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Solution to an open problem: a characterization of conditionally cancellative t-subnorms

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Abstract. In this work we solve an open problem of U. Höhle (Klement et al. Fuzzy Sets Syst 145:471–479, 2004, Problem 11). We show that the solution gives a characterization of all conditionally cancellative t-subnorms. Further, we give an equivalence condition under which a conditionally cancellative t-subnorm has 1 as its neutral element and hence show that conditionally cancellative t-subnorms whose natural negations are strong are, in fact, t-norms.

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1. Introduction

"Triangular norms are, on the one hand, special semigroups and, on the other hand, solutions of some functional equations. This mixture quite often requires new approaches to answer questions about the nature of triangular norms." With this observation, Klement et al. [10], present a collection of open problems posed during the 24th Linz Seminar on fuzzy set theory. They deal with unsolved problems (as of then) related to fuzzy aggregation operations, especially t-norms and t-subnorms. For other collections of open problems related to triangular norms and related operators, see [11,13]

Solutions to these open problems involving triangular norms are of both theoretical and applicational interest. Recently solutions to quite a few open problems in the field of triangular norms have been published [2,4,8,16,19,20,23], including this journal, see for instance, [7,15,22,21]. Since the publication of [10], some problems mentioned therein have been solved—for instance, Problem **1** was solved by Ouyang et al. [18], Problem **5** was solved by Ouyang and

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Li [17] while for some other problems partial solutions have been given, see for instance, the papers of Viceník [24–26] relating to Problem **4**(i).

One of the open problems listed therein was posed by U. Höhle (Problem **11**) which reads as follows:

Problem 1.1 (U. Höhle, [10], Problem **11**). *Characterize all left-continuous t-norms T which satisfy*

$$I(x, T(x, y)) = \max(n(x), y), \quad x, y \in [0, 1], \tag{1.1}$$

where I is the residual operator linked to T, i.e.,

$$I(x, y) = \sup\{t \in [0, 1] | T(x, t) \leq y\}, \quad x, y \in [0, 1]\ , \tag{1.2}$$

$$n(x) = n_T(x) = I(x, 0) \quad \textit{for all } x \in [0, 1]. \tag{1.3}$$

Further, U. Höhle goes on to remark the following:

Remark 1.2. "In the class of continuous t-norms, only nilpotent t-norms fulfill the above property."

In this work we deal with two problems. Firstly, we solve the above open problem of U. Höhle and show that the solution gives a characterization of all conditionally cancellative t-subnorms. From the proven result it does follow that the remark of U. Höhle—Remark 1.2—is not always true and gives an equivalence condition for it to be true, viz., that the natural negation obtained from the t-norm is strong.

Secondly, this quite naturally leads us to consider conditionally cancellative t-subnorms whose natural negations are involutive. Once again, by proving an equivalence condition for a conditionally cancellative t-subnorm to be a t-norm, we show that conditionally cancellative t-subnorms whose natural negations are involutive, in fact, become t-norms.

2. Preliminaries

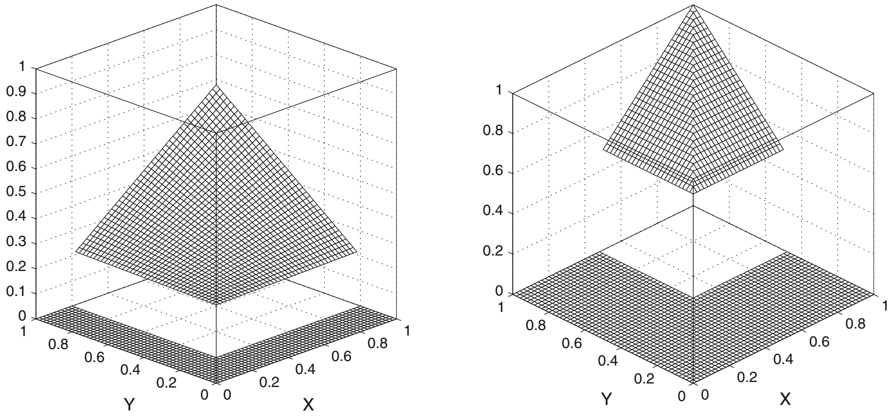
Definition 2.1. A function $N\colon [0, 1] \to [0, 1]$ is called a *fuzzy negation* if N is decreasing and $N(0) = 1$, $N(1) = 0$. N is said to be *involutive* or *strong* if $N \circ N = id_{[0,1]}$.

