

Monotonicity Of SISO Fuzzy Relational Inference with an Implicative Rule Base

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Abstract—A Fuzzy Relational Inference (FRI) mechanism is appraised based on the different desirable properties it possesses. Among these properties, monotonicity of an FRI has not received much attention. In this work, we investigate the monotonicity of a single input single output (SISO) FRI with an implicative form of the rule base. In all the previous works that deal with monotonicity of an FRI with the implicative form of rule base, the employed fuzzy implications come from a residuated lattice. It can be noticed that this rich underlying structure plays a major role in proving the results. Further, they also modify the given monotone rule base. This work differs from the previous works in that (i) the fuzzy implications employed in it do not come from any known residuated structure on $[0, 1]$ and (ii) the original rule base is employed without any alteration. We determine conditions under which monotonicity of an FRI, where the rule base is modeled by a strict fuzzy implication, can be ensured *without transforming* the original rule base. Thus the results in this work further augment the case for considering fuzzy implications, other than those from the residuated setting, to be used in applications.

Index Terms—Fuzzy relational inference, monotone rule base, monotonicity of inference, strict fuzzy implications, Yager's families of fuzzy implications.

I. INTRODUCTION

The term *approximate reasoning* refers to methods and methodologies that enable reasoning with imprecise inputs to obtain meaningful outputs [8]. Fuzzy Inference Systems (FISs) form one particular type of approximate reasoning scheme involving fuzzy sets and are one of the best known applications of fuzzy logic in the wider sense. FISs have many degrees of freedom, namely, the underlying fuzzy partition of the input and output spaces, the fuzzy logic operations employed, the fuzzification and defuzzification procedures used, etc. This freedom gives rise to a variety of FIS with differing capabilities or properties as espoused below. While there exist many types of FIS we focus only on Fuzzy Relational Inference (FRI) systems [19], [33].

A. Monotonicity of a Fuzzy Relational Inference

While dealing with an FRI, the underlying operations can be chosen from a repertoire of fuzzy logic connectives. This choice is not arbitrarily exercised and is done keeping in mind several desirable properties that are expected of an FRI. Some of the well studied desirable properties are the following: (i) Interpolativity, (ii) Continuity, (iii) Robustness, (iv) Approximation Capability and (v) Efficiency, which in the

case of an FRI with an implicative form of rule base have been studied in [11], [13], [15], [16], [18], [25], [26], [28], [29].

Yet another desirable property, which has only recently begun to receive the attention that was due to it, is that of monotonicity. Given a monotone fuzzy rule base (see Definition 5.1) and two *crisp* inputs x, x' such that $x \leq x'$, then one expects the corresponding defuzzified outputs of the FRI y, y' to also exhibit the same ordering, i.e., $y \leq y'$.

Clearly, monotonicity is one of the essential properties of an inference mechanism, unavailability of which leads to an unreliable inference mechanism, see [22], [23], [24], [17], [30]. The often quoted example of a fuzzy controlled water dam serves to highlight the issue [23]. Let the rule base controlling the dam be monotone, i.e., containing rules which capture the monotonicity expected in the control, viz., the amount of water let out is largely and directly proportional to the inflow into the dam. Now an FRI that controls the dam is expected to maintain this monotonicity, failing which would lead to a disastrous situation.

B. Motivation for this work

The study of monotonicity of an FRI forms the main focus of this submission, the motivation for which stems from two roots. On the one hand, monotonicity of an FRI, in fact, of an FIM in general, is a topic that has not received much attention – only some nascent works exist, while other desirable properties have been quite well studied. Thus there is a clear need to study monotonicity of an FRI in its own right. On the other hand, the monotonicity of an FRI, like other properties, depends essentially on the operations employed in the FRI. Typically, the study of the desirable properties of an FRI using implicative form of rules has largely been confined to operations that come from a residuated lattice. Recently, in [16], [18], we had studied the desirable properties listed above on FRIs that use implicative form of rules but whose operations do not come from a residuated lattice setting. In fact, to the best of the authors' knowledge, this was the first such work, wherein the Yager's classes of fuzzy implications were considered and which demonstrated that these FRIs also enjoyed similar desirable properties. Further, some preliminary studies relating to the monotonicity of FRIs that employ the family of Yager's f -implications were discussed in [17]. This work could be seen as yet another logical step in furtherance of studying FRIs whose underlying operators come from a non-residuated setting.

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C. Main contributions of the work

In this work, we investigate the monotonicity property of Single Input Single Output (SISO) FRIs when an implicative model of the rule base is employed, i.e., where the operation between the antecedents and consequents is taken as a fuzzy implication.

As espoused above, so far, in the works dealing with monotonicity of FRIs with implicative or a conditional interpretation of the rules, not only do the operators come from a residuated lattice structure, but also the rule base is transformed into another form and then the monotonic behaviour of the inference mechanism is investigated. However, the inference based on this transformed rule base may lose some of the desirable properties, for instance, interpolativity, see [27], [30].

In this work, firstly, we investigate the monotonicity of an FRI where the underlying operations come from a more generalised class of fuzzy implications and do not come from a residuated structure. Further, by taking the help of the concept of *weak-coherence* introduced in [18], we find some sufficient conditions under which the output of the FRI is monotonic, without having to alter or transform the given rule base. This also ensures that the additional conditions imposed do not affect the other desirable properties the FRI may already possess. Another highlight of this work is that the techniques and the approach employed in proving the results make no assumption on the form or representation of the considered fuzzy implications. Note that in our earlier work [17] we had considered only the family of Yager's f -implications.

D. Outline Of the Work

Firstly in Section II, we present some relevant definitions from both fuzzy set theory and fuzzy logic connectives. In Section III, we introduce the two main types of fuzzy rule bases typically employed in fuzzy systems and the corresponding fuzzy relations representing them. Following this, we discuss fuzzy relational inference mechanisms and their different forms in Section IV. Section V begins by introducing monotone rule bases and goes on to discuss the corresponding monotonicity of the output of the FRI whose underlying rule bases are monotone. Further, we detail all the previous works that deal with this nascent topic and articulate the motivation for our work and clearly specify the main contributions of our work. In the subsequent Section VI, we detail the scope of the current work by recalling the concept of weak-coherence and by specifying the admissible types of fuzzy sets and fuzzy implications. The Section VII contains the main results of this paper and illustrative examples corresponding to the results are presented in Section VIII. Finally, some concluding remarks are given in Section IX.

