T-subnorms with strong associated negation : Some Properties

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Abstract

In this work we investigate t-subnorms M that have strong associated negation. Firstly, we show that such t-subnorms are necessarily t-norms. Following this, we investigate the inter-relationships between different algebraic and analytic properties of such t-subnorms, viz., Archimedeanness, conditional cancellativity, left-continuity, nilpotent elements, etc. In particular, we show that under this setting many of these properties are equivalent. Our investigations lead us to two open problems which are also presented.

Keywords: T-norms, t-subnorms, Archimedeanness, conditional cancellativity, left-continuity, R-implications, residual implications.

1. Introduction

The theory of triangular norms and triangular subnorms have been well studied and their applications well-established. Many algebraic and analytical properties of these operations, viz., Archimedeanness, conditional cancellativity, left-continuity, etc., have been studied and their inter-relationships shown (see for instance, [6]).

Yet another way of categorizing t-subnorms is as follows: Given a t-subnorm M, one can obtain its associated negation n_M (see Definitions 2.2 and 2.4 below). Note that n_M is usually not a fuzzy negation, i.e., $n_M(1) \geq 0$. However, we can broadly consider two sub-classes of t-subnorms based on whether their associated negation n_M is strong or not.

In this work, we study the class of t-subnorms whose associated negation n_M is strong. Firstly, we show that such t-subnorms are necessarily t-norms. Following this, we investigate some particular classes of these and study the inter-relationships between different algebraic and analytic properties of such t-subnorms, viz., Archimedeanness, conditional cancellativity, left-continuity, etc. In particular, we show that under this setting many of these properties are equivalent. Our investigations have led us to two open problems, which are also presented.

2. Preliminaries

To make this short note self-contained, we present some important definitions and properties, which can be found in [6, 1].

Definition 2.1. A fuzzy negation is a function $N\colon [0,1] \to [0,1]$ that is non-increasing and such that $N(1) = 0$ and $N(0) = 1$. Further, it is said to be strong or involutive, if $N \circ N = id_{[0,1]}$.

Definition 2.2. A t-subnorm is a function $M\colon [0,1]^2 \to [0,1]$ such that it is monotonic non-decreasing, associative, commutative and $M(x,y) \leq \min(x,y)$ for all $x, y \in [0,1]$, i.e., 1 need not be the neutral element.

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Definition 2.3. Let M be a t-subnorm.

(i) If 1 is the neutral element of M, then it becomes a t-norm. We denote a t-norm by T in the sequel.

(ii) M is said to satisfy the Conditional Cancellation Law if, for any $x, y, z \in (0, 1]$,

$$M(x, y) = M(x, z) > 0 \text{ implies } y = z . \tag{CCL}$$

Alternately, (CCL) implies that on the positive domain of M, i.e., on the set $\{(x, y) \in (0, 1]^2 \mid M(x, y) > 0\}$, M is strictly increasing.

(iii) M is said to be *Archimedean*, if for all $x, y \in (0, 1)$ there exists an $n \in \mathbb{N}$ such that $x_M^{[n]} < y$.

(iv) An element $x \in (0, 1)$ is a *nilpotent* element of M if there exists an $n \in \mathbb{N}$ such that $x_M^{[n]} = 0$.

(v) A t-norm T is said to be *nilpotent*, if it is continuous and if each $x \in (0, 1)$ is a nilpotent element of T.

Definition 2.4. Let M be any t-subnorm and $x, y \in [0, 1]$.

- The R-implication I_M of M is given by

$$I_M(x, y) = \sup \{t \in [0, 1] \mid M(x, t) \le y\} . \tag{1}$$

- The associated negation n_M of M is given by

$$n_M(x) = \sup\{t \in [0, 1] \mid M(x, t) = 0\}. \tag{2}$$

A brief note on nomenclature is perhaps warranted here. Firstly, the R-implication I_M will be termed a *residual* implication only if the underlying t-subnorm M is left-continous.

