R-implications and the exchange principle: The case of border continuous t-norms

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Abstract

It is well-known that the residual of a left-continuous t-norm satisfies the exchange principle (EP). However, the left-continuity of a t-norm is only sufficient and not necessary, as many examples in the literature illustrate. In this work we study the necessary and sufficient conditions on a t-norm for its residual to satisfy (EP). We present a complete characterization of two classes of t-norms whose residuals satisfy (EP), viz., t-norms that are border-continuous and those that have an ordinal sum representation. Based on the obtained results we characterize t-norms, whose residuals satisfy both the exchange principle and the ordering property.
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1. Introduction

The family of R-implications is one of the most established classes of fuzzy implications. In fact, one of the earliest methods for obtaining implications was from conjunctions as their residuals, when no additional logical connectives are given. In this way Gödel extended the three-valued implication of Heyting, while discussing the possible relationships between many-valued logic on the one hand, and intuitionistic logic on the other. Residuals of conjunctions on a lattice $\mathcal{L}$, be it from t-norms, uninorms, t-subnorms, copulas, etc., have attracted the most attention from researchers, since they can transform the underlying lattice $\mathcal{L}$ into a residuated lattice. In this paper we will consider only R-implications generated from t-norms.

Definition 1.1 (*see Klement et al. [7]*). (i) A function $M: [0, 1]^2 \to [0, 1]$ is called a *t-subnorm*, if it is non-decreasing in both variables, commutative, associative and $M(x, y) \leq \min(x, y)$ for all $x, y \in [0, 1]$.
(ii) A t-norm T is a t-subnorm that has 1 as the neutral element.

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Definition 1.2 (*see Baczyński and Jayaram [2], Gottwald [4]*). A function $I: [0, 1]^2 \rightarrow [0, 1]$ is called an *R-implication*, if there exists a t-norm T such that

$$I(x, y) = \sup\{t \in [0, 1] | T(x, t) \leq y\}, \quad x, y \in [0, 1].$$

If an R-implication is generated from a t-norm T, then we will often denote it by I_T. Obviously, due to the monotonicity of any t-norm T, if $T(x, y) \leq z$ then necessarily $x \leq I_T(y, z)$. Observe that, for a given t-norm T, the pair (T, I_T) satisfies the adjointness property (RP) (see Definition 2.3) if and only if T is left-continuous, see, for instance the monographs [4,1].

R-implications also have a parallel origin other than its logical foundations. They were also obtained from the study of solutions of systems of fuzzy relational equations and have been known under different names, for example, as a Φ-operator in Pedrycz [12], as T-relative pseudocomplement and α_T-operator in [11].

1.1. A first characterization of R-implications generated from left-continuous t-norms

Sanchez [13] showed that the greatest solution of $\sup - \min$ composition of fuzzy relations is the relation obtained from the residual of min. In fact, Miyakoshi and Shimbo [11] generalized this result to any left-continuous t-norm. They also showed that their α_T-operator is equivalent to the Φ-operator of Pedrycz. Most importantly, they gave the first characterization of R-implications obtained from left-continuous t-norms (for the proof see also [1, Theorem 2.5.17]).

Theorem 1.3. *For a function $I: [0, 1]^2 \rightarrow [0, 1]$ the following statements are equivalent*:

(i) *I is an R-implication generated from a left-continuous t-norm.*
(ii) *I is non-decreasing with respect to the second variable, it satisfies the exchange principle, i.e.,*

$$I(x, I(y, z)) = I(y, I(x, z)), \quad x, y, z \in [0, 1], \tag{EP}$$

it satisfies the ordering property, i.e.,

$$x \leq y \iff I(x, y) = 1, \quad x, y \in [0, 1] \tag{OP}$$

and I is right continuous with respect to the second variable.

As we see, there are two important axioms of multivalued implications above: (EP) and (OP). The characterization of t-norms, whose residuals satisfy the ordering property (OP) has been obtained by Baczyński and Jayaram [2].

Definition 1.4 (*Klement et al. [7]*). A function $F: [0, 1]^2 \rightarrow [0, 1]$ is said to be *border-continuous*, if it is continuous on the boundary of the unit square $[0, 1]^2$, i.e., on the set $[0, 1]^2 \backslash]0, 1[^2$.

Proposition 1.5 (*Baczyński and Jayaram [2, Proposition 5.8], Baczyński and Jayaram [1, Proposition 2.5.9]*). *For a t-norm T the following statements are equivalent*:

(i) *T is border-continuous.*
(ii) *I_T satisfies the ordering property* (OP).

Our main goal in this paper is to obtain a similar characterization but for the exchange principle, i.e., we want to characterize those t-norms whose residuals satisfy (EP). To see that this condition is independent from (OP), let us analyze the following examples.

Example 1.6.
(i) Consider the least t-norm, also called the drastic product, given as follows:

$$T_{\mathbf{D}}(x, y) = \begin{cases} 0 & \text{if } x, y \in [0, 1[, \\ \min(x, y) & \text{otherwise.} \end{cases}$$

Observe that it is a non-left-continuous t-norm. The R-implication generated from $T_{\mathbf{D}}$ is given by

$$I_{\mathbf{TD}}(x, y) = \begin{cases} 1 & \text{if } x < 1, \\ y & \text{if } x = 1. \end{cases}$$

$I_{\mathbf{TD}}$ (see Fig. 1(a)) satisfies (EP), but does not satisfy (OP).

