-1)^a\}$	$f(0)=\infty$	$\begin{cases} 0 & \text{if } x \in (0,1] \\ 1 & \text{if } x = 0 \end{cases}$	Only NF
$1-a^{1/\lambda}(1-b)$	$f(0)=1$	$1-a^{1/\lambda}$	NF and NV
$[1-a+ab^p]^{1/p}$	$h(1)=0$	$1-a^{1/p}$	NF and NV
$\min\{[n-na+ab^n]^{1/n},1\}$	$h(1)=1-\frac{1}{n}$	$\min\{[n-na]^{1n},1\}$	Only NV

3.3.2.2. J_h and its natural negation The natural negation of J_h, $N_{J_h}(x) = J_h(x,0) = h^{(-1)}(x \cdot h(0)) = h^{(-1)}(x)$, for all $x \in [0,1]$ is, in general, only a negation. But,

- N_{J_h} is a strict negation if $h(1) = 0$;
- N_{J_h} is a strong negation iff $h = h^{-1}$, in which case $N_{J_h} = h^{(-1)} = h$.

Let $h \not\equiv h^{-1}$. Then, if $h(1) = 0$ we have that the natural negation N_{J_h} is strict and so both upper and lower contrapositivisation techniques are N-compatible, with respect to any strong negation N, depending on whether $N \geqslant N_{J_h}$ or $N \leqslant N_{J_h}$, respectively. On the other hand, if $h(1) > 0$ then N_{J_h} is a non-vanishing but a *filling* negation and only the lower contrapositivisation technique is N-compatible with respect to strong negations $N \leqslant N_{J_h}$.

Table 4 gives a few example classes of J_f and J_h implications whose natural negations are non-vanishing and/or non-filling.

Thus whenever the natural negation N_J of an implication is either a non-filling or a non-vanishing negation (or both) one of the above contrapositivisation techniques is N-compatible with respect to some strong negations N.

Now, let us consider a fuzzy implication J whose natural negation N_J is neither non-filling nor non-vanishing as given below.

Consider the negation

$$N^*(x) = \begin{cases} 1 & \text{if } x \in [0,\alpha], \\ f(x) & \text{if } x \in [\alpha,\beta], \\ 0 & \text{if } x \in [\beta,1], \end{cases}$$

where $f(x)$ is any non-increasing function (possibly discontinuous), $\alpha, \beta \in (0,1)$.

For a t-conorm S, let us define $J^*(x,y) = S(N^*(x),y)$ for $x, y \in [0,1]$. J^* can easily be seen to be a fuzzy implication. Also the natural negation of J^* is N^*. Since N^* is neither non-filling nor non-vanishing, one cannot find any strong negation N such that either $N \leqslant N^*$ or $N \geqslant N^*$ and thus both the upper and lower contrapositivisation techniques are not N-compatible. This motivates the search for a contrapositivisation technique that is independent of the ordering between N and N_J and be N-compatible. The following section explores this idea.

4. An alternate contrapositivisation technique

As can be seen from Propositions 17–20, an ordering between the natural negation $N_J = J(x,0)$ and the strong negation N is essential for the resulting contrapositivised implication to be N-compatible, when we employ either $\overset{L:N}{\Longrightarrow}$ or $\overset{U:N}{\Longrightarrow}$. In this section, we introduce a new contrapositivisation technique, denoted by $\overset{M:N}{\Longrightarrow}$, whose N-compatibility is independent of the ordering between N and N_J. For notational simplicity, we denote $\overset{M:N}{\Longrightarrow}$ by $\overset{M}{\Longrightarrow}$ in this section.

4.1. The M-contrapositivisation

Definition 26. Let J be any fuzzy implication and N a *strict* negation. The M-contrapositivisation of J with respect to N, denoted herein as $\overset{M}{\Longrightarrow}$, is defined as follows:

$$x \overset{M}{\Longrightarrow} y = \min\{J(x,y) \vee N(x),\ J(N(y),N(x)) \vee y\} \tag{10}$$

for any $x, y \in [0,1]$.

Proposition 27. *Let J be any fuzzy implication, N a* strict *negation and* $\stackrel{M}{\Longrightarrow}$ *the M-contrapositivisation of J with respect to N. Then*

(i) $\stackrel{M}{\Longrightarrow}$ *is a fuzzy implication.*
(ii) *If in addition, N is involutive, i.e., N is strong, then* $\stackrel{M}{\Longrightarrow}$ *has* CPS(N).