Secondly, while n_M is clearly a non-increasing function and $n_M(0) = 1$, note that it need not be a fuzzy negation, since $n_M(1)$ can be greater than 0. Hence, only in the case n_M is a fuzzy negation we call n_M the *natural negation* of M in this work. However, many results hold even if $n_M(1) > 0$, see for instance [3, 9], and hence to preserve this generality in such situations we term n_M as the *associated negation*.

For instance, the following result is true even when $n_M(1) > 0$.

Proposition 2.5 (cf. [1], Proposition 2.3.4). *Let M be any t-subnorm and n_M its associated negation. Then we have the following:*

(i) $M(x, y) = 0 \Longrightarrow y \le n_M(x)$.

(ii) $y < n_M(x) \Longrightarrow M(x, y) = 0$.

(iii) *If M is left-continuous then $y = n_M(x) \Longrightarrow M(x, y) = 0$, i.e., the reverse implication of (i) also holds.*

Proposition 2.6. *Let M be any t-subnorm with n_M being a natural negation with e as its fixed point, i.e., $n_M(e) = e$. Then*

(i) *Every $x \in (0, e)$ is a nilpotent element; in fact, $x_M^{[2]} = 0$ for all $x \in [0, e)$.*

(ii) *In addition, if M is either conditionally cancellative or left-continuous, then e is also a nilpotent element.*

Proof. (i) By definition,

$$n_M(e) = \sup\{t \in [0, 1] \mid M(e, t) = 0\} = e,$$

implies that $M(e, e^-) = 0$, from whence we get $M(x, x) \le M(e, e^-) = 0$ for all $x \in [0, e)$. In other words, $x_M^{[2]} = 0$ for all $x \in [0, e)$.

(ii) Let M be conditionally cancellative. If $e_M^{[2]} = 0$ then clearly e is a nilpotent element. If not, then we have $M(e, e) = x < M(1, e) \le e$ and from (ii) above we have $M(x, x) = 0$. Now,

$$e_M^{[4]} = M(M(e, e), M(e, e)) = M(x, x) = 0.$$

If M is left-continuous, then $n_M(e) = \max\{t \in [0, 1] \mid M(e, t) = 0\} = e$, i.e., $e \in \{t \in [0, 1] \mid M(e, t) = 0\}$ and hence $M(e, e) = 0$, i.e., e is also a nilpotent element.

□

Remark 2.7. (i) In the case n_M is a strong natural negation we can show that if M is conditionally cancellative then every $x \in (0,1)$ is also a nilpotent element, see Remark 5.9(ii).

(ii) Note that without any further assumptions, the set of nilpotent elements need not be the whole of $(0,1)$. For instance, for the nilpotent minimum t-norm

$$T_{\mathbf{nM}}(x,y) = \begin{cases} 0, & \text{if } x+y \leq 1, \\ \min(x,y), & \text{otherwise,} \end{cases} \quad x,y \in [0,1],$$

which is left-continuous but not conditionally cancellative, its set of nilpotent elements is $(0, .5]$, while its set of zero divisors is $(0,1)$.

However, Theorem 6.1 gives an equivalence condition for the whole of $(0,1)$ to be the set of nilpotent elements under a suitable condition on n_M.

3. T-subnorms with strong associated negation = T-norms

There are works showing that some classes of t-subnorms M whose associated negations n_M are involutive do become t-norms. Jenei [4], also see [5], showed it for the class of left-continuous M, while Jayaram [2] did the same for conditionally cancellative M. The main result of this section shows that the above results are true in general, i.e., any t-subnorm with a strong natural negation is a t-norm.

The following result was firstly proven by Jenei in [4]. However, we give a very simple proof of this result without resorting to the rotation-invariance property.