(ii) Consider the non-left-continuous t-norm given in [7, Example 1.24 (i)] as follows:

$$T_{\mathbf{B}^*}(x, y) = \begin{cases} 0 & \text{if } x, y \in]0, 0.5[, \\ \min(x, y) & \text{otherwise.} \end{cases}$$

Then the R-implication generated from $T_{\mathbf{B}^*}$ is

$$I_{\mathbf{TB}^*}(x, y) = \begin{cases} 1 & \text{if } x \le y, \\ 0.5 & \text{if } x > y \text{ and } x \in [0, 0.5[, \\ y & \text{otherwise.} \end{cases}$$

Obviously, $I_{\mathbf{TB}^*}$ (see Fig. 1(b)) satisfies (OP) but not (EP), since

$$I_{\mathbf{TB}^*}(0.4, I_{\mathbf{TB}^*}(0.5, 0.3)) = 0.5,$$

while

$$I_{\mathbf{TB}^*}(0.5, I_{\mathbf{TB}^*}(0.4, 0.3)) = 1.$$

(iii) Consider now the non-left-continuous t-norm given in [7, Example 1.24 (ii)] as follows:

$$T_{\mathbf{B}}(x, y) = \begin{cases} 0 & \text{if } (x, y) \in]0, 1[^2 \backslash [0.5, 1[^2, \\ \min(x, y) & \text{otherwise.} \end{cases}$$

Then the R-implication generated from $T_{\mathbf{B}}$ is

$$I_{\mathbf{TB}}(x, y) = \begin{cases} 1 & \text{if } x \le y \text{ or } x, y \in [0, 0.5[, \\ 0.5 & \text{if } x \in [0.5, 1[\text{ and } y \in [0, 0.5[, \\ y & \text{otherwise.} \end{cases}$$

It is obvious that $I_{\mathbf{TB}}$ (see Fig. 1(c)) does not satisfy (OP). $I_{\mathbf{TB}}$ also does not satisfy (EP) since

$$I_{\mathbf{TB}}(0.8, I_{\mathbf{TB}}(0.5, 0.3)) = I_{\mathbf{TB}}(0.8, 0.5) = 0.5,$$

while

$$I_{\mathbf{TB}}(0.5, I_{\mathbf{TB}}(0.8, 0.3)) = I_{\mathbf{TB}}(0.5, 0.5) = 1.$$

(iv) Finally, consider the largest t-norm, $T_{\mathbf{M}}(x, y) = \min(x, y)$ whose residual is the Gödel implication (see Fig. 1(d))

$$I_{\mathbf{GD}}(x, y) = \begin{cases} 1 & \text{if } x \le y, \\ y & \text{if } x > y, \end{cases}$$

which satisfies both (EP) and (OP).

1.2. Left-continuity of T for (EP) of I_T: sufficient but necessary?

Left-continuity of a t-norm T is sufficient for I_T to satisfy (EP), but is not necessary, see Example 1.6(i). As another counterexample consider the non-left-continuous nilpotent minimum t-norm, which is border-continuous (see [9, p. 851]):

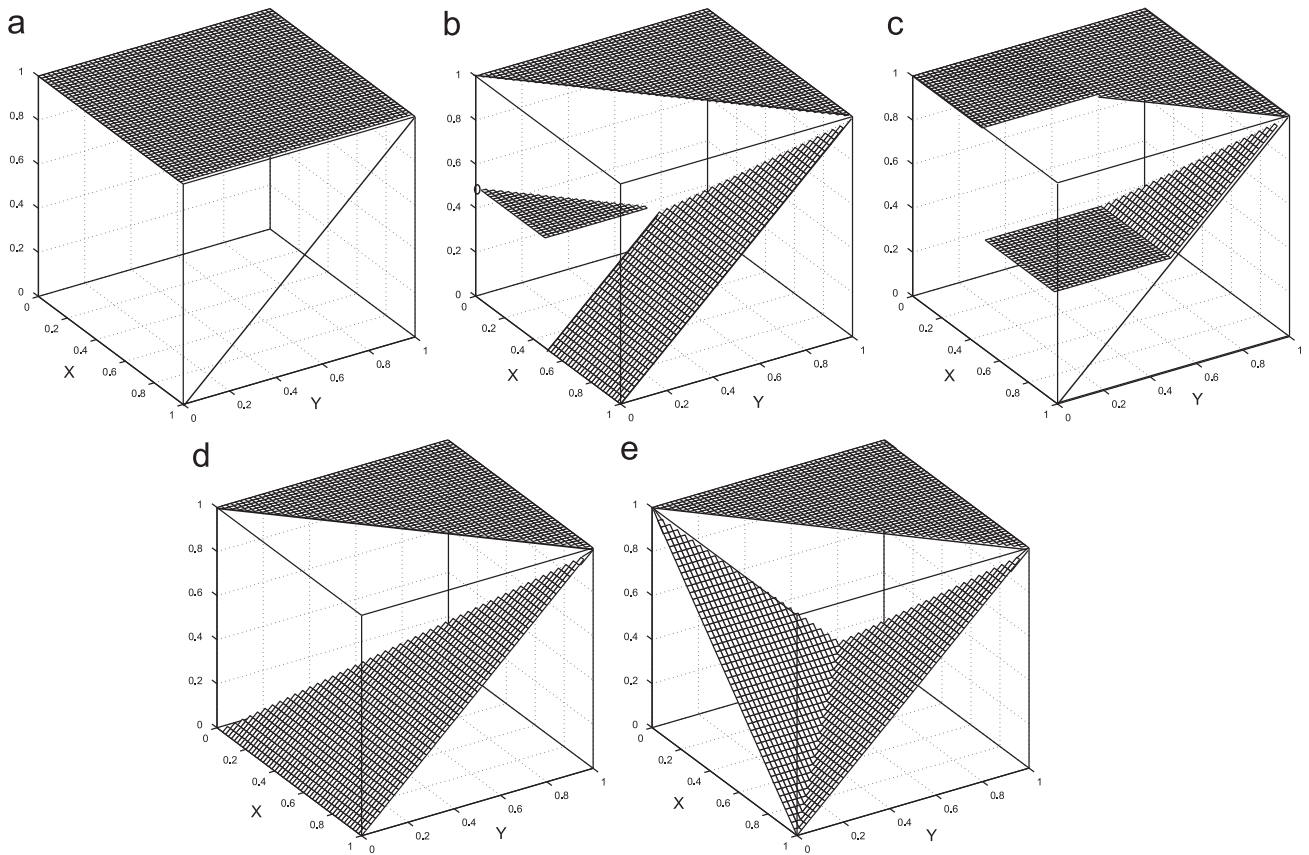$$T_{\mathbf{nM}^*}(x, y) = \begin{cases} 0 & \text{if } x + y < 1, \\ \min(x, y) & \text{otherwise.} \end{cases}$$

Fig. 1. R-implications from t-norms: border-continuity, left-continuity and the exchange principle (see Example 1.6): (a) $I_{\mathbf{TD}}$; (b) $I_{\mathbf{TB}^*}$; (c) $I_{\mathbf{TB}}$; (d) $I_{\mathbf{GD}}$; (e) $I_{\mathbf{FD}}$.