Definition 2.2 ([9], *Definition* 1.7). A *t-subnorm* is a function $M\colon [0, 1]^2 \to [0, 1]$ such that it is monotonic non-decreasing, associative, commutative and $M(x, y) \leq \min(x, y)$ for all $x, y \in [0, 1]$.

Note that for a t-subnorm 1 need not be the neutral element, unlike in the case of a t-norm.

Definition 2.3 (cf. [9], *Definition* 2.9 (iii)). A t-subnorm M satisfies the *Conditional Cancellation Law* if, for any $x, y, z \in (0, 1]$,

$$M(x, y) = M(x, z) > 0 \quad \text{implies } y = z\ . \tag{CCL}$$



(a) $M_{\mathbf{C}}$ **(b)** $M_{\mathbf{B}}$

FIGURE 1. While $M_{\mathbf{C}}$ is a conditionally cancellative t-subnorm, $M_{\mathbf{B}}$ is not

Alternately, (CCL) implies that on the positive domain of M, i.e., on the set $\{(x, y) \in (0,1]^2 \mid M(x, y) > 0\}$, M is strictly increasing. For instance, the Łukasiewicz t-norm $T_{\mathbf{LK}}(x, y) = \max(0, x+y-1)$ is a conditionally cancellative t-norm, while the nilpotent minimum $T_{\mathbf{nM}}$ given below is not:

$$T_{\mathbf{nM}}(x, y) = \begin{cases} 0, & \text{if } 1 - x \leq y, \\ \min(x, y), & \text{otherwise.} \end{cases}$$

The following proper t-subnorms, i.e., t-subnorms that are not t-norms, are examples of a conditionally cancellative t-subnorm ($M_{\mathbf{C}}$, see Fig. 1a) and one that is not ($M_{\mathbf{B}}$, see Fig. 1b):

$$M_{\mathbf{C}}(x, y) = \begin{cases} 0, & \text{if } \min(x, y) \leq 0.2, \\ 0.2 + \frac{3}{4}(y - 0.2)(x - 0.2), & \text{otherwise,} \end{cases}$$

$$M_{\mathbf{B}}(x, y) = \begin{cases} 0, & \text{if } \min(x, y) \leq 0.5, \\ \min(x, y), & \text{otherwise.} \end{cases}$$

For more examples, see [14, 12].

Definition 2.4 (cf. [1], *Definition* 2.3.1). Let M be any t-subnorm. Its natural negation n_M is given by

$$n_M(x) = \sup\{t \in [0,1] \mid M(x, t) = 0\}, \quad x \in [0,1] . \tag{2.1}$$

Note that though $n_M(0) = 1$, it need not be a fuzzy negation, since $n_M(1)$ can be greater than 0. However, we have the following result.

Lemma 2.5 (cf. [1], Proposition 2.3.4). *Let M be any t-subnorm and n_M its natural negation. Then we have the following:*

(i) $M(x,y) = 0 \Longrightarrow y \leq n_M(x)$.
(ii) $y < n_M(x) \Longrightarrow M(x,y) = 0$.
(iii) *If M is left-continuous then $y = n_M(x) \Longrightarrow M(x,y) = 0$, i.e., the reverse implication of (i) also holds.*

Proof. Let $\mathcal{A}_x = \{t \in [0,1] | M(x,t) = 0\}$. Clearly, $0 \in \mathcal{A}_x$ since $M(x,0) = 0$. Moreover, by the monotonicity of M we have that if $t \in \mathcal{A}_x$ then $[0,t] \subseteq \mathcal{A}_x$. Thus, for every $x \in (0,1)$ we have that $\mathcal{A}_x = [0,\alpha_x)$ or $[0,\alpha_x]$ for some $\alpha_x \in [0,1]$. Now, $n_M(x) = \sup \mathcal{A}_x = \alpha_x$.

(i) Let $M(x,y) = 0$. Then $y \in \mathcal{A}_x$ and hence $y \leq \sup \mathcal{A} = n_M(x)$.
(ii) Conversely, let $y < \sup \mathcal{A}_x = n_M(x)$. Then $y \in \mathcal{A}_x$ and hence $M(x,y)=0$.
(iii) If M is left-continuous then $y = n_M(x) = \alpha_x \in \mathcal{A}_x$ and $M(x,y) = 0$.