Proof.
(i) Since J is a fuzzy implication and N a strict negation we have the following:

- $1 \stackrel{M}{\Longrightarrow} 0 = \min\{J(1,0) \vee N(1), J(N(0), N(1)) \vee 0\} = \min\{J(1,0), J(1,0)\} = J(1,0) = 0$.
- $0 \stackrel{M}{\Longrightarrow} 1 = \min\{J(0,1) \vee N(0), J(N(1), N(0)) \vee 1\} = \min\{1,1\} = 1$. Similarly, $0 \stackrel{M}{\Longrightarrow} 0 = 1 \stackrel{M}{\Longrightarrow} 1 = 1$.
- $x \stackrel{M}{\Longrightarrow} 1 = \min\{J(x,1) \vee N(x), J(N(1), N(x)) \vee 1\} = \min\{1,1\} = 1$.
- $0 \stackrel{M}{\Longrightarrow} y = \min\{J(0,y) \vee N(0), J(N(y), N(0)) \vee y\} = \min\{1,1\} = 1$.
- Let $x \leqslant z$. Then $J(x,y) \geqslant J(z,y)$ and since $N(x) \geqslant N(z)$ we have $J[N(y), N(x)] \geqslant J[N(y), N(z)]$, from which we obtain that

$$\begin{aligned} x \stackrel{M}{\Longrightarrow} y &= \min\{J(x,y) \vee N(x), J(N(y), N(x)) \vee y\} \\ &\geqslant \min\{J(z,y) \vee N(z), J(N(y), N(z)) \vee y\} \\ &= z \stackrel{M}{\Longrightarrow} y. \end{aligned}$$

- Similarly, it can be shown that $x \stackrel{M}{\Longrightarrow} y \leqslant x \stackrel{M}{\Longrightarrow} z$ whenever $y \leqslant z$. Thus $\stackrel{M}{\Longrightarrow}$ is a fuzzy implication.

(ii) Let N be strong. That $\stackrel{M}{\Longrightarrow}$ has CPS(N) can be seen from the following equalities:

$$\begin{aligned} N(y) \stackrel{M}{\Longrightarrow} N(x) &= \min\{J(N(y), N(x)) \vee N(N(y)), J(x,y) \vee N(x)\} \\ &= \min\{J(N(y), N(x)) \vee y, J(x,y) \vee N(x)\} \\ &= \min\{J(x,y) \vee N(x), J(N(y), N(x)) \vee y\} \\ &= x \stackrel{M}{\Longrightarrow} y. \qquad \square \end{aligned}$$

Fig. 2 gives the plot of (a) the Baczynski Implication J_{B} and (b) the plots of the different contrapositivisations applied on J_{B}.

Proposition 28. *If J is a neutral fuzzy implication and N a* strict *negation then the M-contrapositivisation of J with respect to N is such that*

(i) $x \stackrel{M}{\Longrightarrow} 0 = N(x), \ \forall x \in [0,1]$;
(ii) $1 \stackrel{M}{\Longrightarrow} y = y, \ \forall y \in [0,1]$.

Proof. (i) By definition of $\stackrel{M}{\Longrightarrow}$, for all $x \in [0,1]$, we have

$$\begin{aligned} x \stackrel{M}{\Longrightarrow} 0 &= \min\{J(x,0) \vee N(x), J(1, N(x)) \vee 0\} \quad [\text{ by definition }] \\ &= \min\{N_J(x) \vee N(x), N(x)\} \quad [\because \text{J is neutral}] \\ &= \min\{N_J(x) \vee N(x), N(x)\} \\ &= [N_J(x) \vee N(x)] \wedge N(x) = N(x). \end{aligned}$$

(ii) Again by definition, for any $y \in [0,1]$, we have

$$\begin{aligned} 1 \stackrel{M}{\Longrightarrow} y &= \min\{J(1,y) \vee N(1), J(N(y), 0) \vee y\} \\ &= \min\{y \vee 0, N_J(N(y)) \vee y\} \\ &= y \wedge [N_J(N(y)) \vee y] = y. \qquad \square \end{aligned}$$