Then the R-implication generated from $T_{\mathbf{nM}^*}$ is the following Fodor implication (see Fig. 1(e)):

$$I_{\mathbf{FD}}(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ \max(1 - x, y) & \text{if } x > y, \end{cases}$$

which satisfies both (EP) and (OP). This leads us to the following natural question:

> *What is(are) the most general condition(s) on t-norm T to ensure that I_T has* (EP)?

In this work, we take up this study and present a complete characterization of the class of t-norms whose residuals satisfy (EP). Towards this end, we firstly partition the class of t-norms into those that are border-continuous and those that are not and deal with each of them separately.

2. Preliminaries

We assume that the reader is familiar with the classical results concerning basic fuzzy logic connectives, but to make this work more self-contained, we introduce some notations used in the text and we briefly mention some of the concepts and results employed in the rest of the work.

Definition 2.1 (*see Baczyński and Jayaram [1]*). A function $I: [0, 1]^2 \rightarrow [0, 1]$ is called a fuzzy implication if it satisfies the following conditions:

I is non-increasing in the first variable,   (I1)

I is non-decreasing in the second variable,   (I2)

$I(0, 0) = 1, \quad I(1, 1) = 1, \quad I(1, 0) = 0.$ (I3)

The set of all fuzzy implications will be denoted by $\mathcal{FI}$.

Proposition 2.2 (*see Jayaram and Mesiar [5, Theorem 7.6]*). *If a function $F : [0, 1]^2 \to [0, 1]$ is border-continuous, commutative, monotonic non-decreasing with neutral element 1, then the residual $I_F \in \mathcal{FI}$, it satisfies* (OP) *and it is right-continuous with respect to the second variable.*

Proof. By [5, Theorem 7.6] it is enough to show that I_F is right-continuous with respect to the second variable. On the contrary, let us assume that F is not right-continuous with respect to the second variable for some point $(x_0, y_0) \in [0, 1] \times [0, 1[$. We know that $I_F \in \mathcal{FI}$, so F is non-decreasing with respect to the second variable. Therefore there exist $\varepsilon > 0$ and decreasing sequence $(y_n)_{n \in \mathbb{N}}$ in $]y_0, 1]$ such that $\lim_{n \to \infty} y_n = y_0$ and

$$I_F(x_0, y_n) > I_F(x_0, y_0) + \varepsilon \quad \text{for all } n \in \mathbb{N}.$$

By the definition of I_F we get

$$\sup\{t \in [0, 1] | F(x_0, t) \le y_n\} > I_F(x_0, y_0) + \varepsilon \quad \text{for all } n \in \mathbb{N},$$

thus

$$F(x_0, I_F(x_0, y_0) + \varepsilon) \le y_n \quad \text{for all } n \in \mathbb{N}.$$

In the limit $n \to \infty$ we get

$$F(x_0, I_F(x_0, y_0) + \varepsilon) \le y_0,$$

hence

$$F(x_0, y_0) \ge I_F(x_0, y_0) + \varepsilon,$$

which is a contradiction. $\square$

Definition 2.3 (*Gottwald [4]*). Two functions $F, G : [0, 1]^2 \to [0, 1]$ form an adjoint pair if they satisfy the residuation property, i.e.,

$$F(x, z) \le y \iff G(x, y) \ge z, \quad x, y, z \in [0, 1]. \tag{RP}$$

Theorem 2.4 (*cf. Baczyński and Jayaram [1, Proposition 2.5.2,Theorem 2.5.7], Król [8, Theorems 2, 5]*). *If M is a left-continuous t-subnorm, then*

  (i) $I_M(x, y) = \max\{t \in [0, 1] | M(x, t) \le y\}$, *for all* $x, y \in [0, 1]$,
  (ii) *M and I_M form an adjoint pair*,
 (iii) *I_M satisfies* (EP).

Theorem 2.5 (*Baczyński and Jayaram [1, Theorem 2.5.14]*). *If a function $I : [0, 1]^2 \to [0, 1]$ satisfies* (EP), (OP) *and is both monotonic non-decreasing and right-continuous with respect to the second variable, then T_I defined as below*:

$$T_I(x, y) = \min\{t \in [0, 1] | I(x, t) \ge y\}, \quad x, y \in [0, 1],$$

is a left-continuous t-norm, where the right side exists for all $x, y \in [0, 1]$.

Definition 2.6. Let $F : [0, 1]^2 \to [0, 1]$ be monotonic non-decreasing and commutative. Then the function $F^* : [0, 1]^2 \to [0, 1]$ defined as below:

$$F^*(x, y) = \begin{cases} \sup\{F(u, v) | u < x, v < y\} & \text{if } x, y \in]0, 1[, \\ F(x, y) & \text{otherwise,} \end{cases} \tag{1}$$

is called the *conditionally left-continuous completion* of F.

Lemma 2.7. *If $F : [0, 1]^2 \to [0, 1]$ is monotonic non-decreasing and commutative, then the function F^* as defined in* (1) *is monotonic non-decreasing and commutative.*

Proof. By the monotonicity of F we have

$$F^*(x, y) = \begin{cases} F(x^-, y^-) & \text{if } x, y \in]0, 1[, \\ F(x, y) & \text{otherwise,} \end{cases}$$

for any $x, y \in [0, 1]$, where the value $F(x^-, y^-)$ denotes the left-hand limit. Clearly, $F^*(x, y) = F^*(y, x)$ and F^* is monotonic non-decreasing. $\square$

Remark 2.8. Let T be a t-norm.

(i) T^* is monotonic non-decreasing, commutative, it has 1 as its neutral element and $T^*(0, 0) = 0$, so it is a fuzzy conjunction in the sense of Fodor and Keresztfalvi [3] (see also [8]).

(ii) If T is border-continuous, then T^* is left-continuous (in particular it is also border-continuous).

(iii) One can easily check that I_{T^*} is a fuzzy implication.

(iv) By the monotonicity of T we have $T^* \leq T$ and hence $I_{T^*} \geq I_T$.

(v) If $x \leq y$, then $I_{T^*}(x, y) = I_T(x, y) = 1$.