Theorem 3.1 (Jenei, [4], Theorem 3). *If M is a left-continuous t-subnorm with n_M being strong, then M is a t-norm.*

Proof. Firstly, note that if M is a left-continuous t-subnorm, then its residual implication satisfies the exchange principle, i.e.,

$$I_M(x, I_M(y,z)) = I_M(y, I_M(x,z)).$$

It follows from the fact that the neutral element of M does not play any role in the proof, see, for instance the proof given for Theorem 2.5.7 in [1].

If n_M is strong, then for every $y \in [0,1]$ there exists $y' \in [0,1]$ such that $n_M(y) = y'$. Now,

$$I_M(1,y') = I_M(1, I_M(y,0)) = I_M(y, I_M(1,0)) = I_M(y,0) = y'.$$

Thus, for all $y' \in [0,1]$,

$$I_M(1,y') = \max\{t \mid M(1,t) \leq y'\} = y' \Longrightarrow M(1,y') = y' .$$

□

Theorem 3.2 (Jayaram [2], Theorem 4.4). *Let M be any conditionally cancellative t-subnorm. If n_M is a strong natural negation then M is a t-norm.*

Now, we prove the main result of this section which shows that the above results are true in general.

Theorem 3.3. *Let M be any t-subnorm with n_M being a strong natural negation. M is a t-norm.*

Proof. Note, firstly, that since $n_M(x) = \sup\{t \in [0,1] \mid M(x,t) = 0\}$, is a strong negation, we have that $n_M(z) = 1 \Longleftrightarrow z = 0$ and $n_M(z) = 0 \Longleftrightarrow z = 1$. Equivalently, $M(1,z) = 0 \Longleftrightarrow z = 0$.

On the contrary, let us assume that $M(1,x) = x' \lneq x$ for some $x \in (0,1]$. Since n_M is strong, the following are true:

(i) $n_M(x') > n_M(x)$

(ii) if $p > n_M(x)$ then $M(x,p) > 0$,

(iii) there exists a $y \in (0,1)$ such that $n_M(x') > y > n_M(x)$ and $M(y, x) = q > 0$ while $M(y, x') = 0$.

Now, by associativity we have

$$\left.\begin{array}{l} M(y, M(x,1)) = M(y, x') = 0 \\ M(M(y,x),1) = M(q,1) \end{array}\right\} \Longrightarrow M(q,1) = 0,$$

a contradiction. Thus $M(1, x) = x$ for **all** $x \in [0, 1]$ and hence we have the result. □

In the following sections, we deal with t-subnorms whose associated negations are strong, or equivalently t-norms whose associated negations are strong. We discuss the inter-relationships between the different algebraic and analytical properties for this subclass of t-norms; in particular, Archimedeanness, Conditional Cancellativity, (Left-)continuity and Nilpotence that are relevant to our context. We begin with listing out some established results and go on to present some new ones.

4. Continuity and Nilpotence

Let T be a t-norm and n_T a strong negation. The following result, whose proof is straight-forward, shows the equivalence between continuity and nilpotence:

Theorem 4.1 (Klement et al. [6]). *Let T be a t-norm with n_T being strong. Then the following are equivalent:*

(i) *T is continuous.*
(ii) *T is a nilpotent t-norm.*

Further, we know that every nilpotent t-norm is both Archimedean and Conditionally cancellative, since every nilpotent t-norm is isomorphic to the Lukasiewicz t-norm and the Archimedeanness and Conditionally cancellativity of T are preserved under isomorphism, see [6], Examples 2.14(iv) and 2.15(v). Trivially, every nilpotent t-norm is also left-continuous.

5. Conditional Cancellativity, Left Continuity and Nilpotence

Recenty, in Jayaram [2], the following problem of U.Höhle, given in Klement et al. [7] has been solved. Further it was shown that it characterizes the set of all conditionally cancellative t-subnorms.

Problem 5.1. (U.Höhle, [7], Problem 11) Characterize all left-continuous t-norms T which satisfy

$$I_T(x, T(x, y)) = \max(n_T(x), y), \quad x, y \in [0, 1] \,, \tag{3}$$

where I_T, n_T are as given in (1) and (2) with $M = T$.