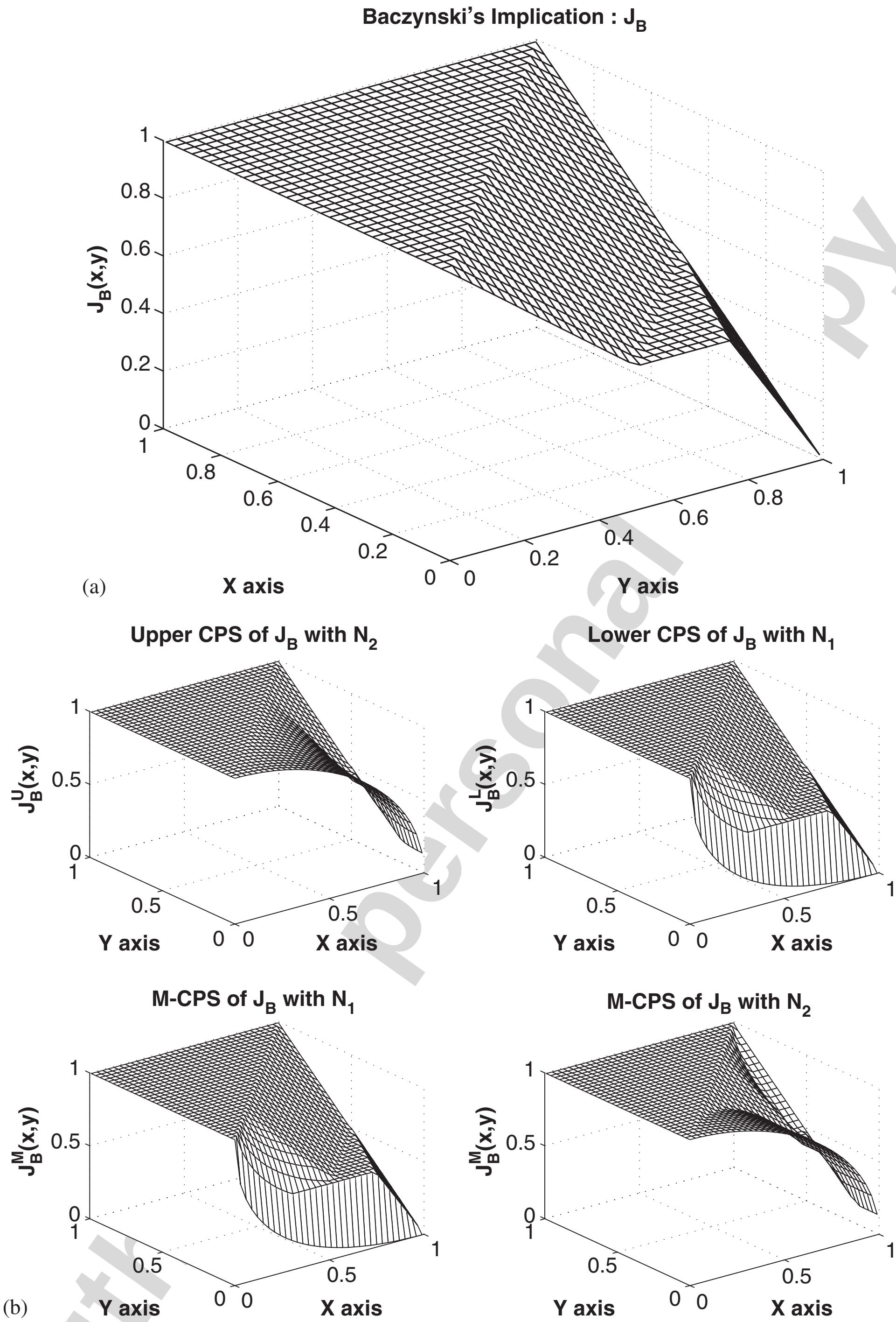

Fig. 2. (a) The Baczynski implication J_B (see Example 1) with (b) its upper, lower and M-contrapositivisations where $N_1(x) = (1 - \sqrt{x})^2$ and $N_2(x) = \sqrt{1 - x^2}$.