□

3. Solution to the open problem of U. Höhle

It should be noted that in the case when T is left-continuous—as stated in Problem **1**—the sup in (1.2) actually becomes max. It is worth mentioning that the residual can be determined for more generalised conjunctions and the conditions under which this residual becomes a fuzzy implication can be found in, for instance, [3,5]. Hence we further generalise the statement of Problem **1** by considering a t-subnorm instead of a t-norm and also dropping the condition of left-continuity. As we show below the solution characterizes the set of all conditionally cancellative t-subnorms.

Theorem 3.1. *Let M be any t-subnorm and I be the residual operation linked to M by (1.2). Then the following are equivalent:*

(i) *M satisfies (1.1) with I.*
(ii) *M is a Conditionally Cancellative t-subnorm.*

Proof. Let M be any t-subnorm, not necessarily left-continuous. Note that we denote n_M simply by n.

(i) $\Longrightarrow$ (ii): Let M satisfy (1.1) with I. On the contrary, let us assume that there exist $x,y,z \in (0,1)$ such that $M(x,y) = M(x,z) > 0$ but $y < z$. Then we have that

$$\text{LHS } (1.1) = I(x, M(x,y)) = \sup\{t \in [0,1] \mid M(x,t) \leq M(x,y)\} \geq z > y\,.$$

However, note that, from Lemma 2.5 (i) we have that $y \geq n(x)$, since $M(x,y) > 0$. Thus

$$\text{RHS }(1.1) = \max(n(x), y) = y < \text{LHS }(1.1),$$

a contradiction to the fact that M satisfies (1.1) with I. Hence M satisfies (CCL).

(ii) $\Longrightarrow$ (i): Now, let M satisfy (CCL). Consider an arbitrary $x, y \in [0, 1]$. Then either $n(x) > y$ or $n(x) \leq y$.

If $n(x) > y$, then by Lemma 2.5 (ii) we see that $M(x, y) = 0$ and hence

$$\text{LHS }(1.1) = I(x, M(x, y)) = I(x, 0) = n(x) = \max(n(x), y) = \text{RHS }(1.1).$$

If $n(x) \leq y$ and $M(x, y) = 0$ then by Lemma 2.5 (i) we have that $n(x) \geq y$ and hence $n(x) = y$ and it reduces to the above case. Hence let $M(x, y) > 0$. Then

$$\text{RHS }(1.1) = \max(n(x), y) = y.$$

We claim now that LHS (1.1) $= I(x, M(x, y)) = y$. If this were not true, then there would exist $1 \geq z > y$ ($z \not< y$ by the monotonicity of M) such that

$$I(x, M(x, y)) = \sup\{t \in [0, 1] \mid M(x, t) \leq M(x, y)\} = z.$$

This implies that there exists a $w \in (0, 1)$ such that $z > w > y$ and $M(x, w) \leq M(x, y)$, which by the monotonicity of t-subnorms implies that $M(x, w) = M(x, y)$ with $w > y$, a contradiction to the fact that M satisfies (CCL). Thus M satisfies (1.1) with I. □

Example. Consider the product t-norm $T_{\mathbf{P}}(x, y) = xy$, which is a strict t-norm and hence continuous and Archimedean, whose residual is the Goguen implication given by

$$I_{\mathbf{GG}}(x, y) = \begin{cases} 1, & \text{if } x \leq y, \\ \frac{y}{x}, & \text{if } x > y. \end{cases}$$

It can be easily verified that $T_{\mathbf{P}}$ does indeed satisfy (1.1) with $I_{\mathbf{GG}}$ whereas the natural negation of $T_{\mathbf{P}}$ is the Gödel negation

$$n_{T_{\mathbf{P}}}(x) = I_{\mathbf{GG}}(x, 0) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x > 0. \end{cases}$$

This example clearly shows that the remark of U. Höhle, Remark 1.2, is not always true. In the following we give an equivalence condition under which it is true.

Theorem 3.2. *Let T be a continuous t-norm that satisfies* (1.1) *along with its residual. Then the following are equivalent:*

(i) *T is nilpotent.*
(ii) *n_T is strong.*

Proof. (i) $\Longrightarrow$ (ii): If T is nilpotent then it is isomorphic to the Łukasiewicz t-norm, i.e., there exists an increasing bijection $\varphi : [0,1] \to [0,1]$ such that $T(x,y) = \varphi^{-1}(\max(\varphi(x) + \varphi(y) - 1, 0))$. It can be easily verified that $n_T(x) = \varphi^{-1}(1 - \varphi(x))$ which is an involutive negation.