Theorem 5.2 (cf. Jayaram [2], Theorem 3.1). *Let M be any t-subnorm, not necessarily left-continuous. Then the following are equivalent:*

(i) *The pair (I_M, M) satisfies (3).*
(ii) *M is a Conditionally Cancellative t-subnorm.*

Remark 5.3. The following statements follow from Theorem 5.2 with $M = T$, a t-norm:

(i) If a (right) continuous T satisfies (3) along with its R-implication then T is necessarily Archimedean, see [6], Proposition 2.15(ii).

(ii) However, if a left-continuous T satisfies (3) along with its residual implication then T need not be Archimedean and hence not continuous. An example is Hajék's t-norm or the following t-norm $T_{\mathbf{OY}}$ of Ouyang et al [11], Example 3.4, which is a (CCL) t-norm (and hence a t-subnorm too) that is left-continuous but not continuous at $(0.5, 0.5)$ and hence is not Archimedean (see Figure 1(a)):

$$T_{\mathbf{OY}}(x,y) = \begin{cases} 2(x-0.5)(y-0.5)+0.5, & \text{if } (x,y) \in (0.5,1]^2 \\ 2y(x-0.5), & \text{if } (x,y) \in (0.5,1] \times [0,0.5] \\ 2x(y-0.5), & \text{if } (x,y) \in [0,0.5] \times (0.5,1] \\ 0, & \text{otherwise} \end{cases}.$$

Theorem 5.4 (Jenei. [4], Theorem 2)**.** *Let T be a left-continous t-norm with n_T being strong. Then the following are equivalent:*

(i) *T is a conditionally cancellative t-norm.*

(ii) *T is a nilpotent t-norm.*

In fact, for a conditionally cancellative t-subnorm M we can give an equivalent condition for it to be left-continuous.

Theorem 5.5. *Let M be a* (CCL) *t-subnorm. Then the following are equivalent:*

(i) $M(x, n_M(x)) = 0, \quad x \in [0,1]$.

(ii) *M is left-continuous.*

Proof. (i) $\Longrightarrow$ (ii): Let $M(x, n_M(x)) = 0$, for all $x \in [0,1]$. On the contrary, let us assume that M is not left-continuous. Then there exist $x_0 \in (0,1]$, $y_0 \in (0,1]$ and an increasing sequence $(x_n)_{n\in\mathbb{N}}$, where $x_n \in [0,1)$, such that $\lim_{n\to\infty} x_n = x_0$, but

$$\lim_{n\to\infty} M(x_n, y_0) = M(x_0^-, y_0) = z' < z_0 = M(x_0, y_0).$$

Observe that

$$I_M(y_0, z') = \sup\{t \in [0,1] \mid M(y_0, t) \le z'\} = x_0, \tag{4}$$

since from the monotonicity of M we have $M(y_0, x_n) \le z'$ for every $n \in \mathbb{N}$ and $M(y_0, x_0) = z_0 > z'$. Since M is (CCL), we have from (3)

$$I_M(y_0, z') = I_M(y_0, M(y_0, x_0^-)) = \max(n(y_0), x_0^-).$$

Now, we have two cases. On the one hand, if $I_M(y_0, z') = x_0^- \lneq x_0$, then it is a contradiction to (4). On the other hand, if $I_M(y_0, z') = n(y_0)$, then this implies that $n(y_0) = x_0$ from (4) and hence