II. PRELIMINARIES

In this work we consider $X \subseteq \mathbb{R}$ to be a closed and bounded interval and hence X is also totally ordered and linear. However, many of the concepts below are applicable to more general sets and hence the definitions are given accordingly.

A. Fuzzy Sets

Let $X \neq \emptyset$. $\mathcal{F}(X)$ will denote the fuzzy power set of X , i.e., $\mathcal{F}(X) = \{A|A : X \rightarrow [0, 1]\}$.

Definition 2.1: A fuzzy set $A \in \mathcal{F}(X)$ is said to be

- *normal* if there exists an $x \in X$ such that $A(x) = 1$,
- *convex* if X is a compact (closed and bounded) subset of a linear space and for any $\lambda \in [0, 1]$, $x, y \in X$, $A(\lambda x + (1 - \lambda)y) \geq \min\{A(x), A(y)\}$.

Definition 2.2: For an $A \in \mathcal{F}(X)$, the *Support*, *Height*, *Kernel*, *Ceiling* and α -*cut* for an $\alpha \in (0, 1]$ are, respectively, defined as:

- $\text{Supp}(A) = \{x \in X | A(x) > 0\}$,
- $\text{Hgt}(A) = \sup\{A(x) | x \in X\}$,
- $\text{Ker}(A) = \{x \in X | A(x) = 1\}$,
- $\text{Ceil}(A) = \{x \in X | A(x) = \text{Hgt}(A)\}$,
- $[A]_\alpha = \{x \in X | A(x) \geq \alpha\}$.

A is said to be *bounded* if $\text{Supp}(A)$ is a bounded set. Note that for a normal fuzzy set $\text{Ker}(A) = \text{Ceil}(A)$ and $\text{Hgt}(A) = 1$.

Definition 2.3 ([20], **Definition 3**): For two convex fuzzy sets A_1 and A_2 , we say that $A_1 \prec A_2$ if for any $\alpha \in (0, 1]$ it holds that $\inf_\alpha [A_1]_\alpha \leq \inf_\alpha [A_2]_\alpha$ and $\sup_\alpha [A_1]_\alpha \leq \sup_\alpha [A_2]_\alpha$.

Definition 2.4: Let $\mathcal{P}$ be a finite collection of fuzzy sets of X , i.e., $\mathcal{P} = \{A_k\}_{k=1}^n \subseteq \mathcal{F}(X)$. $\mathcal{P}$ is said to form a *fuzzy partition* on X if $X \subseteq \bigcup_{k=1}^n \text{Supp}(A_k)$.

In the literature, a partition $\mathcal{P}$ of X as defined above is also called a *complete partition*.

Definition 2.5: A fuzzy partition $\mathcal{P} = \{A_k\}_{k=1}^n \subseteq \mathcal{F}(X)$ is said to be *consistent* and a *Ruspini Partition*, respectively, if

- whenever for some k , $A_k(x) = 1$ then $A_j(x) = 0$ for $j \neq k$,
 - $\sum_{k=1}^n A_k(x) = 1$ for each $x \in X$.
- (RP)

B. Defuzzification

Often there is a need to convert a fuzzy set to a crisp value, a process which is called *Defuzzification*. This process of defuzzification can be seen as a mapping $d : \mathcal{F}(X) \rightarrow X$.

Example 2.6: For an $A \in \mathcal{F}(X)$, with bounded $\text{Ceil}(A)$, the *Mean of Maxima* (MOM) defuzzifier returns the mean of all those values in X with the highest membership value, i.e.,

$$\text{MOM}(A) = \frac{\int_{\text{Ceil}(A)} A(x) dx}{\int_{\text{Ceil}(A)} 1 dx}, \text{ if } \int_{\text{Ceil}(A)} 1 dx \neq 0. \quad (1)$$

The other commonly employed *Smallest of Maxima* (SOM), *Largest of Maxima* (LOM), *Center of Gravity* (COG) and the *Bisector* (BIS) defuzzifiers can be mathematically expressed as

$$\text{SOM}(A) = \min \{x | x \in \text{Ceil}(A)\}, \quad (2)$$

$$\text{LOM}(A) = \max \{x | x \in \text{Ceil}(A)\}, \quad (3)$$

$$\text{COG}(A) = \frac{\int_{\text{Supp}(A)} A(x) dx}{\int_{\text{Supp}(A)} 1 dx}, \text{ if } \int_{\text{Supp}(A)} 1 dx \neq 0, \quad (4)$$

$$\text{BIS}(A) = \left\{ x^{**} \left| \int_{\inf \text{Supp}(A)}^{x^{**}} A(x) dx = \int_{x^{**}}^{\sup \text{Supp}(A)} A(x) dx \right. \right\}. \quad (5)$$

C. Fuzzy Logic Connectives

Note that in this work, we use the term *decreasing* and *increasing* in a non-strict sense.

Definition 2.7 ([2]): A function $I: [0, 1]^2 \rightarrow [0, 1]$ is called a *fuzzy implication* if it is decreasing in the first variable, increasing in the second variable and satisfies $I(0, 0) = 1$, $I(1, 1) = 1$, $I(1, 0) = 0$. The set of all fuzzy implications will be denoted by $\mathbb{I}$.

Definition 2.8 ([2]): A fuzzy implication $I: [0, 1]^2 \rightarrow [0, 1]$ is said to

- satisfy the *left neutrality property*, if

$$I(1, y) = y, \quad y \in [0, 1], \quad (\text{NP})$$

- be *positive* if $I(x, y) > 0$, when $x, y \in (0, 1]$. (POS)
- be *strict* if I is strictly monotonic in both the variables. (ST)

Remark 2.9: If I satisfies (ST), then it satisfies (POS) but the converse is not true [2].

Definition 2.10 ([2]): A function $N: [0, 1] \rightarrow [0, 1]$ is called a *fuzzy negation* if $N(0) = 1$, $N(1) = 0$ and N is decreasing.