(ii) $\Longrightarrow$ (ii): If T is continuous and satisfies (1.1) along with its residual then, from Theorem 3.1, T is conditionally cancellative and hence necessarily Archimedean by [9], Proposition 2.15 (ii). Thus T is either nilpotent or strict. If T is continuous with a strong natural negation, clearly, T has zero-divisors and hence T is nilpotent. $\square$

4. Conditional cancellativity and unit element

From the above remarks we note that when the natural negation of the underlying conjunction (a continuous t-norm, in the above case) is strong the class of conjunctions that satisfy (1.1) along with its residual gets restricted. Hence we study the class of t-subnorms M that satisfy (1.1) along with its residual and whose natural negations are strong. In other words, we seek the characterization of the class of conditionally cancellative t-subnorms with strong natural negations.

Let us recall from the remark following Definition 2.4 that the natural negation of a t-subnorm n_M need not be a fuzzy negation. If a t-subnorm has 1 as its neutral element, i.e., if it is a t-norm, then we have

$$M(1,y) = 0 \Longleftrightarrow y = 0,$$
$$\text{i.e., } y = \sup\{t | M(1,t) = 0\} = n_M(1) = 0.$$

Equivalently, by the monotonicity of M we have that n_M is a fuzzy negation. However, this is only a necessary and not a sufficient condition.

It was Jenei [6] who proposed some sufficiency conditions and showed that left-continuous t-subnorms with strong natural negations are t-norms, i.e., 1 does become a neutral element.

In the following we show that if a conditionally cancellative t-subnorm is such that $M(1,y) = y$ for *some* $y \in (0,1]$ and if the associated negation is a fuzzy negation then 1 is a neutral element of M, i.e., M is a t-norm. Based on this, we show that in the case when n_M is a strong negation then M always is a t-norm. In other words, there does not exist any conditionally cancellative *proper* t-subnorm whose natural negation is involutive.

Lemma 4.1. *Let M be a conditionally cancellative t-subnorm. Let $M(1,y_0) = y_0$, for some $y_0 \in (0,1]$.*

(i) *Then $M(1,y) = y$ for all $y \in [y_0,1]$.*

(ii) *Let $y^* = \sup\{t|M(1,t) = 0\} = n_M(1)$. Then $M(1,y) = y$ for all $y \in (y^*, y_0]$.*

Proof. Let $M(1, y_0) = y_0$, for some $y_0 \in (0, 1]$.

(i) Let $y_0 < y \le 1$. Clearly, $y_0 = M(1, y_0) < M(1, y) \le y$. If $M(1, y) = y' < y$, then by associativity and conditional cancellativity we have

$$\left.\begin{array}{l} M(M(1, y_0), y) = M(y_0, y) \\ M(M(1, y), y_0) = M(y', y_0) \end{array}\right\} \Longrightarrow M(y_0, y) = M(y_0, y') \Longrightarrow y = y',$$

i.e., $M(1, y) = y$ for all $y \ge y_0$.

(ii) Let $y^* < y \le y_0$. Clearly, $y_0 = M(1, y_0) > y \ge M(1, y) = y'$. If $M(1, y) = y' < y$, then, once again, by associativity and conditional cancellativity we have

$$\left.\begin{array}{l} M(M(1, y_0), y) = M(y_0, y) \\ M(M(1, y), y_0) = M(y', y_0) \end{array}\right\} \Longrightarrow M(y_0, y) = M(y_0, y') \Longrightarrow y = y',$$

i.e., $M(1, y) = y$ for all $y \in (y^*, y_0]$. □

Based on the above result, we now have the following equivalence condition for a conditionally cancellative t-subnorm to be a t-norm:

Theorem 4.2. *Let M be any conditionally cancellative t-subnorm. Then the following are equivalent:*

(i) *M is a t-norm.*

(ii) *n_M is a negation and $M(1, y_0) = y_0$, for some $y_0 \in (0, 1]$.*

Proof. Sufficiency is obvious. Necessity follows from the fact that if n_M is a negation then $y^* = 0$ in Lemma 4.1 above. □

Remark 4.3. Note that both conditions in Theorem 4.2(ii) are mutually exclusive.