(vi) Also, if $x = 1$, then by the neutrality of T we have $I_{T^*}(1, y) = I_T(1, y) = y$.

(vii) In general T^* may not be left-continuous. For example when $T = T_D$, the drastic t-norm, then $T^* = T$, but T_D is not left-continuous. This explains why T^* is called the *conditionally left-continuous completion* of T. Further, T^* may not satisfy the associativity (see Example 2.9).

Example 2.9.

(i) Consider the following non-left continuous but border-continuous Viceník t-norm (see [14,15]) given by the formula

$$T_{VC}(x, y) = \begin{cases} 0.5 & \text{if } \min(x, y) \geq 0.5 \text{ and } x + y \leq 1.5, \\ \max(x + y - 1, 0) & \text{otherwise.} \end{cases}$$

Then the conditionally left-continuous completion of T_{VC} is given by

$$T_{VC}^*(x, y) = \begin{cases} 0.5 & \text{if } \min(x, y) > 0.5 \text{ and } x + y < 1.5, \\ \max(x + y - 1, 0) & \text{otherwise.} \end{cases}$$

One can easily check that T_{VC}^* is not a t-norm since it is not associative. Indeed, we have

$$T_{VC}^*(0.55, T_{VC}^*(0.95, 0.95)) = 0.5,$$

while

$$T_{VC}^*(T_{VC}^*(0.55, 0.95), 0.95) = 0.45.$$

For the plots of both the functions see Fig. 2.

(ii) Consider the non-left-continuous nilpotent minimum t-norm T_{nM^*}, which is border-continuous. Then the conditionally left-continuous completion of T_{nM^*} is the left-continuous nilpotent minimum t-norm (see [7, Remark 1.21]), given by

$$T_{nM}(x, y) = \begin{cases} 0 & \text{if } x + y \leq 1, \\ \min(x, y) & \text{otherwise.} \end{cases}$$

Definition 2.10 (*cf. Jenei [6, Definition 5.7.2]*). A monotonic non-decreasing, commutative and associative function $F: [0, 1]^2 \to [0, 1]$ is said to satisfy the (*CLCC-A*)-*property*, if its conditionally left-continuous completion F^*, as defined by (1), is associative.

3. Border-continuous t-norms

This section contains the main contribution of this work. We consider the class of border-continuous t-norms and determine its sub-class whose residuals satisfy (EP). Note that the t-norm T_{B^*} in Example 1.6(ii) is a border-continuous but non-left-continuous t-norm whose residual does not satisfy (EP).
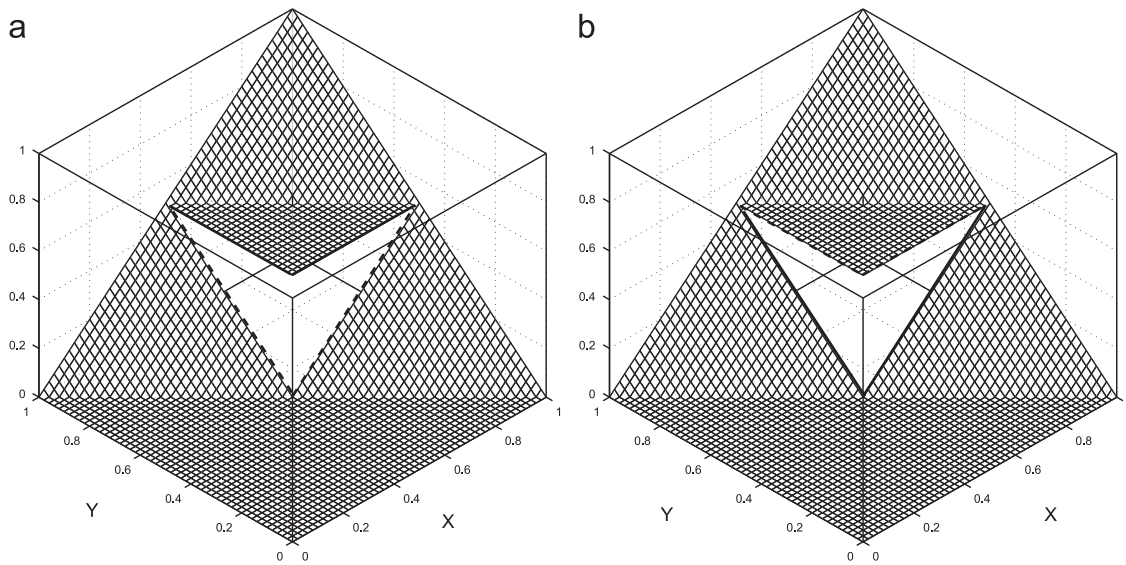
Fig. 2. The Viceník t-norm $T_{\mathbf{VC}}$ and its conditionally left-continuous completion $T^*_{\mathbf{VC}}$ (see Example 2.9): (a) $T_{\mathbf{VC}}$; (b) $T^*_{\mathbf{VC}}$.

Lemma 3.1. *Let T be a border-continuous t-norm such that I_T satisfy* (EP). *Then $I_T = I_{T^*}$.*

Proof. From formula for T^* and Remark 2.8 we know that $I_T(x, y) = I_{T^*}(x, y)$ when $x \leq y$ or $(x, y) \in [0, 1]^2 \setminus]0, 1[^2$. Therefore assume that there exist $x_0, y_0 \in]0, 1[$ such that $x_0 > y_0$ and

$$\beta = I_{T^*}(x_0, y_0) > I_T(x_0, y_0) = \alpha.$$

Since T^* is left-continuous we have that $\beta = I_{T^*}(x_0, y_0) \Longrightarrow T^*(x_0, \beta) \leq y_0$. Thus, $\beta < 1$ and for every $\delta \in (\alpha, \beta)$ we have

$$y_0 \geq T^*(x_0, \beta) = T(x_0^-, \beta^-) \geq T(x_0^-, \delta) \geq T(x_0^-, \alpha). \tag{2}$$

Fix arbitrarily $\delta \in (\alpha, \beta)$. Now, we have two cases:

1. $\alpha \in \{t \mid T(x_0, t) \leq y_0\}$, in which case

$$T(x_0, \alpha) \leq y_0 < T(x_0, \delta).$$

2. $\alpha \notin \{t \mid T(x_0, t) \leq y_0\}$, in which case

$$T(x_0, \alpha^-) \leq y_0 < T(x_0, \alpha) \leq T(x_0, \delta).$$

From (2) and any of the above two cases we have

$$(T(x_0^-, \delta) \leq y_0 < T(x_0, \delta)) \Longrightarrow (I_T(\delta, y_0) = \sup\{t \mid T(\delta, t) \leq y_0\} = x_0).$$

Now, since I_T satisfies (EP) and (OP) we get

$$I_T(x_0, I_T(\delta, y_0)) = I_T(x_0, x_0) = 1 = I_T(\delta, I_T(x_0, y_0)) = I_T(\delta, \alpha),$$

thus $\delta \leq \alpha$, by (OP); a contradiction. Hence $\beta = I_{T^*}(x_0, y_0) = I_T(x_0, y_0) = \alpha$. $\square$

Lemma 3.2. *Let T be a border-continuous t-norm such that I_T satisfy* (EP). *Then T satisfies the* (CLCC-A)-*property, i.e., its conditionally left-continuous completion T^* is associative.*

Proof. To prove the associativity of T^* it is enough to show that T^* is equal to the t-norm $T_{I_{T^*}}$ obtained from its residual I_{T^*}. Let us define

$$T_{I_{T^*}}(x, y) = \inf\{t \mid I_{T^*}(x, t) \geq y\}, \quad x, y \in [0, 1]. \tag{3}$$

Since T^* is border-continuous, by Proposition 2.2 we see that I_{T^*} satisfies (OP) and it is right-continuous with respect to the second variable. By Lemma 3.1, we obtain that $I_T = I_{T^*}$ and hence I_{T^*} satisfies (EP). Thus, by Theorem 2.5, we get the fact that $T_{I_{T^*}}$ is a left-continuous t-norm. Finally, observe that by Remark 2.8 T^* fulfills assumptions on a conjunction in [8, Theorem 13], so $T^* = T_{I_{T^*}}$, which ends the proof. $\square$

Theorem 3.3. *For a border-continuous t-norm T the following statements are equivalent*:

(i) I_T *satisfies* (EP).
(ii) T *satisfies the* (CLCC-A)*-property (i.e., T^* is a associative), and $I_T = I_{T^*}$.*

Proof. (i) $\Longrightarrow$ (ii): Follows from Lemmas 3.2 and 3.1.

(ii) $\Longrightarrow$ (i): If T satisfies the (CLCC-A)-property, then T^* is a left-continuous t-norm. Therefore I_{T^*} satisfies (EP). But $I_T = I_{T^*}$, so I_T also satisfies (EP). $\square$

Based on the obtained results we are able to present the characterization of t-norms, whose residuals satisfy both the exchange principle and the ordering property.

Corollary 3.4. *For a t-norm T the following statements are equivalent*:

(i) I_T *satisfies* (EP) *and* (OP).
(ii) T *is border-continuous, satisfies the* (CLCC-A)*-property and $I_T = I_{T^*}$.*

4. Ordinal sums of t-norms

In this section, we study the above problem, that of determining the necessary and sufficient conditions for a T such that its residual satisfies (EP), but for t-norms T that have an ordinal sum representation. Towards this end, we firstly, determine the formula for the residual generated from the t-norm T whose ordinal summands are also t-norms.

Just as there exists a complete representation of continuous t-norms in terms of an ordinal sum representation (see [7, Theorem 5.11]), the following representation of left-continuous t-norms as the ordinal sum of t-subnorms can be given.

Theorem 4.1 (*Mesiar and Mesiarová [10, Theorem 1]*). *A function $T:[0, 1]^2 \to [0, 1]$ is a left-continuous t-norm if and only if there exist a family of pairwise disjoint open sub-intervals $\{]\alpha_k, \beta_k[\}_{k \in \mathcal{K}}$ of $[0,1]$ and a family of left-continuous t-subnorms $(M_k)_{k \in \mathcal{K}}$ such that if either $\beta_k = 1$ for some $k \in \mathcal{K}$ or $\beta_k = \alpha_{k^*}$ for some $k, k^* \in \mathcal{K}$ and M_{k^*} has zero-divisors, then M_k is a t-norm, so that*

$$T(x, y) = \begin{cases} \alpha_k + (\beta_k - \alpha_k) \cdot M_k \left(\dfrac{x - \alpha_k}{\beta_k - \alpha_k}, \dfrac{y - \alpha_k}{\beta_k - \alpha_k} \right) & \text{if } x, y \in]\alpha_k, \beta_k], \\ \min(x, y) & \text{otherwise.} \end{cases}$$

Theorem 4.2 (*Mesiar and Mesiarová [10, Theorem 5]*). *If T is a left-continuous t-norm with the ordinal sum structure as given in Theorem 4.1, then*

$$I_T(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ \alpha_k + (\beta_k - \alpha_k) \cdot I_{M_k} \left(\dfrac{x - \alpha_k}{\beta_k - \alpha_k}, \dfrac{y - \alpha_k}{\beta_k - \alpha_k} \right) & \text{if } \alpha_k < y < x \leq \beta_k, \\ y & \text{otherwise,} \end{cases}$$

$$= \begin{cases} \alpha_k + (\beta_k - \alpha_k) \cdot I_{M_k} \left(\dfrac{x - \alpha_k}{\beta_k - \alpha_k}, \dfrac{y - \alpha_k}{\beta_k - \alpha_k} \right) & \text{if } \alpha_k < y < x \leq \beta_k, \\ I_{\mathbf{GD}}(x, y) & \text{otherwise,} \end{cases}$$

where $I_{\mathbf{GD}}$ is the Gödel implication (see Example 1.6(iv)).

Obviously, I_T given in Theorem 4.2 satisfies (EP) and thus the formula in Theorem 4.1 can be used as a construction method for t-norms yielding residual implications possessing (EP). This method of construction (based on left-continuous triangular subnorms) of t-norms for which the residual implication satisfies (EP) can be further generalized, not requiring the left-continuity of single summands in the ordinal sum. We show such a generalization considering t-norms summands only (i.e., we will deal with ordinal sums of t-norms only).