Example 2.11: One such fuzzy negation is the Gödel negation

$$N_{\text{D1}}(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x > 0, \end{cases} \quad x \in [0, 1]. \quad (6)$$

Definition 2.12 ([2]): Let $I \in \mathbb{I}$ be any fuzzy implication. The function $N_I: [0, 1] \rightarrow [0, 1]$ defined by $N_I(x) = I(x, 0)$ is a fuzzy negation and is called the *natural negation* of I .

We denote the class of fuzzy implications that are positive and whose natural negation $N_I = N_{\text{D1}}$, i.e., $N_I(x) = N_{\text{D1}}(x)$ for all $x \in [0, 1]$, by $\mathbb{I}_{N_{\text{D1}}}^+ \subsetneq \mathbb{I}$.

III. FUZZY IF-THEN RULE BASE AND FUZZY RELATION

Given two non-empty crisp sets $X, Y \subseteq \mathbb{R}$, a Single-Input Single-Output (SISO) fuzzy IF-THEN rule base consists of rules of the form:

$$\text{IF } \tilde{x} \text{ is } A_i \text{ THEN } \tilde{y} \text{ is } B_i, \quad (7)$$

where $\tilde{x}, \tilde{y}$ are the *linguistic* variables that are assigned the *linguistic* values $A_i, B_i, i = 1, 2, \dots, n$ which are represented by fuzzy sets in their corresponding domains, i.e., $A_i \in \mathcal{F}(X), B_i \in \mathcal{F}(Y)$. A fuzzy rule base can be viewed in two different ways [9], [10]. When each of the rules is viewed as a *constraint*, i.e., when the rules are combined together as (7), we have the conditional form (IF-THEN) of the rules. In fuzzy relational inference mechanisms (see Section IV below), fuzzy relations $R: X \times Y \rightarrow [0, 1]$ are employed to represent the rule base (7). One of the commonly employed fuzzy relations is the following: For any $x \in X, y \in Y$,

$$\hat{R}_{\rightarrow}(x, y) = \bigwedge_{i=1}^n (A_i(x) \rightarrow B_i(y)), \quad (8)$$

where $\rightarrow$ is taken as a fuzzy implication. Note that the fuzzy relation $\hat{R}_{\rightarrow}$ captures the conditional form (7) of the given rules. For more details, please refer to [9], [10].

IV. FUZZY RELATIONAL INFERENCE MECHANISM

Given a rule base of the form (7) and an input " $\tilde{x}$ is A' ", the main objective of a fuzzy inference mechanism is to find a *meaningful* B' such that " $\tilde{y}$ is B' ". While many types of fuzzy inference mechanisms have been proposed in the literature we restrict this study only to fuzzy relation based inference mechanisms.

The inference mechanism in a fuzzy relational inference (FRI) can be expressed as follows:

$$B' = f_R^{\textcircled{A}}(A') = A' \textcircled{A} R, \quad (\text{FRI-R})$$

where $A' \in \mathcal{F}(X)$ is the input, the relation $R \in \mathcal{F}(X \times Y)$ represents or models the rule base, $B' \in \mathcal{F}(Y)$ is the obtained output and $\textcircled{A}$ is called the *composition operator*, which is a mapping $\textcircled{A}: \mathcal{F}(X) \times \mathcal{F}(X \times Y) \rightarrow \mathcal{F}(Y)$.

A. Two main types of FRIs

One of the two main FRIs is the Compositional Rule of Inference (CRI) proposed by Zadeh [32], where $\star$ is a t-norm (for definition of a t-norm, please see [3], [7]):

$$\begin{aligned} B'(y) &= f_R^{\circ}(A')(y) = (A' \circ R)(y) \\ &= \bigvee_{x \in X} [A'(x) \star R(x, y)], \quad y \in Y. \end{aligned} \quad (\text{CRI-R})$$

Later Pedrycz [19] proposed another FRI mechanism based on the Bandler-Kohout Subproduct composition given as:

$$\begin{aligned} B'(y) &= f_R^{\triangleleft}(A')(y) = (A' \triangleleft R)(y) \\ &= \bigwedge_{x \in X} [A'(x) \rightarrow R(x, y)], \quad y \in Y, \end{aligned} \quad (\text{BKS-R})$$

with $\rightarrow$ interpreted as a fuzzy implication. The operator $\triangleleft$ is also known as the *inf* $-I$ composition, where I is a fuzzy implication.

B. Singleton Inputs and FRIs with Reducible Composition and Its System Function

Often one needs to deal with crisp inputs, viz., an $x_0 \in X$. In such a case, it is suitably *fuzzified*, i.e., a fuzzy set $A' \in \mathcal{F}(X)$ is suitably constructed from x_0 . Commonly, the following *singleton* fuzzifier $\mu_s: X \rightarrow \mathcal{F}(X)$ is employed to obtain a fuzzy input $A' \in \mathcal{F}(X)$. For any $x_0 \in X$,

$$\mu_s(x_0) = A'(x) = \begin{cases} 1, & x = x_0, \\ 0, & x \neq x_0. \end{cases}$$

With the above input A' , the FRI mechanism (FRI-R) reduces to

$$B'(y) = R(x_0, y), \quad y \in Y, \quad (\text{FRI-R-Singleton})$$

for any t-norm $\star$ in case of (CRI-R) and any implication I satisfying (NP) in case of (BKS-R). Thus, in the case of a singleton input, the output of both the (CRI-R) and (BKS-R)

are essentially the same (provided $\rightarrow$ in (BKS- R) satisfies (NP)) and is fully dependent on the model of the rule base R . In other words, $f_R^\circ \equiv f_R^{\triangleleft} \equiv f_R$ and hence the composition $\circ$ or $\triangleleft$ - when the I in $\triangleleft = \inf - I$ composition satisfies (NP) - does not play any role.

An FRI whose output, for singleton inputs with singleton fuzzification μ_s , does not depend on the underlying composition is said to be an *FRI with reducible composition* and hence $f_R^\circ \equiv f_R$.

We denote an *FRI with reducible composition* as a quadruple $\mathbb{F} = (\mathcal{P}_X, \mathcal{P}_Y, R, d)$, where $\mathcal{P}_X = \{A_i\}$ and $\mathcal{P}_Y = \{B_i\}$ correspond to the input and output fuzzy partitions on X and Y , respectively, R is the fuzzy relation modeling the rule base and d is the defuzzifier used to obtain a crisp output from the obtained B' in (FRI- R -Singleton). Thus given an $\mathbb{F}$ the overall inference can be seen as a function $g : X \rightarrow Y$ as follows:

$$g(x') = d(B'(\cdot)) = d(R(x', \cdot)), \quad x' \in X. \quad (9)$$

g is also known as the system function of a given $\mathbb{F}$, see for instance, [14], [15].