Corollary 29. *If J is a neutral fuzzy implication and N a strong negation then the M-contrapositivisation of J with respect to N is N-compatible.*

Proposition 30. *If J has the ordering property* (OP) *then so does the M-contrapositivisation of J with respect to a* strict *negation N.*

Proof. If $x \leqslant y$ then by (OP) $J(x, y) = 1$. Also $N(y) \leqslant N(x)$ and $J(N(y), N(x)) = 1$. Now,

$$\begin{aligned} x \stackrel{M}{\Longrightarrow} y &= \min\{J(x, y) \vee N(x), J(N(y), N(x)) \vee y\} \quad [\text{ by definition }] \\ &= \min\{1 \vee N(x), 1 \vee y\} \quad [\because \text{J has (OP)}] \\ &= \min\{1, 1\} = 1. \end{aligned}$$

On the other hand, if $x \stackrel{M}{\Longrightarrow} y = 1$ then

$$\begin{aligned} x \stackrel{M}{\Longrightarrow} y = 1 &\Rightarrow \min\{J(x, y) \vee N(x), J(N(y), N(x)) \vee y\} = 1 \\ &\Rightarrow J(x, y) \vee N(x) = 1 \quad \text{and} \quad J(N(y), N(x)) \vee y = 1. \end{aligned}$$

Now, if $J(x, y) \vee N(x) = 1$ then either $J(x, y) = 1$ or $N(x) = 1$. Since J satisfies (OP), $J(x, y) = 1 \Leftrightarrow x \leqslant y$. Also $J(x, y) = 1$ implies that $J(N(y), N(x)) = 1$. On the other hand, if $N(x) = 1$, since N is strict we have $x = 0$ in which case $x \leqslant y$ for all $y \in [0, 1]$. Similarly, if $y = 1$ then obviously $x \leqslant y$ for all $x \in [0, 1]$. Thus $x \stackrel{M}{\Longrightarrow} y = 1$ implies $x \leqslant y$, i.e, $\stackrel{M}{\Longrightarrow}$ has the ordering property (OP) if J does. □

Remark 31. (i) If $J(x, y) \geqslant y$ then it can be shown that $x \stackrel{M}{\Longrightarrow} y \geqslant y$.

(ii) If J is continuous, by the continuity of min and max we have that $\stackrel{M}{\Longrightarrow}$ is also continuous.

(iii) With regard to upper and lower contrapositivisation, when J has CPS(N) with respect to the strong N, we have that $x \stackrel{L:N}{\Longrightarrow} y \equiv x \stackrel{U:N}{\Longrightarrow} y \equiv J(x, y)$, i.e., $J^* \equiv J$ (see also Corollary 21). Whereas, in the case of M- contrapositivisation we have that

$$\begin{aligned} x \stackrel{M}{\Longrightarrow} y &= \min\{J(x, y) \vee N(x), J(N(y), N(x)) \vee y\} \\ &= \min\{J(x, y) \vee N(x), J(x, y) \vee y\} \quad [\text{since } J \text{ has CPS}(N)] \\ &= J(x, y) \vee [N(x) \wedge y] \not\equiv J(x, y). \end{aligned}$$

Thus even when J has CPS(N) the M-contrapositivisation of J with respect to N is a different implication and allows us to construct newer fuzzy implications that have CPS(N).

Also the above process does not continue indefinitely. In fact, in the case when J has CPS(N) let J^* be the M-contrapositivisation of J with respect to N, i.e, $J^*(x, y) = x \stackrel{M_J:N}{\Longrightarrow} y$. Then the M-contrapositivisation of J^* with respect to the same N is

$$\begin{aligned} (J^*)^*(x, y) &= (x \stackrel{M_J:N}{\Longrightarrow} y) \vee (N(x) \wedge y) \\ &= [J(x, y) \vee (N(x) \wedge y)] \vee (N(x) \wedge y) \\ &= [J(x, y) \vee (N(x) \wedge y)] \equiv J^*(x, y) \quad \text{i.e., } (J^*)^* = J^*. \end{aligned}$$

5. Contrapositivisations as S-implications

Since all S-implications $J_{S,N}$ possess CPS(N_J), with respect to their natural negation N_J—which is also the strong negation N used in its definition—and also since all the above contrapositivisation techniques can be applied to any general fuzzy implication, it is only appropriate that we study $\stackrel{U:N}{\Longrightarrow}$, $\stackrel{L:N}{\Longrightarrow}$ and $\stackrel{M}{\Longrightarrow}$ as S-implications for some fuzzy disjunctions $\circ$, $\diamond$ and $\triangle$, respectively. An investigation into this forms the rest of this section.