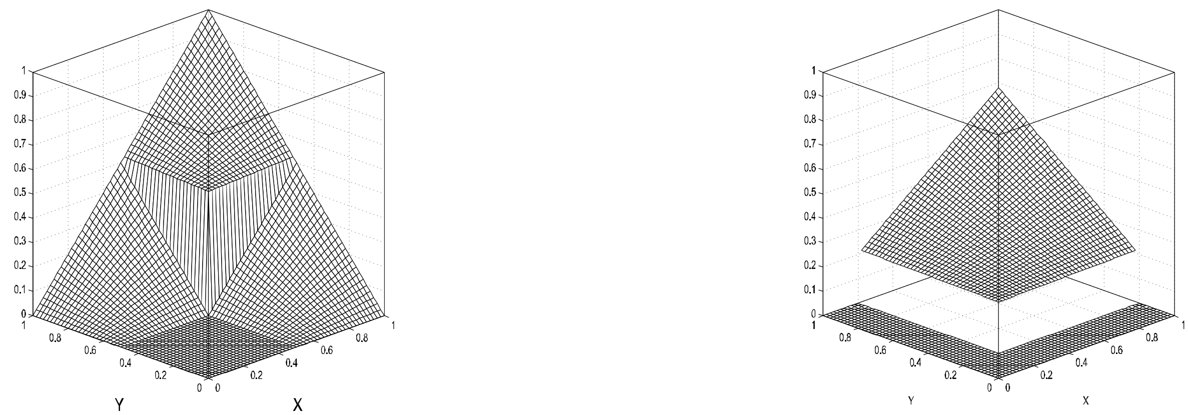
$$M(x_0, y_0) = M(n(y_0), y_0) = z_0 = 0,$$

by the hypothesis and hence there does not exist any $z' < z_0$ and hence M is left-continuous.

(ii) $\Longrightarrow$ (i): Follows from Proposition 2.5(iii).

□

In other words, Theorem 5.5 states that for a (CCL) t-subnorm M, the only points at which M may not be left-continuous is the boundary of the zero region $Z_M = \{(x,y) \in [0,1]^2 | M(x,y) = 0\}$ which does not contain the origin.



(a) $T_{\mathbf{OY}}$ (b) $M_{\mathbf{P_f}}$

Figure 1: A t-norm and a t-subnorm that are conditionally cancellative

Remark 5.6. In Theorem 5.5 we do not need n_M to be a negation, i.e., $n_M(1) \geq 0$. Consider the following t-subnorm $M_{\mathbf{P_f}}$ (cf. Example 3.15 of [6], see Figure 1(b)),

$$M_{\mathbf{P_f}} = \begin{cases} 0.2 + \dfrac{3(x-0.2)(y-0.2)}{4}, & \text{if } (x,y) \in (0.2, 1]^2 \\ 0, & \text{otherwise} \end{cases}$$

which is a left-continuous (CCL) t-subnorm but $n_{M_{\mathbf{P_f}}}$ is not a negation since $n_{M_{\mathbf{P_f}}}(1) = 0.2$.

Theorem 5.7. *Let M be a (CCL) t-subnorm whose n_M is strong. Then M is left-continuous.*

Proof. If possible, let $M(x_0, n(x_0)) = p > 0$ for some $x_0 \in (0,1)$. Since M is (CCL), we have $M(1^-, x_0) < x_0$ and hence by associativity we have

$$M(1^-, M(x_0, n(x_0))) = M(1^-, p)$$
$$M(M(1^-, x_0), n(x_0)) = 0$$

from whence it follows $M(1^-, p) = 0$, i.e., $n(p) = 1$, a contradiction to the fact that n_M is strong. Thus $p = 0$ and the result follows from Theorem 5.5. □

Theorem 5.8. *Let M be a t-subnorm such that n_M is strong. Then the following are equivalent:*

(i) *M is conditionally cancellative.*
(ii) *M is a nilpotent t-norm.*

Proof. If M satisfies (CCL) then M is left-continuous, from Theorem 5.7 and now, using Theorem 5.4 we have the result. □

Remark 5.9. (i) The nilpotent minimum t-norm $T_{\mathbf{nM}}$ is an example of a t-subnorm M whose n_M is involutive and M satisfies (LEM) with n_M but is not conditionally cancellative and hence is not a nilpotent t-norm.

(ii) In the case n_M is a strong natural negation, from Theorem 5.7 we see that conditionally cancellativity implies left-continuity and from Theorem 5.8 that every $x \in (0,1)$ is a nilpotent element.