In this paper we deal only with the implicative form of the rule base, i.e., the antecedents of the rules are related to their consequents using a fuzzy implication and hence fix $R = \hat{R}_\rightarrow$ in the sequel. Thus this work deals with FRIs of the form $\mathbb{F}_\rightarrow = (\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_\rightarrow, d) \subseteq \mathbb{F}$.

V. MONOTONICITY OF RULE BASES AND INFERENCE

Fuzzy rule bases can be classified along many lines, for instance, complete, sparse, implicative, conjunctive, etc. In this work we look at yet another classification, that of monotone rule bases. We begin this section by introducing monotone rule bases that are essential to capture the monotonicity present in the system that an FRI is trying to model. Following this, we discuss the monotonicity of an FRI with reducible composition, by showing that not all FRIs are automatically monotone and depend in an essential way on many factors, chief among them being the underlying fuzzy logic operations and the nature of fuzzy sets used in the rule base. This possible lack of monotonicity has led researchers to study the conditions under which it could be ensured, from whence we derive our motivation.

A. Monotone Rule Base

Definition 5.1 ([23]): A fuzzy rule base (7) is called *monotone* if for any two rules

$$\begin{aligned} \text{IF } \tilde{x} \text{ is } A_i \text{ THEN } \tilde{y} \text{ is } B_i, \\ \text{IF } \tilde{x} \text{ is } A_j \text{ THEN } \tilde{y} \text{ is } B_j, \end{aligned}$$

such that $A_i \prec A_j$, it also holds that $B_i \prec B_j$, where $\prec$ is as defined in Definition 2.3.

We denote a monotone rule base of Definition 5.1 in the following form:

$$\mathcal{R}_M(A_i, B_i) : \text{IF } \tilde{x} \text{ is } A_i \text{ THEN } \tilde{y} \text{ is } B_i, i = 1, \dots, n. \quad (10)$$

In the following we give an example illustrating a monotone rule base.

Example 5.2: Let the input and output space be $X = [0, 1]$ and $Y = [0, 1]$, respectively. Let us consider the fuzzy sets $A_1 = \langle 0, 0, 0.2, 0.3 \rangle$, $A_2 = \langle 0.2, 0.3, 0.5, 0.9 \rangle$, $A_3 = \langle 0.5, 0.9, 1, 1 \rangle$ and $B_1 = \langle 0, 0, 0.2, 0.6 \rangle$, $B_2 = \langle 0.2, 0.6, 0.8, 1 \rangle$, $B_3 = \langle 0.8, 1, 1, 1 \rangle$, where a quadruple $\langle a, b, c, d \rangle$ represents a trapezoidal fuzzy set that increases and decreases linearly on the intervals $[a, b]$, $[c, d]$ and remains constant at 1 on the interval $[b, c]$, respectively. Clearly, $A_1 \prec A_2 \prec A_3$ and $B_1 \prec B_2 \prec B_3$ and hence the rule base in (11) is monotone:

$$\text{IF } \tilde{x} \text{ is } A_i \text{ THEN } \tilde{y} \text{ is } B_i, i = 1, 2, 3. \quad (11)$$

B. Monotonicity of the Output of an FRI

Even if the given rule base is monotone, the defuzzified output of an FRI need not always be monotone. In the following, we give some illustrative examples wherein the monotone rule base employed is the one given in (11) above.

Example 5.3: Let us consider the FRI with *reducible composition* $\mathbb{F}_\rightarrow = (\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_\rightarrow, d)$, where $\rightarrow = I_{\mathbf{KD}}$ is defined as, $I_{\mathbf{KD}}(x, y) = \max(1 - x, y)$, $x, y \in [0, 1]$. Then the system function $g(\cdot)$ is as shown in Fig. 1 for two different types of defuzzification methods, viz., (i) COG and (ii) MOM (see Fig. 1). For the formulae of COG and MOM, please see equations (4) and (1), respectively. From the Fig. 1, it can

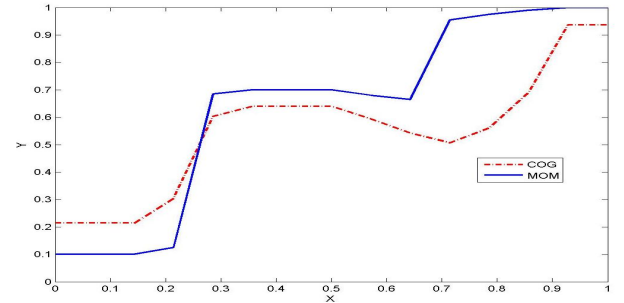


Fig. 1. System function of the FRI $\mathbb{F}_\rightarrow$ given in Example 5.3 with COG and MOM defuzzifier and $\rightarrow = I_{\mathbf{KD}}$, the Kleene-Dienes implication.

be noticed that the system functions in both the cases are not monotonic, hence the FRIs $\mathbb{F}_\rightarrow = (\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_\rightarrow, \text{COG})$ and $\mathbb{F}_\rightarrow = (\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_\rightarrow, \text{MOM})$ with $\rightarrow = I_{\mathbf{KD}}$ are not monotonic.

Example 5.4: Let us once again consider the same rule base as in Example 5.2 but a different FRI with reducible composition, viz., $\mathbb{F}_\rightarrow = (\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_\rightarrow, d)$, where $\rightarrow = I_{\mathbf{LK}}$ is defined as, $I_{\mathbf{LK}}(x, y) = \min(1, 1 - x + y)$, $x, y \in [0, 1]$. Then the system function $g(\cdot)$ is as shown in Fig. 2 for two different types of defuzzification methods, viz., (i) COG and (ii) MOM (see Fig. 2).

From the Fig. 2, it can be noticed that the system function in one of the cases is monotonic and in another case it is not. Hence the FRI $\mathbb{F}_\rightarrow = (\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_\rightarrow, \text{MOM})$ with $\rightarrow = I_{\mathbf{LK}}$ is monotonic, whereas $\mathbb{F}_\rightarrow = (\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_\rightarrow, \text{COG})$ with $\rightarrow = I_{\mathbf{LK}}$ is not.