5.1. Upper contrapositivisation as an S-implication of a binary operator

Taking cue from Lemma 2, given a fuzzy implication J and a negation N, we define a binary operator $\circ$ on [0, 1] as follows:

$$x \circ y = \max\{J(N(x), y), J(N(y), x)\}. \tag{11}$$

Theorem 3. *Let J be a neutral fuzzy implication and N a strong negation such that the natural negation $J(x, 0) = N_J(x) \leqslant N(x)$, $\forall x \in [0, 1]$. Then the upper contrapositivisation of J with respect to N, $\stackrel{U:N}{\Longrightarrow}$, is such that*

(i) $x \stackrel{U:N}{\Longrightarrow} y = N(x) \circ y, \ \forall x \in [0, 1]$;

(ii) $x \circ 0 = x,\ \forall x \in [0,1]$;
(iii) $x \circ y = y \circ x,\ \forall x \in [0,1]$;
(iv) $\circ$ *is non-decreasing in both the variables.*

If, in addition, $\overset{U:N}{\Longrightarrow}$, *has the exchange property* (EP), *then* $\circ$ *is a* t-*conorm.*

Proof. Since $N_J(x) \leqslant N(x)$, $\forall x \in [0,1]$, and J is neutral, from Proposition 17 we know that $\overset{U:N}{\Longrightarrow}$ is N-compatible. Thus $\overset{U:N}{\Longrightarrow}$ has CPS(N) and from Corollary 11 it has (NP) and by Lemma 2 it follows that $\circ$ has the above four properties. If, in addition, $\overset{U:N}{\Longrightarrow}$ has the exchange property (EP) then again by Lemma 2(v) it follows that $\circ$ is a t-conorm. □

Theorem 4. *Let J be any fuzzy implication with natural negation N_J being involutive. If $J(1,y) \leqslant y$, for all $y \in [0,1]$, and the upper contrapositivisation of J with respect to N_J,* $\overset{U:N_J}{\Longrightarrow}$, *has the exchange property* (EP), *then* $\circ$ *is a* t-*conorm.*

Proof. Let us consider the upper contrapositivisation of J with $N \equiv N_J$. Then obviously $\overset{U:N_J}{\Longrightarrow}$ has CPS(N) and we also have that

$$\begin{aligned} 1 \overset{U:N_J}{\Longrightarrow} y &= \max\{J(1,y), J(N_J(y),0)\} \\ &= \max\{J(1,y), N_J(N_J(y))\} \\ &= \max\{J(1,y), y\} \quad [\because N_J \text{ is involutive }] \\ &= y \quad [\because J(1,y) \leqslant y\ ]. \end{aligned}$$

Thus $\overset{U:N_J}{\Longrightarrow}$ has CPS(N), (EP) and (NP) and by Lemma 2 it follows that $\circ$ is a t-conorm. □

Example 7. Consider the reciprocal Goguen's implication J_{GG} (see Example 2). $J_{GG}(1,y) = 0$, for all $y \in [0,1)$ and the natural negation of $N_{J_{GG}}(x) = 1 - x$, a strong negation, but J_{GG} does not have either CPS($1-x$) or (EP). Also the upper contrapositivisation of J_{GG} with $N(x) = 1 - x$, as can be verified, does not have the exchange property (EP).

From Theorems 3 and 4 we see the importance of satisfaction of (EP) by $\overset{U:N}{\Longrightarrow}$ for $\circ$ to be a t-conorm.

5.2. Lower contrapositivisation as an S-implication of a binary operator

Similarly, given a fuzzy implication J and a negation N, defining a binary operator $\diamond$ on [0, 1] as follows:

$$x \diamond y = \min\{J(N(x),y), J(N(y),x)\}. \tag{12}$$

we have the following counterparts of Theorems 3 and 4, which can be proven along similar lines.

Theorem 5. *Let J be a neutral fuzzy implication and N a strong negation such that the natural negation $J(x,0) = N_J(x) \geqslant N(x)$, $\forall x \in [0,1]$. Then the lower contrapositivisation of J with respect to N is such that*

(i) $x \overset{L:N}{\Longrightarrow} y = N(x) \diamond y,\ \forall x \in [0,1]$;
(ii) $x \diamond 0 = x,\ \forall x \in [0,1]$;
(iii) $x \diamond y = y \diamond x,\ \forall x \in [0,1]$;
(iv) $\diamond$ *is non-decreasing in both the variables.*

If, in addition, $\overset{L:N}{\Longrightarrow}$, *has the exchange property* (EP), *then* $\diamond$ *is a* t-*conorm.*

Theorem 6. *Let J be any fuzzy implication with natural negation N_J being involutive. If $J(1,y) \geqslant y$, for all $y \in [0,1]$ and the lower contrapositivisation of J with respect to N_J,* $\overset{L:N_J}{\Longrightarrow}$, *has the exchange property* (EP), *then* $\diamond$ *is a* t-*conorm.*