(i) Consider the conditionally cancellative (proper) t-subnorm (see Fig. 2a)

$$M_{\mathbf{P}}(x, y) = \frac{xy}{2}$$

whose associated negation is a negation but $M(1, y) \neq y$ for any $y \in (0, 1]$.

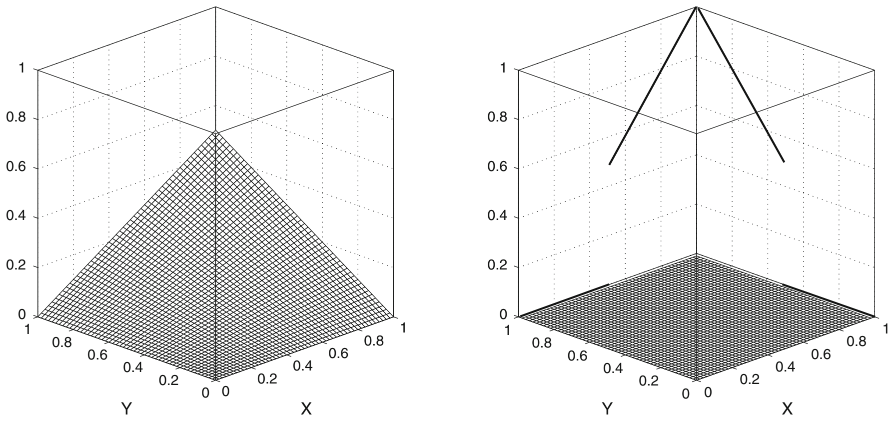
(ii) Consider the conditionally cancellative (proper) t-subnorm (see Fig. 2b)

$$M_{\mathbf{D}}(x, y) = \begin{cases} x, & \text{if } y = 1 \text{ and } x \in (0.5, 1] \\ y, & \text{if } x = 1 \text{ and } y \in (0.5, 1] \\ 0, & \text{otherwise} \end{cases},$$

whose associated negation

$$N_M(x) = \begin{cases} 1, & \text{if } x \in [0, 1) \\ 0.5, & \text{if } x = 1 \end{cases}$$

is not a fuzzy negation, since $N_M(1) \neq 0$. Note however that $M(1, y) = M(y, 1) = y$ for any $y \in (0.5, 1]$.



(a) $M_{\mathbf{P}}$ **(b)** $M_{\mathbf{D}}$

Figure 2. The conditionally cancellative (proper) t-subnorms $M_{\mathbf{P}}$ and $M_{\mathbf{D}}$ from Remark 4.3

The final result of this work shows that in the case when n_M is a strong negation then M is always a t-norm.

Theorem 4.4. *Let M be any conditionally cancellative t-subnorm. If n_M is a strong natural negation then M is a t-norm.*

Proof. Our approach will be to show that $M(1,1) = 1$ and then the result follows easily from Theorem 4.2. Note also that since n_M is a strong negation, we have that $n_M(x) = 1 \iff x = 0$ and $n_M(x) = 0 \iff x = 1$. Equivalently, $M(1,x) = 0 \iff x = 0$.

On the contrary, let us assume that $M(1,y) < y$ for all $y \in (0,1]$. In particular, $M(1,1) = z$ such that $0 < z < 1$. Since n_M is strong, there exists $z' \in (0,1)$ such that $z = n_M(z')$. We claim that $z' = 0$ and hence $z = 1$.

If not, then there exists $0 < z'' < z'$ and by the definition of n_M we have that $M(z,z'') = 0$. Also, by our assumption $0 < M(1,z'') = z^* < z''$. Now, by associativity and conditional cancellativity we have

$$\left.\begin{array}{l} M(M(1,1),z'') = M(z,z'') \\ M(M(1,z''),1) = M(z^*,1) \end{array}\right\} \Longrightarrow M(z,z'') = 0 = M(z^*,1)$$
$$\Longrightarrow z^* = 0\ ,$$

a contradiction. Thus $z = 1$ and hence we have the result. $\square$

5. Concluding remarks

In this work we have solved a more generalised version of an open problem of U. Höhle and have shown that the solution gives a characterization of all conditionally cancellative t-subnorms. Further, by proving an equivalence condition for a conditionally cancellative t-subnorm to be a t-norm, we have shown that conditionally cancellative t-subnorms with involutive natural negations are t-norms.

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