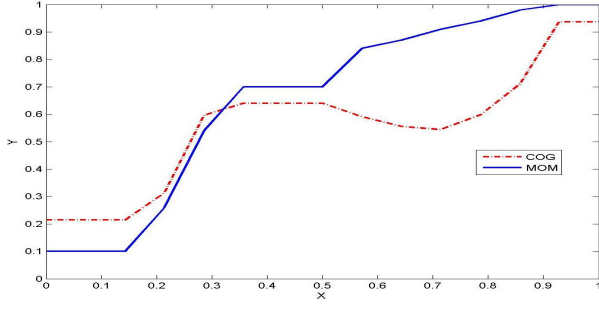


Fig. 2. System function of the FRI $\mathbb{F}_{\rightarrow}$ given in Example 5.4 with COG and MOM defuzzifier and $\rightarrow = I_{LK}$, the Łukasiewicz implication.

C. Works dealing with Monotonicity of the Output of an FRI

The earliest works to appear on this topic dealt with FRIs where a Cartesian product interpretation of the fuzzy rules was employed, see Broekhoven and De Baets [21], [22]. Later Štěpnička and De Baets in [23] and [24] considered an FRI with $R = \hat{R}_{\rightarrow}$, (see (8)), where $\rightarrow$ is any residuated implication obtained from a left-continuous t-norm. They transform the rule base by modifying the antecedent and consequent fuzzy sets into the at-least (ATL) and at-most (ATM) fuzzy rules as proposed by Bodenhofer [4], [5], denoted by, $\hat{R}^{\uparrow}$, $\hat{R}^{\downarrow}$ and $\hat{R}^{\dagger}$ and have shown that with the modified rule bases, the FRIs $\mathbb{F} = (\mathcal{P}_X, \mathcal{P}_Y, \hat{R}^{\uparrow}, \text{FOM})$, $\mathbb{F} = (\mathcal{P}_X, \mathcal{P}_Y, \hat{R}^{\downarrow}, \text{LOM})$ and $\mathbb{F} = (\mathcal{P}_X, \mathcal{P}_Y, \hat{R}^{\dagger}, \text{MOM})$ are monotonic.

Note, however, that due to the above transformation of the rules, some of the properties that were satisfied by the original rule base and hence by the FRI itself could be lost. For instance, the antecedents of the untransformed rule base may have formed a Ruspini partition on the underlying domain which is important to ensure interpolativity and continuity of the FRI, see [16], [25], [28], [29]. Now, due to the above transformation this property could be lost leading to situations where the obtained FRI may not be interpolative [27], [30].

D. Contributions and Scope of the Work

In this work, we show that SISO FRIs of the form $\mathbb{F}_{\rightarrow} = (\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_{\rightarrow}, d)$ can be made monotonic for suitable choice of operations. Unlike in earlier works dealing with monotonicity of implicative models, we do not modify the antecedent and consequent fuzzy sets of the rule base and, hence, the partitions formed by the antecedents $A_i \in \mathcal{P}_X$ and consequents $B_i \in \mathcal{P}_Y$ fuzzy sets are of the Ruspini type, thus still remaining within the practical setting that is common in applications.

Quite interestingly, there has been no published work, to the best of the authors' knowledge, that discussed the monotonicity of the system function of an FRI when an implicative model of the monotone rule base is employed, *without transforming* the rule base. The only works dealing with this topic, both in the SISO and MISO cases, have discussed the monotonicity only under the setting of transformed rule bases. In fact, Examples 5.3 and 5.4 presented above seem to be the first formal attempt to check if, even with an untransformed monotone rule base, monotonicity can be obtained.

As already mentioned, we consider a large class of fuzzy implications outside of the class of residuated implications. However, this forces us to deal with FRIs that may not have the very important *coherence* property. To overcome this, we have employed the concept of weak-coherence [18], which plays an important role in enlarging the class of fuzzy implications that can be considered. The given proofs are sufficiently general without depending on the form or representation of the fuzzy implications considered. Thus, we believe that these results are very much applicable in most of the practical and desirable contexts [9].

VI. WEAK-COHERENCE AND SOME REQUIREMENTS ON THE MONOTONE RULE BASE

The purpose of this section is to clearly specify the scope and reach of the results contained in this work. We begin by recalling the concepts of coherence and weak-coherence, based on which we restrict the scope of the work by determining the subclass of fuzzy implications, for which at least weak-coherence can be ensured. Following this, we discuss the type of admissible antecedents and consequents in a given monotone rule base. However, our results are valid for a large class of fuzzy implications, that also contains the Yager's families of fuzzy implications [31].

A. Coherence and Implicative Models

Dubois *et al.* [9] defined the concept of coherence for an implicative model $\hat{R}_{\rightarrow}$ (see (8)) of a rule base as follows, which is suitably modified to fit into our notation.

Definition 6.1 ([6], [9]): Given an implicative rule base (7), a fuzzy relation $\hat{R}_{\rightarrow}(x, y)$ – as in (8) – modeling this rule base, is coherent if for any $x \in X$ there exist $y \in Y$ such that $\hat{R}_{\rightarrow}(x, y) = 1$.

The coherence property states that for any x , the final fuzzy output B' should be normal, i.e., $\text{Ker}(B') \neq \emptyset$. Coherence of an implicative model of a rule base is very much dictated by the semantics involved [9]. Further, this property is essential when using defuzzification techniques that are dependent on the kernel to be non-empty. However, there exist *reasonable* defuzzification methods that do not depend on the kernel of the output fuzzy set.

B. A Weaker form of Coherence

Relaxing the coherence property the following weaker form of coherence has been defined in [18] which will be useful in the sequel.

Definition 6.2: For a given implicative rule base (7), a fuzzy relation $\hat{R}_{\rightarrow}(x, y)$ is said to be *weakly coherent* if for any $x \in X$ there exist $y \in Y$ such that $\hat{R}_{\rightarrow}(x, y) > 0$.