Example 8. Consider the $J(x,y) = 1 - x(1-\sqrt{y})^2$. $J(1,y) = 1 - (1-\sqrt{y})^2 = 2\cdot\sqrt{y} - y$, for all $y \in [0,1]$. Since $y \in [0,1]$ we have that $\sqrt{y} \geqslant y \Rightarrow 2\cdot\sqrt{y} \geqslant 2\cdot y \Rightarrow 2\cdot\sqrt{y} - y \geqslant y$ and hence $J(1,y) \geqslant y$. Also the natural negation $N_J(x) = 1 - x$ is involutive, but J does not have either CPS($1-x$) or (EP).

5.3. *M*-contrapositivisation as an S-implication of a binary operator

Finally, given a fuzzy implication J and a negation N, defining a binary operator $\triangle$ on [0, 1] as follows:

$$x \triangle y = \min\{J(N(x), y) \vee x, J(N(y), x) \vee y\}. \tag{13}$$

We now have the following:

Theorem 7. *Let J be any neutral fuzzy implication such that the contrapositivisation of J with respect to a strong N,* $\stackrel{M:N}{\Longrightarrow}$, *has the exchange property* (EP). *Then* $\triangle$ *is a* t-*conorm.*

6. Contrapositivisation and the residuation principle

A t-norm T and J_T the R-implication obtained from T are said to have the residuation principle if they satisfy the following:

$$T(x, y) \leqslant z \quad \text{iff } J_T(x, z) \geqslant y, \quad x, y \in [0, 1]. \tag{RP}$$

It is important to note that (RP) is a characterising condition for a left-continuous t-norm T (see [8, p. 25], [9, Proposition 5.4.2]).

6.1. Upper contrapositivisation as a residuation of a binary operator

Let T be a strict t-norm and J_T the corresponding R-implication obtained from T such that they satisfy the residuation principle (RP). Fodor in [7] has defined a binary operation $*_T$ on [0, 1] by

$$x *_T y = \min\{T(x, y), N(J_T(y, N(x))\}. \tag{14}$$

Fodor has shown that the upper contrapositivisation $\stackrel{U:N}{\Longrightarrow} = \rightarrow_T$ is the fuzzy implication generated by the residuation of $*_T$ (see [7, Theorem 2(d), p. 146]), i.e., $\rightarrow_T$ and $*_T$ satisfy the residuation principle (RP). The $*_T$ operator is not a t-norm in general, and in [7] the following sufficient condition on T for $*_T$ to be a t-norm is given.

Theorem 8 (*Fodor [7, Theorem 3, p. 147]*). *For a* t-*norm T and a strong negation N, if* $T(x, y) \leqslant N[J_T(y, N(x))]$ *for* $y > N(x)$ *then* $*_T$ *is a* t-*norm and is given by*

$$x *_T y = \begin{cases} T(x, y) & \text{if } y > N(x) \\ 0 & \text{if } y \leqslant N(x). \end{cases} \tag{15}$$

$*_T$ in addition to having very attractive properties has also opened up avenues for many subsequent research works. Also the nilpotent minimum proposed therein has led to some interesting research—transformations of t-norms called N-annihilation in [10], characterisation of R_0-implications in [5], study of generalisation of nilpotent minimum t-norms in [4].

It is only natural to ask whether the lower contrapositivisation of an R-implication J_T can also be obtained as a residuation of a binary operator. Along the same lines of Fodor [7], in Section 6.2, we propose a suitable binary operation $*_t$ such that the lower contrapositivisation of the R-implication J_T has the residuation principle with respect to $*_t$.

6.2. Lower contrapositivisation as a residuation of a binary operator

Taking cue from $*_T$ of Fodor we define a binary operator $*_t$ on [0, 1] as follows:

$$x *_t y = \max\{T(x, y), N(J_T(y, N(x)))\}. \tag{16}$$

where J_T is the corresponding R-implication obtained from the t-norm T. Now the following can be easily shown, along the lines of Theorem 2 in [7]:

Theorem 9. *Let T be a left continuous* t-*norm*, J_T *its corresponding R-implication*, N *is a strong negation such that* $N(x) \leqslant N_{J_T}(x)$, *for all* $x \in [0,1]$, *and operations* $\overset{L:N}{\Longrightarrow}$ *and* $*_t$ *are as defined in* (6) *and* (16). *Then the following conditions are satisfied*:

(i) $1 *_t y = y *_t 1 = y$;
(ii) $x *_t 0 = 0 *_t x = 0$;
(iii) $*_t$ is non-decreasing in both the variables;
(iv) $x *_t z \leqslant y \Leftrightarrow x \overset{L:N}{\Longrightarrow} y \geqslant z$.