6. Archimedeanness , Left Continuity and Nilpotence

We begin with a result that shows that if n_M is strong, then the Archimedeanness is equivalent to every element $x \in (0,1)$ being nilpotent. However, unless M is also left-continuous, M is not a nilpotent t-norm.

Theorem 6.1. *Let M be any t-subnorm such that n_M is not completely vanishing, i.e., there exists $z \in (0,1)$ such that $n_M(z) > 0$. The following are equivalent:*

(i) *Every $x \in (0,1)$ is a nilpotent element.*
(ii) *M is Archimedean.*

Proof. (i) $\Longrightarrow$ (ii): Follows from Proposition 2.15 (iv) in [6].

(ii) $\Longrightarrow$ (i): Let M be any Archimedean t-subnorm such that n_M is not completely vanishing, i.e., there exists $z \in (0,1)$ such that $n_M(z) > 0$. By Proposition 2.5(ii) we see that for any $0 < z' < n_M(z)$ we have $M(z', z) = 0$.

For any $x \in [0,1)$, by the Archimedeanness of M, there exists an $n, p \in \mathbb{N}$ such that $x_M^{[n]} < z'$ and $x_M^{[p]} < z$ from whence we have that

$$x_M^{[n+p]} = M\left(x_M^{[n]}, x_M^{[p]}\right) \leq M(z', z) = 0.$$

□

Corollary 6.2. *Let M be any t-subnorm such that n_M is a strong negation. Then the following are equivalent:*

(i) *Every $x \in (0,1)$ is a nilpotent element.*
(ii) *M is Archimedean.*

The following result is due to Kolesárová [8]:

Theorem 6.3. *Let T be any Archimedean t-norm. Then the following are equivalent:*

(i) *T is left-continuous.*
(ii) *T is continuous.*

Corollary 6.4. *A left-continuous Archimedean t-subnorm M whose n_M is involutive is a nilpotent t-norm.*

Proof. From Theorem 3.1 we see that M is a left-continuous t-norm. From Theorem 6.3, since M is Archimedean it is continuous. Also by Theorem 6.1, we have that every $x \in (0,1)$ is a nilpotent element. Thus T is nilpotent, i.e., isomorphic to $T_{\mathbf{LK}}(x,y) = \max(0, x+y-1)$. □

Remark 6.5. (i) Note that there exist left-continuous Archimedean t-subnorms M that are not continuous and hence their n_M is not involutive. For instance, consider the t-subnorm

$$M(x,y) = \begin{cases} x+y-1, & \text{if } x+y > \frac{3}{2}, \\ 0, & \text{otherwise }, \end{cases} \quad x, y \in [0,1].$$

(ii) The nilpotent minimum t-norm $T_{\mathbf{nM}}$ is an example of a left-continuous t-subnorm M whose n_M is involutive but is not Archimedean and hence is not a nilpotent t-norm.

(iii) However, it is not clear whether there exists any non-nilpotent Archimedean t-subnorm M whose n_M is involutive. Clearly such t-(sub)norms are not left-continuous.

Problem 1. Does there exist any non-nilpotent Archimedean t-subnorm M whose n_M is involutive. In other words, is an Archimedean t-subnorm M whose n_M is involutive necessarily left-continuous?