From (FRI-R-Singleton) and (8), we have the following:

$$B'(y) = \hat{R}_{\rightarrow}(x_0, y) = \bigwedge_{i=1}^n (A_i(x_0) \rightarrow B_i(y)) .$$

Now if the antecedent fuzzy sets are normal and form a Ruspini partition (See (RP)), then x_0 intersects atmost two fuzzy sets say, A_m, A_{m+1} . Then the above reduces to

$$B'(y) = (B'_m(y)) \wedge (B'_{m+1}(y)) ,$$

where B'_m and B'_{m+1} are the fuzzy sets B_m and B_{m+1} modified by the fuzzy implication $\longrightarrow$ with $A_m(x_0)$, $A_{m+1}(x_0)$, i.e., $B'_k(y) = A_k(x_0) \longrightarrow B_k(y)$, $y \in Y, k = m, m+1$.

It is clear that for B' to be non-empty the supports of B'_m and B'_{m+1} should intersect, i.e., $\text{Supp}(B'_m) \cap \text{Supp}(B'_{m+1}) \neq \emptyset$. While coherence insists that the kernels of B'_m and B'_{m+1} should intersect, weak-coherence as defined above relaxes this to a mere intersection of their supports. It should be noted that while relaxing coherence to weak-coherence does expand the set of fuzzy implications that can be considered in $\hat{R}_{\rightarrow}$, it still does not encompass the whole set of fuzzy implications $\mathbb{I}$.

C. Class of Admissible Fuzzy Sets in the Rule Base

Let $\mathcal{F}^*(X)$ denote the space of fuzzy sets on X which are normal, convex and strict on both sides of the ceiling. In the rest of this work, we only consider monotone rule bases $\mathcal{R}_M(A_i, B_i)$ where $A_i \in \mathcal{F}^*(X)$, $B_i \in \mathcal{F}^*(Y)$ and form Ruspini partitions on the underlying domains X, Y , respectively.

D. Classes of Admissible Fuzzy Implications

In the following, we discuss the class of fuzzy implications that can be considered for an FRI with $\hat{R}_{\rightarrow}$ to be at least weakly coherent. This leads to study the effect of using fuzzy implications to modify fuzzy sets. The study corresponding to modification of fuzzy sets using fuzzy implications can be found in [18].

From Section 6.2 of [18], it is clear that for an FRI with *reducible composition* (see, Section IV-B), $\mathbb{F}_{\rightarrow} = (\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_{\rightarrow}, d)$, to obtain a nonempty output, we at least need to ensure weak-coherence (as defined in Definition 6.2). While coherence insists that the kernels of B'_m and B'_{m+1} should intersect, the weak-coherence defined in Definition 6.2 relaxes this to a mere intersection of their supports. From Section VI-B we know that for a fuzzy relation $\hat{R}_{\rightarrow}$ to ensure weak-coherence at the least, the class of fuzzy implications I that can be considered should be restricted. Since in most practical settings we deal only with fuzzy sets that are bounded, continuous, convex and that which often form a Ruspini partition, to ensure weak-coherence or non emptiness of the output, it is sufficient to consider fuzzy implications $I \in \mathbb{I}$ that either

- satisfy the ordering property (OP), (i.e., $I(x, y) = 1 \iff x \leq y, x, y \in [0, 1]$), in which case often we can ensure even coherence [25], or
- are positive i.e., $I \in \mathbb{I}^+$, in which case we can ensure at least weak-coherence [18].

It is clear from Proposition 6.6 of [18] (i) that if we use a non-positive implication, then the support of B'_m and B'_{m+1} may shrink, giving rise to an empty fuzzy set as B' , which is not at all desirable. Thus, in this work, we limit the study of monotonicity to FRIs that employ fuzzy implications that come from the class $\mathbb{I}^+$. Further, among fuzzy implications $I \in \mathbb{I}^+$ we only consider those that are strict (ST) (see Definition 2.8) and denote this class by $\mathbb{I}^{\text{st}} \subsetneq \mathbb{I}^+$.

Towards better clarity and readability of the proofs presented later, we partition $\mathbb{I}^{\text{st}}$ into two subclasses, viz., (i) $\mathbb{I}^{\text{st}}_{N_{D1}}$, which

contain fuzzy implications I that are strict (ST) with $N_I = N_{D1}$ and (ii) $\mathbb{I}^{\text{st}}_{N_{D1}^c}$, which contain fuzzy implications I that are strict (ST) but with $N_I \neq N_{D1}$.

Remark 6.3: Note that $\mathbb{I}_{\text{OP}}$ and $\mathbb{I}^{\text{st}}$ are mutually exclusive. Table I lists some fuzzy implications illustrating $\mathbb{I}_{\text{OP}} \cap \mathbb{I}^{\text{st}} = \emptyset$, $\mathbb{I}_{\text{OP}} \cap \mathbb{I}_{N_{D1}} \neq \emptyset$ and $\mathbb{I}^{\text{st}} \cap \mathbb{I}_{N_{D1}} \neq \emptyset$.

E. Some families of Fuzzy Implications that belong to $\mathbb{I}^{\text{st}}_{N_{D1}} \cup \mathbb{I}^{\text{st}}_{N_{D1}^c} = \mathbb{I}^{\text{st}}$

In fact, many established families of fuzzy implications fall in either of the above two classes. For the definitions and the properties these families satisfy, please refer to the monograph [2]. Two such specific families are defined as follows:

Definition 6.4 ([2], Definition 3.1.1): Let $f: [0, 1] \rightarrow [0, \infty]$ be a strictly decreasing and continuous function with $f(1) = 0$. The function $I_f: [0, 1]^2 \rightarrow [0, 1]$ defined by

$$I_f(x, y) = f^{-1}(x \cdot f(y)), \quad x, y \in [0, 1], \quad (12)$$

with the understanding $0 \cdot \infty = 0$, is a fuzzy implication and called an *f-implication*.

Definition 6.5 ([2], Definition 3.2.1): Let $g: [0, 1] \rightarrow [0, \infty]$ be a strictly increasing and continuous function with $g(0) = 0$. The function $I_g: [0, 1]^2 \rightarrow [0, 1]$ defined by

$$I_g(x, y) = g^{(-1)}\left(\frac{1}{x} \cdot g(y)\right), \quad x, y \in [0, 1], \quad (13)$$

with the understanding $\frac{1}{0} = \infty$ and $\infty \cdot 0 = \infty$, is a fuzzy implication and called a *g-implication*, where the function $g^{(-1)}$ in (13) is given by $g^{(-1)}(x) = g^{-1}(\min(x, g(1)))$.