Proof. (i) Since $N(y) \leqslant N_{J_T}(y)$, for all $y \in [0,1]$, we have $N(N(y)) \geqslant N(N_{J_T}(y))$ and from which we obtain the following:

$$\begin{aligned} 1 *_t y &= \max\{T(1,y), N(J_T[y,0])\} \\ &= \max\{y, N(N_{J_T}(y))\} \\ &= \max\{N(N(y)), N(N_{J_T}(y))\} = N(N(y)) = y. \end{aligned}$$

Similarly,

$$\begin{aligned} y *_t 1 &= \max\{T(y,1), N(J_T[1, N(y)])\} \\ &= \max\{y, N(N(y))\} \\ &= \max\{y, y\} = y. \end{aligned}$$

(ii) As shown in (i) one can also show that $x *_t 0 = 0 *_t x = 0$.

(iii) Obvious by noting that N is strictly decreasing and J_T is non-increasing in the first variable and non-decreasing in the second variable.

(iv) Since T is left-continuous (T, J_T) satisfies (RP). Consider first the case when $x \overset{L:N}{\Longrightarrow} y \geqslant z$ which leads to the following two cases:

Case 1: $J_T(x,y) \geqslant J_T(N(y), N(x))$.

Then $x \overset{L:N}{\Longrightarrow} y = J_T(N(y), N(x))$ and we have

$$\begin{aligned} x \overset{L:N}{\Longrightarrow} y \geqslant z \quad &\text{iff } J_T(N(y), N(x)) \geqslant z, \\ &\text{iff } T(N(y), z) \leqslant N(x) \text{ [by (RP)]}, \\ &\text{iff } J_T(z, N(x)) \geqslant N(y), \\ &\text{iff } N[J_T(z, N(x))] \leqslant y. \end{aligned} \tag{17}$$

Case 2: $J_T(x,y) \leqslant J_T(N(y), N(x))$.

Then $x \overset{L:N}{\Longrightarrow} y = J_T(x,y)$ and

$$x \overset{L:N}{\Longrightarrow} y \geqslant z \quad \text{iff } J_T(x,y) \geqslant z \quad \Leftrightarrow \quad T(x,z) \leqslant y. \tag{18}$$

From (17) and (18) we see that

$$x *_t z = \max\{T(x,z), N(J_T(z, N(x)))\} \leqslant y.$$

On the other hand, let $x *_t z \leqslant y$. This implies by (16) that $T(x,z) \leqslant y$ and $N[J_T(z, N(x))] \leqslant y$. By (RP)

$$T(x,z) \leqslant y \quad \text{iff } J_T(x,y) \geqslant z, \tag{19}$$

and we also have that

$$\begin{aligned} N[J_T(z, N(x))] \leqslant y \ &\text{iff } \ J_T(z, N(x)) \geqslant N(y), \\ &\text{iff } \ T(z, N(y)) \leqslant N(x) \ [\because (T, J_T) \text{ satisfy (RP)}], \\ &\text{iff } \ T(N(y), z) \leqslant N(x) \ [\because T \text{ is commutative}], \\ &\text{iff } \ J_T(N(y), N(x)) \geqslant z. \end{aligned} \tag{20}$$

Again, we have from (19) and (20) that $x \overset{L:N}{\Longrightarrow} y \geqslant z$. Thus

$$x *_t z \leqslant y \quad \text{iff } x \overset{L:N}{\Longrightarrow} y \geqslant z. \qquad \square$$

Given a left-continuous t-norm $*$ and a non-involutive negation n, in [4] the authors have studied the following binary operation on [0, 1]:

$$x *_n y = \begin{cases} x * y & \text{if } y > n(x), \\ 0 & \text{if } y \leqslant n(x). \end{cases} \tag{21}$$

Evidently, (21) is a generalisation of (15). The authors have proven that $*_n$ has all the properties of a t-norm, except for associativity (see [4, Lemma 1, p. 287]). Moreover, left-continuity of $*$ implies that of $*_n$. If $\rightarrow$ is the residuum of $*$, the residuum $\Rightarrow_n$ of $*_n$ is given as follows:

$$x \Rightarrow_n y = \begin{cases} 1 & \text{if } x \leqslant y, \\ n(x) \vee (x \rightarrow y) & \text{if } x > y. \end{cases} \tag{22}$$

For a characterisation, some examples and graphs of such negations see [4, pp. 285–286].