Theorem 4.3 (*Klement et al. [7, Theorem 3.43]*). *Let $(T_k)_{k \in \mathcal{K}}$ be a family of t-norms and $\{]\alpha_k, \beta_k[\}_{k \in \mathcal{K}}$ be a family of non-empty, pairwise disjoint open subintervals of* $[0,1]$. *Then the following function is a t-norm*:

$$T(x, y) = \begin{cases} \alpha_k + (\beta_k - \alpha_k) \cdot T_k \left(\dfrac{x - \alpha_k}{\beta_k - \alpha_k}, \dfrac{y - \alpha_k}{\beta_k - \alpha_k} \right) & \textit{if } x, y \in]\alpha_k, \beta_k], \\ \min(x, y) & \textit{otherwise}, \end{cases} \quad x, y \in [0, 1]. \tag{4}$$

The t-norm T defined as above is called the ordinal sum of the summands $\{]\alpha_k, \beta_k[\}_{k \in \mathcal{K}}$ and we shall write $T = (\langle \alpha_k, \beta_k, T_k \rangle)_{k \in \mathcal{K}}$.

Proposition 4.4. *Let $(T_k)_{k \in \mathcal{K}}$ be a family of t-norms and let $T = (\langle \alpha_k, \beta_k, T_k \rangle)_{k \in \mathcal{K}}$ be an ordinal sum t-norm as given in Theorem* 4.3. *Then*

$$I_T(x, y) = \begin{cases} \alpha_k + (\beta_k - \alpha_k) \cdot I_{T_k} \left(\dfrac{x - \alpha_k}{\beta_k - \alpha_k}, \dfrac{y - \alpha_k}{\beta_k - \alpha_k} \right) & \textit{if } \alpha_k \leq y < x \leq \beta_k, \\ I_{\mathbf{GD}}(x, y) & \textit{otherwise}, \end{cases} \quad x, y \in [0, 1].$$

Proof. Let us fix arbitrarily $x, y \in [0, 1]$. It is obvious that if $x \leq y$, then $I_T(x, y) = 1 = I_{\mathbf{GD}}(x, y)$. Therefore let us assume that $x > y$ and we consider several cases now.

- If $\alpha_k \leq y < x \leq \beta_k$, for some $k \in \mathcal{K}$, then from the representation (4) of a t-norm T, for any $t \in [0, 1]$ we have the following subcases.
  1. If $t > \beta_k$, then $T(x, t) = \min(x, t) = x > y$.
  2. If $t < \alpha_k$, then $T(x, t) = \min(x, t) = t < \alpha_k \leq y$, thus

  $$\sup\{t \in [0, \alpha_k] | T(x, t) \leq y\} = \sup\{t \in [0, \alpha_k] | t \leq y\} = \alpha_k.$$

  3. If $\alpha_k \leq t \leq \beta_k$, then

  $$\sup\{t \in [\alpha_k, \beta_k] | T(x, t) \leq y\} = \sup\left\{ t \in [\alpha_k, \beta_k] | \alpha_k + (\beta_k - \alpha_k) \cdot T_k \left(\dfrac{x - \alpha_k}{\beta_k - \alpha_k}, \dfrac{t - \alpha_k}{\beta_k - \alpha_k} \right) \leq y \right\}$$

  $$= \sup\left\{ t \in [\alpha_k, \beta_k] | T_k \left(\dfrac{x - \alpha_k}{\beta_k - \alpha_k}, \dfrac{t - \alpha_k}{\beta_k - \alpha_k} \right) \leq \dfrac{y - \alpha_k}{\beta_k - \alpha_k} \right\}$$

  $$= \alpha_k + (\beta_k - \alpha_k) \cdot I_{T_k} \left(\dfrac{x - \alpha_k}{\beta_k - \alpha_k}, \dfrac{y - \alpha_k}{\beta_k - \alpha_k} \right) \geq \alpha_k.$$

  Hence

  $$I_T(x, y) = \alpha_k + (\beta_k - \alpha_k) \cdot I_{T_k} \left(\dfrac{x - \alpha_k}{\beta_k - \alpha_k}, \dfrac{y - \alpha_k}{\beta_k - \alpha_k} \right)$$

  in this case.
- If $\alpha_k \leq x \leq \beta_k$, for some $k \in \mathcal{K}$ and $y < \alpha_k$, then again from the representation 4 of a t-norm T, for any $t \in [0, 1]$ we have the following subcases.
  1. If $t > \beta_k$, then $T(x, t) = \min(x, t) = x > y$.
  2. If $t < \alpha_k$, then $T(x, t) = \min(x, t) = t$, thus

  $$\sup\{t \in [0, \alpha_k] | T(x, t) \leq y\} = \sup\{t \in [0, \alpha_k] | t \leq y\} = y.$$

3. If $\alpha_k \leq t \leq \beta_k$, then

$$T(x, t) = \alpha_k + (\beta_k - \alpha_k) \cdot T_k \left(\frac{x - \alpha_k}{\beta_k - \alpha_k}, \frac{t - \alpha_k}{\beta_k - \alpha_k} \right) \geq \alpha_k > y$$

so in this case does not exist any t such that $T(x, t) \leq y$.

From the above discussion we get

$$I_T(x, y) = y = I_{\mathbf{GD}}(x, y)$$

in this case.

- If x does not belong to any closed interval $[\alpha_k, \beta_k]$ for $k \in \mathcal{K}$, then $T(x, t) = \min(x, t)$ for any $t \in [0, 1]$. Since $y < x$, in this case we also obtain

$$I_T(x, y) = \sup\{t \in [0, 1] | T(x, t) \leq y\} = \sup\{t \in [0, 1] | \min(x, t) \leq y\} = y = I_{\mathbf{GD}}(x, y). \qquad \square$$

Firstly, we consider t-norms obtained as an ordinal sum with a single summand.

Theorem 4.5. *Let T be a t-norm and let* $V = (\langle a, b, T \rangle)$. *Then the following statements are equivalent*:

(i) I_V *satisfies* (EP).
(ii) I_T *satisfies* (EP) *and if* $b < 1$ *then T is border-continuous*.

Proof. Let $V = (\langle a, b, T \rangle)$ be an ordinal sum t-norm with a unique summand t-norm T.

(i) $\Longrightarrow$ (ii): We consider two cases: $b = 1$ and $b < 1$.