From the above two definitions, the following observations can be made:

- Let $\mathbb{I}_{\mathbb{F}}$ denote the set of all *f-implications*. Further, let us denote by $\mathbb{I}_{\mathbb{F}, \infty} \subsetneq \mathbb{I}_{\mathbb{F}}$ the set of *f-implications* that are generated from generators such that $f(0) = \infty$. Every $I \in \mathbb{I}_{\mathbb{F}, \infty}$ is strict and its natural negation is the Gödel negation (see [12], [1]), i.e., $N_I = N_{D1}$.
- Let $\mathbb{I}_{\mathbb{F}, 1} \subsetneq \mathbb{I}_{\mathbb{F}}$ be the set of *f-implications* that are generated from generators such that $f(0) = 1$. Every $I \in \mathbb{I}_{\mathbb{F}, 1}$ is strict but its natural negation is a strict negation (see [12], [1]), i.e., $N_I \neq N_{D1}$. Thus $\mathbb{I}_{\mathbb{F}, 1} \subsetneq \mathbb{I}^{\text{st}}_{N_{D1}^c}$, while $\mathbb{I}_{\mathbb{F}, \infty} \subsetneq \mathbb{I}^{\text{st}}_{N_{D1}}$.
- If $\mathbb{I}_{\mathbb{G}}$ denotes the set of all *g-implications*, then every $I \in \mathbb{I}_{\mathbb{G}}$ is positive and $N_I = N_{D1}$ (see [1], Proposition 4). Thus $\mathbb{I}_{\mathbb{G}} \subsetneq \mathbb{I}^{\text{st}}_{N_{D1}}$.

Note that $I_{Y_{\mathbb{G}}} \in \mathbb{I}_{\mathbb{F}, \infty} \cap \mathbb{I}_{\mathbb{G}}$ while $I_{R_{\mathbb{C}}} \in \mathbb{I}_{\mathbb{F}, 1}$, see Table I.

In the following sections we will only deal with rules modeled by fuzzy relations $\hat{R}_{\rightarrow}$ where the fuzzy implication $\longrightarrow$ satisfies (ST). Clearly, the presented results are valid for the Yager's families of *f-* and *g-implications* too.

VII. MONOTONICITY OF FRI $\mathbb{F}_{\rightarrow \text{ST}}$

Herein, we discuss the monotonicity of the output of an FRI with reducible composition

$$\mathbb{F}_{\rightarrow \text{ST}} = (\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_{\rightarrow \text{ST}}, \text{MOM}),$$

Implication $I \in \mathbb{I}$	$I \in \mathbb{I}_{OP}$	$I \in \mathbb{I}^{st}$	$I \in \mathbb{I}_{ND1}$
$\times$	$\checkmark$	$\checkmark$	$\checkmark$
$\times$	$\checkmark$	$\checkmark$	$\times$
$I_G(x, y) = \min(1, \frac{y}{x})$	$\checkmark$	$\times$	$\checkmark$
$I_{LK}(x, y) = \min(1, 1 - x + y)$	$\checkmark$	$\times$	$\times$
$I_{YG}(x, y) = \min(1, y^x)$	$\times$	$\checkmark$	$\checkmark$
$I_{RC}(x, y) = 1 - x + xy$	$\times$	$\checkmark$	$\times$
$I(x, y) = \begin{cases} 0, & \text{if } x > 0 \text{ and } y = 0 \\ 1, & \text{otherwise} \end{cases}$	$\times$	$\times$	$\checkmark$
$I_{KD}(x, y) = \max(1 - x, y)$	$\times$	$\times$	$\times$

TABLE I
ON THE MUTUAL EXCLUSIVITY OF $\mathbb{I}_{OP}$, $\mathbb{I}^{st}$ AND $\mathbb{I}_{ND1}$

with $\hat{R}_{\rightarrow_{ST}} = \bigwedge_{i=1}^n (A_i \rightarrow_{ST} B_i)$, where $\rightarrow_{ST} \in \mathbb{I}^{st}$. While investigating this FRI we partition the set $\mathbb{I}^{st}$ into two parts (i) $\mathbb{I}_{ND1}^{st}$ and (ii) $\mathbb{I}_{ND1^c}^{st}$ as given in Section VI-D and investigate the following FRIs for monotonicity:

$$\mathbb{F}_{\rightarrow_{D1}} = (\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_{\rightarrow_{D1}}, \text{MOM}),$$

$$\mathbb{F}_{\rightarrow_{D1^c}} = (\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_{\rightarrow_{D1^c}}, \text{MOM}),$$

with $\hat{R}_{\rightarrow_{D1}} = \bigwedge_{i=1}^n (A_i \rightarrow_{D1} B_i)$ where $\rightarrow_{D1} \in \mathbb{I}_{ND1}^{st}$ and $\hat{R}_{\rightarrow_{D1^c}} = \bigwedge_{i=1}^n (A_i \rightarrow_{D1^c} B_i)$ where $\rightarrow_{D1^c} \in \mathbb{I}_{ND1^c}^{st}$. In the following two results we propose some sufficient conditions under which the corresponding system functions of $\mathbb{F}_{\rightarrow_{D1}}$ and $\mathbb{F}_{\rightarrow_{D1^c}}$ are monotonic.

Theorem 7.1: Let us be given a fuzzy IF-THEN rule base $\mathcal{R}_M(A_i, B_i)$ as in (10) which is monotone and $A_i \in \mathcal{P}_X$, $i = 1, 2, \dots, n$, form a Ruspini partition on X and $B_i \in \mathcal{P}_Y$, $i = 1, 2, \dots, n$, form a Ruspini partition on Y , respectively. Further, let every element of $\mathcal{P}_X$ and $\mathcal{P}_Y$ be normal, convex and strictly monotone on both sides of the ceiling, i.e., $\mathcal{P}_X \subseteq \mathcal{F}^*(X)$ and $\mathcal{P}_Y \subseteq \mathcal{F}^*(Y)$. Then the system function g of the FRI with reducible composition $\mathbb{F}_{\rightarrow_{D1}} = (\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_{\rightarrow_{D1}}, \text{MOM})$ is monotonic, where $\rightarrow_{D1} \in \mathbb{I}_{ND1}^{st}$.