The authors have characterised continuous t-norms $*$ such that, given a (non-involutive) negation n, $*_n$ defined by (21) is a t-norm and the natural negation of the corresponding residuum $\Rightarrow_n$ of $*_n$, defined by (22), coincides with n, i.e., $n(x) = x \Rightarrow_n 0$.

Let Θ denote the above class of (left-)continuous t-norms $*_n$. Now, for any $T \in \Theta$, the natural negation N_{J_T} of the corresponding R-implication J_T is not a strong negation and thus J_T does not have CPS(N_{J_T}) as per Definition 9. If N_{J_T} is also a non-vanishing negation then given a strong negation $N \leqslant N_{J_T}$, the lower contrapositivisation of J_T is not only N-compatible but also has the residuation principle with respect to a binary operation $*_t$ as defined in (16).

Again, $*_t$ is not a t-norm in general. In the following we give a sufficient condition for $*_t$ to be a t-norm:

Theorem 10. *For a* t-*norm T and a strong negation N, if whenever $y > N(x)$, $T(x, y) \geqslant N[J_T(y, N(x))]$, then $*_t$ is a* t-*norm. In fact $*_t \equiv T$.*

Proof. By the definition of an R-implication J_T obtained from a t-norm T we have that $x \leqslant y \Rightarrow J_T(x, y) = 1$.

- If $y \leqslant N(x)$ then $J_T(y, N(x)) = 1$ and $N[J_T(y, N(x))] = 0$. Now, $x *_t y = \max\{T(x, y), N(J_T(y, N(x)))\} = T(x, y)$.
- If $y > N(x)$ then by the hypothesis $T(x, y) \geqslant N[J_T(y, N(x))]$ and hence $x *_t y = T(x, y)$.

Thus both when $y \leqslant N(x)$ and $y > N(x)$, $*_t \equiv T$, a t-norm. $\square$

Remark 32. Theorem 10 is not satisfactory as $*_t \equiv T$ implies $\overset{L:N}{\Longrightarrow} = J_T$, which in turn means that J_T has CPS(N) and thus by the neutrality of J_T we have that $N \equiv N_{J_T}$. An alternate sufficient condition for $*_t$ to be a t-norm is worth exploring in the light of the importance $*_T$ received in the literature.

7. Concluding remarks

In this work we have investigated the N-compatibility of two contrapositivisation techniques proposed by Bandler and Kohout [2] towards imparting contrapositive symmetry to a given fuzzy implication, viz., upper and lower contrapositivisations. A contrapositivisation technique is said to be N-compatible if the natural negation of the contrapositivised fuzzy implication is equal to the strong negation N employed therein. This is equivalent to the neutrality of the contrapositivised implication.

We have shown that given a strong negation N and a fuzzy implication J, the upper contrapositivisation of J with respect to N, $\overset{U:N}{\Longrightarrow}$, is N-compatible if and only if $N \geqslant N_J$, where N_J is the natural negation of J. Such a strong N exists only if N_J is non-filling. Similarly, the lower contrapositivisation of J with respect to N, $\overset{L:N}{\Longrightarrow}$, is N-compatible if and only if $N \leqslant N_J$. Such a strong N exists only if N_J is non-vanishing.

Also we have proposed a new contrapositivisation technique, called M- contrapositivisation, $\overset{M}{\Longrightarrow}$, whose N-compatibility is independent of the ordering between N and N_J and thus is applicable for any implication J and strong negation N. Some interesting properties of this new contrapositivisation technique are also discussed.

Fodor [7] has shown that for a special case of J the upper contrapositivisation is a residuation of a binary operator. We have shown that the lower contrapositivisation can be seen as the residuation of a suitable binary operator.

We have also proposed binary operators (fuzzy disjunctions) such that the lower, upper and M-contrapositivisation can be seen as S-implications obtained from them. Also some sufficient conditions for these fuzzy disjunctions to become t-conorms are given. It is interesting to note that the exchange principle of the contrapositivisations seems very much necessary for the disjunctions $\diamond$, $\circ$ and $\triangle$ to become t-conorms. Thus it would be worthwhile to investigate when does $\overset{U/L/M}{\Longrightarrow}$ have the exchange property (EP).

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