- Suppose that $b = 1$. Then by Proposition 4.4 we have

$$I_V(x, y) = \begin{cases} a + (1 - a) \cdot I_T \left(\dfrac{x - a}{1 - a}, \dfrac{y - a}{1 - a} \right) & \text{if } a \leq y < x \leq 1, \\ I_{\mathbf{GD}}(x, y) & \text{otherwise,} \end{cases} \qquad x, y \in [0, 1]. \qquad (5)$$

Let us fix arbitrarily $x, y, z \in [0, 1]$. We will show that I_T satisfies (EP) by considering the following cases:

○ if $x \leq z$, then clearly, $I_T(x, z) = 1$ and since $I_T(p, q) \geq q$ for any $p, q \in [0, 1]$ and any t-norm T, from the monotonicity of I_T with respect to the second variable we have

$$1 \geq I_T(x, I_T(y, z)) \geq I_T(x, z) = 1 = I_T(y, 1) = I_T(y, I_T(x, z)),$$

so I_T satisfies (EP) in this case.

○ if $y \leq z$, then the proof is similar to above.

○ if $x > z$ and $y > z$, then let us define $x_0 = a + (1 - a)x$, $y_0 = a + (1 - a)y$ and $z_0 = a + (1 - a)z$. Observe that $x_0, y_0, z_0 \in [a, 1]$ and $x_0 > z_0$ and $y_0 > z_0$. Now observe that

$$I_V(y_0, z_0) < x_0 \Longleftrightarrow I_V(x_0, z_0) < y_0.$$

To see the above, let us assume that $I_V(x_0, z_0) < y_0$ but $I_V(y_0, z_0) \geq x_0$. From (5) we get $I_V(x_0, I_V(y_0, z_0)) = 1 = I_V(y_0, I_V(x_0, z_0))$, since I_V satisfies (EP). However, once again the definition of I_V in (5) now implies that $y_0 \leq I_V(x_0, z_0)$, a contradiction. The reverse implication can be shown similarly.

Thus, on the one hand, if $x_0 \leq I_V(y_0, z_0)$, then from (5) we see that also $x \leq I_T(y, z)$ and similarly, $y_0 \leq I_V(x_0, z_0)$ implies that $y \leq I_T(x, z)$. From these two inequalities, we get

$$I_T(x, I_T(y, z)) = 1 = I_T(y, I_T(x, z)),$$

thus I_T satisfies (EP) in this subcase.

On the other hand, if $x_0 > I_V(y_0, z_0) \geq z_0 \geq a$, then from (5) we get

$$I_V(x_0, I_V(y_0, z_0)) = a + (1 - a) \cdot I_T \left(\frac{x_0 - a}{1 - a}, \frac{I_V(y_0, z_0) - a}{1 - a} \right)$$

$$= a + (1 - a) \cdot I_T \left(\frac{x_0 - a}{1 - a}, \frac{a + (1 - a)I_T \left(\frac{y_0 - a}{1 - a}, \frac{z_0 - a}{1 - a} \right) - a}{1 - a} \right)$$

$$= a + (1 - a) \cdot I_T \left(\frac{x_0 - a}{1 - a}, I_T \left(\frac{y_0 - a}{1 - a}, \frac{z_0 - a}{1 - a} \right) \right)$$

$$= a + (1 - a) \cdot I_T(x, I_T(y, z)).$$

Similarly, we have

$$I_V(y_0, I_V(x_0, z_0)) = a + (1 - a) \cdot I_T(y, I_T(x, z)).$$

Since I_V satisfies (EP), we easily get that I_T also satisfies (EP) in this subcase.

- Suppose now that $b < 1$. Then

$$I_V(x, y) = \begin{cases} a + (b - a) \cdot I_T \left(\frac{x - a}{b - a}, \frac{y - a}{b - a} \right) & \text{if } a \leq y < x \leq b, \\ I_{\mathbf{GD}}(x, y) & \text{otherwise,} \end{cases} \quad x, y \in [0, 1].$$

○ If T is not border continuous, then from Proposition 1.5 we get that I_T does not satisfy (OP). Therefore there exist $0 \leq u < v \leq 1$ such that $I_T(v, u) = 1$. Let us take $x = a + (b - a)v$ and $z = a + (b - a)u$. Then $a \leq z < x \leq b$ and

$$I_V(x, z) = a + (b - a) \cdot I_T \left(\frac{x - a}{b - a}, \frac{y - a}{b - a} \right)$$

$$= a + (b - a) \cdot I_T(v, u)$$

$$= b.$$

Thus, on the one hand we get

$$I_V(b, I_V(x, z)) = I_V(b, b) = 1.$$

On the other hand I_T satisfies the neutral property (see Theorem 2.5.4 in [1]), so

$$I_V(b, z) = a + (b - a) \cdot I_T \left(\frac{b - a}{b - a}, \frac{z - a}{b - a} \right)$$

$$= a + (b - a) \cdot I_T \left(1, \frac{z - a}{b - a} \right)$$

$$= a + (b - a) \frac{z - a}{b - a} = z,$$

and thus

$$I_V(x, I_V(b, z)) = I_V(x, z) = b < 1,$$

and hence I_V does not satisfy (EP); a contradiction.

○ If T is border continuous but I_T does not satisfy (EP), then there exist $x_0, y_0, z_0 \in [0, 1]$ such that

$$I_T(x_0, I_T(y_0, z_0)) \neq I_T(y_0, I_T(x_0, z_0)). \tag{6}$$

By the properties of R-implications (see [1, Theorem 2.5.4]), one can easily check that these numbers must satisfy the following inequalities:

$$0 \leq z_0 < x_0 < 1 \quad \text{and} \quad 0 \leq z_0 < y_0 < 1.$$

Indeed, due to the neutrality principle, see Remark 2.8(vi), if $\max(x_0, y_0) = 1$, then LHS(6) = RHS(6) = $I_T(\min(x_0, y_0), z)$. Moreover, if $\min(x_0, y_0) = 0$ or $z_0 = 1$, then evidently LHS(6) = RHS(6) = 1. Now it is

enough to choose $x = a + (b - a)x_0$, $y = a + (b - a)y_0$ and $z = a + (b - a)z_0$ to violate (EP), leading to a contradiction. Of course $z < x$ and $z < y$ and $x, y \in]a, b[$, so by the definition of I_V we see that

$$a \leq I_V(y, z) = a + (b - a)I_T \left(\frac{y - a}{b - a}, \frac{z - a}{b - a} \right) < b,$$

and similarly, $a \leq I_V(x, z) < b$. Now observe that

$$x \leq I_V(y, z) \iff y \leq I_V(x, z).$$

Indeed, let $x \leq I_V(y, z)$. Then, by the definition of I_V and since I_V satisfies (EP), we have $I_V(x, I_V(y, z)) = 1 = I_V(y, I_V(x, z))$. Note that V is border continuous as it is ordinal sum without summand boundary $b = 1$, and thus we have (OP) of I_V. Hence $y \leq I_V(x, z)$.