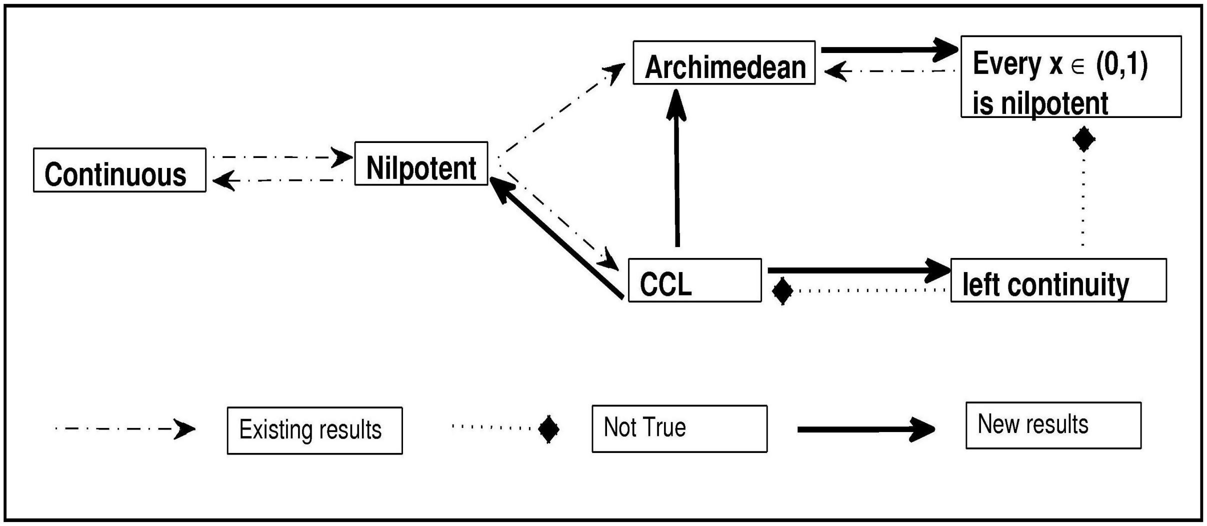


Figure 2: A Summary of the results available so far when n_T is strong

7. Archimedeanness and Conditional Cancellativity

In general, there does not exist any inter-relationships between Archimedeanness and conditional cancellativity, as the following examples show.

Example 7.1. (i) The Ouyang t-norm $T_{\mathbf{OY}}$ is an example of a t-(sub)norm which is not Archimedean but is both left-continuous and conditionally cancellative.

(ii) The following t-norm is neither Archimedean nor left-continuous but is conditionally cancellative:

$$T(x,y) = \begin{cases} 0, & \text{if } xy \leq \frac{1}{2} \& \max(x,y) < 1 \\ xy, & \text{if } xy > \frac{1}{2} \\ \min(x,y), & \text{otherwise} \end{cases} .$$

(iii) The following t-subnorm is Archimedean and continuous, but not conditionally cancellative:

$$M(x,y) = \max(0, \min(x+y-1, x-a, y-a, 1-2a)),$$

where $a \in (0, 0.5)$. For instance, with $a = 0.25$ we have $M(0.75, 0.75) = M(0.75, 0.8) = 0.5$.

(iv) The nilpotent minimum $T_{\mathbf{nM}}$, whose n_M is strong, is neither Archimedean nor conditionally cancellative, but is left-continuous.

(v) The Lukasiewicz t-norm $T_{\mathbf{LK}}(x,y) = \max(0, x+y-1)$ is both Archimedean and conditionally cancellative. Further, $n_{T_{\mathbf{LK}}}$ is strong.

In fact, in the case when n_M is strong we have the following partial implication.

Lemma 7.2. *Let M be any t-subnorm whose n_M is strong. If M is conditionally cancellative then M is Archimedean.*

Proof. From Theorem 5.8, we have that if M is conditionally cancellative then M is a nilpotent t-norm from whence it follows that M is Archimedean. □

Problem 2. Does there exist any Archimedean t-subnorm M whose n_M is involutive but is not conditionally cancellative? In other words, is an Archimedean t-subnorm M whose n_M is involutive necessarily conditionally cancellative?

In fact, from Theorem 3.3, it can be easily seen that the above two problems are an alternate formulation of Problem 2.1 in [10].

8. Concluding Remarks

In this work, we have shown that t-subnorms whose associated negations are strong are necessarily t-norms. Further, we have studied the inter-relationships between some algebraic and analytical properties of such t-(sub)norms. Figure 2 gives a pictorial summary of the results that exist so far. Our study has also opened up two interesting open problems.

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