Thus, on the one hand if $x \leq I_V(y, z)$, then, from the definition of I_V, we obtain

$$
\begin{aligned}
x &= a + (b - a)x_0 \leq a + (b - a)I_T \left(\frac{y - a}{b - a}, \frac{z - a}{b - a} \right) \\
&\implies x_0 \leq I_T(y_0, z_0) \\
&\implies I_T(x_0, I_T(y_0, z_0)) = 1.
\end{aligned}
$$

Similarly, from $y \leq I_V(x, z)$ we get $I_T(y_0, I_T(x_0, z_0)) = 1 = I_T(x_0, I_T(y_0, z_0))$, contradicting (6) in this subcase. On the other hand, if $x > I_V(y, z)$, then from the definition of I_V, we obtain

$$
I_V(x, I_V(y, z)) = a + (b - a)I_T \left(\frac{x - a}{b - a}, \frac{a + (b - a)I_T \left(\frac{y - a}{b - a}, \frac{z - a}{b - a} \right) - a}{b - a} \right)
$$
$$
= a + (b - a)I_T(x_0, I_T(y_0, z_0)).
$$

Similarly

$$I_V(y, I_V(x, z)) = a + (b - a)I_T(y_0, I_T(x_0, z_0)).$$

Once again, by (EP) of I_V, we get $I_T(x_0, I_T(y_0, z_0)) = I_T(y_0, I_T(x_0, z_0))$, contradicting (6).

(ii) $\implies$ (i): Let us fix arbitrarily $x, y, z \in [0, 1]$. We will show that I_V satisfies (EP) by considering the following cases:

- if $x \leq z$ or $y \leq z$, then it is obvious that I_V satisfies (EP).
- if $x > z$ and $y > z$ and $z < a$ or $z > b$, then (EP) holds because it is valid for the Gödel implication $I_{\mathbf{GD}}$. Indeed, we get

$$I_V(x, I_V(y, z)) = I_V(x, z) = z = I_V(y, z) = I_V(y, I_V(x, z)).$$

- if $x > b$ and $y > b$ and $a \leq z \leq b$, then similarly as above we get

$$I_V(x, I_V(y, z)) = I_V(x, z) = z = I_V(y, z) = I_V(y, I_V(x, z)).$$

- if $x > b$ and $a \leq z < y \leq b$, then $I_V(y, z) \in [a, b]$ and hence

$$I_V(x, I_V(y, z)) = I_{\mathbf{GD}}(x, I_V(y, z)) = I_V(y, z) = I_V(y, I_V(x, z)).$$

- if $y > b$ and $a \leq z < x \leq b$, then $I_V(x, z) \in [a, b]$ and similarly as above

$$I_V(y, I_V(x, z)) = I_{\mathbf{GD}}(y, I_V(x, z)) = I_V(x, z) = I_V(x, I_V(y, z)).$$

- if $a \leq z < x \leq b$ and $a \leq z < y \leq b$, then (EP) is valid because of (EP) of I_T. $\square$

A generalization of the above result to t-norms with countable ordinal summands is straight-forward.

Corollary 4.6. *Let* $T = (\langle \alpha_k, \beta_k, T_k \rangle)_{k \in \mathcal{K}}$ *be an ordinal sum t-norm. Then the following statements are equivalent*:

(i) I_T *satisfies* (EP).
(ii) *For every* $k \in \mathcal{K}$, I_{T_k} *satisfies* (EP) *and either* T_k *is border-continuous or* $\beta_k = 1$.

Proof. (i) $\Longrightarrow$ (ii): As per the above notation, the necessity goes similarly as in Theorem 4.5—if $\beta_k = 1$ and I_{T_k} does not satisfy (EP), then we show that also I_T does not satisfy (EP). In all other cases, if T_k is not border continuous, we show that (EP) does not hold for I_T, and if T_k is border continuous but I_T does not satisfy (EP), then again we show the violation of (EP) for I_T.

(ii) $\Longrightarrow$ (i): The sufficiency will be shown based on $z \in [0, 1]$.

- If z is not in any of $[\alpha_k, \beta_k]$, then (EP) holds because I_T holds for $I_{\mathbf{GD}}$.
- If $\alpha_k \leq z \leq \beta_k$ for some $k \in \mathcal{K}$, then it is necessary to check the case $z < x$ and $z < y$ only; we repeat the arguments above, namely:
  - If $x > \beta_k$ and $y > \beta_k$, then (EP) turns into $z = z$.
  - If $x > \beta_k$ and $y \leq \beta_k$, then $I_T(y, z) \in [\alpha_k, \beta_k]$ and (EP) turns into $I_T(y, z) = I_T(y, z)$.
  - If $y > \beta_k$ and $x \leq \beta_k$, then similarly as above (EP) turns into $I_T(x, z) = I_T(x, z)$.
  - If $x \leq \beta_k$ and $y \leq \beta_k$, then (EP) is valid for I_T, because of (EP) of I_{T_k}. $\square$

5. Concluding remarks

In this work we have studied the problem of characterizing t-norms whose residuals satisfy the exchange principle. We have presented a complete characterization of two classes of t-norms whose residuals satisfy the exchange principle, viz., the border-continuous t-norms and those t-norms that have an ordinal sum representation. The study reveals that the concept of conditionally left-continuous completion of a t-norm plays an important role. In fact, it can be seen that unless a t-norm can be embedded into a left-continuous t-norm, in some rather precise manner as presented in the work, its residual does not satisfy the exchange principle.

Acknowledgments

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