Proof: While the proof is valid for any fuzzy sets which are normal, convex and strict on both sides of the ceiling, for better readability we prove this result only for triangular fuzzy sets. For an input $x' \in X$ the fuzzy relational inference mechanism (FRI- R -Singleton) with $R = \hat{R}_{\rightarrow_{D1}}$ is of the form, $B'(y) = \hat{R}_{\rightarrow_{D1}}(x', y)$, $y \in Y$. Since A_i 's and B_i 's are convex and both form (RP) on their corresponding domains, clearly, only adjacent A_i 's and B_i 's can overlap. Hence, w.l.o.g., let the supports of A_i, B_i be such that $\text{Supp}(A_i) = [x_{i-1}, x_{i+1}]$, $\text{Supp}(B_i) = [y_{i-1}, y_{i+1}]$, $i = 2, 3, \dots, n-1$, $\text{Supp}(A_1) = [x_1, x_2]$, $\text{Supp}(A_n) = [x_{n-1}, x_n]$, $\text{Supp}(B_1) = [y_1, y_2]$, $\text{Supp}(B_n) = [y_{n-1}, y_n]$. Further, let $A_i(x_i) = 1$ and $B_i(y_i) = 1$ for $i = 1, 2, \dots, n$.

Let $x' \in X$ be any given input. Clearly, $x' \in [x_m, x_{m+1}]$ for some $m \in \{1, 2, \dots, n-1\}$. Since $\{A_i\}_{i=1}^n$ are normal and form a Ruspini partition, $A_j(x') = 0$, for all $j \neq m, m+1$. Let $A_m(x') = s'_m$ and $A_{m+1}(x') = s'_{m+1}$. Since A_i 's form

an (RP), $s'_m + s'_{m+1} = 1$ and

$$B'(y) = [s'_m \rightarrow_{D1} B_m(y)] \wedge [s'_{m+1} \rightarrow_{D1} B_{m+1}(y)] \\ = B'_m(y) \wedge B'_{m+1}(y). \quad (14)$$

Clearly, since B_m, B_{m+1} are convex and normal, B'_m, B'_{m+1} are also convex and normal (see Proposition 6.4 of [18]). Hence $B' = B'_m \cap B'_{m+1}$ is also convex. Let $y' = \text{MOM}(B')$. **Claim 1:** If $x' \in [x_m, x_{m+1}]$, then $y' = \text{MOM}(B') \in [y_m, y_{m+1}]$ for $m \in \{1, 2, \dots, n-1\}$. Further, if $B_m = B_{m+1}$ then $y' = g(x') = y_m \in [y_m, y_{m+1}]$.

For a better understanding of the proof we refer to Fig. 3 where the implication used is $I_{YG}(x, y) = \min(1, y^x)$, the Yager's implication. Note that $I_{YG} \in \mathbb{I}_{ND1}^{st}$.

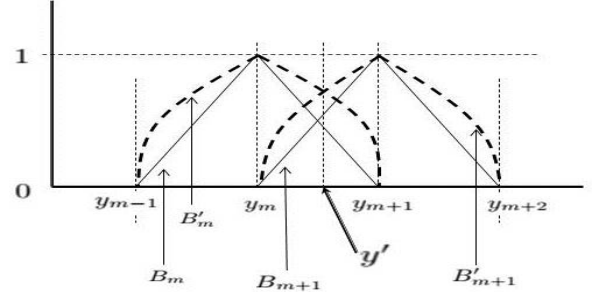


Fig. 3. The Modified Fuzzy Sets, using $I_{YG}(x, y) = \min(1, y^x)$

From Remark 6.7 of [18] we can verify that, since $x \rightarrow_{D1} 0 = 0$ for any $x \in (0, 1]$, we have that the supports of both the modified fuzzy sets $B'_m = s'_m \rightarrow_{D1} B_m$ and $B'_{m+1} = s'_{m+1} \rightarrow_{D1} B_{m+1}$ are the same as those of B_m, B_{m+1} .

$$\text{Hence, } \text{Supp}(B') = \text{Supp}(B'_m) \cap \text{Supp}(B'_{m+1}) \\ = \text{Supp}(B_m \cap B_{m+1}) = [y_m, y_{m+1}]. \quad (15)$$

Since (15) holds, $y' = \text{MOM}(B') \in [y_m, y_{m+1}]$.

Now, let $B_m = B_{m+1}$. From (14), we have

$$B'(y) = [s'_m \rightarrow_{D1} B_m(y)] \wedge [s'_{m+1} \rightarrow_{D1} B_m(y)] \\ = [(s'_m \vee s'_{m+1}) \rightarrow_{D1} B_m(y)].$$

Since $B_m(y_m) = 1$, $B'(y_m) = (s'_m \vee s'_{m+1}) \rightarrow_{D1} B_m(y_m) = (s'_m \vee s'_{m+1}) \rightarrow_{D1} 1 = 1$. From the fact that $\rightarrow_{D1}$ is strict, B_m is strictly increasing on $[y_{m-1}, y_m]$ and strictly decreasing on $[y_m, y_{m+1}]$, we have $(s'_m \vee s'_{m+1}) \rightarrow_{D1} B_m$ is strictly increasing on $[y_{m-1}, y_m]$ and strictly decreasing on

$[y_m, y_{m+1}]$. So B' reaches 1 only at y_m . Hence $\text{Ker}(B') = \{y_m\}$, consequently, $y' = g(x') = \text{MOM}(B') = y_m$. $\square$

Claim 2: The system function g is monotonic, i.e., if $x' \leq x''$ then $g(x') = y' \leq y'' = g(x'')$.

We prove the above claim by discussing different cases.

Case 1: Let $x' \in [x_m, x_{m+1}]$ and $x'' \in [x_{m+p}, x_{m+p+1}]$, where $p \geq 1$. By the **Claim 1** above, irrespective of the orderings between B_m, B_{m+1} and B_{m+p}, B_{m+p+1} , we have that $y' \in [y_m, y_{m+1}]$ and $y'' \in [y_{m+p}, y_{m+p+1}]$ and hence $y' \leq y''$.

Case 2: Let $x', x'' \in [x_m, x_{m+1}]$, i.e., $p = 0$ in the above **Case 1**.

Case 2a: If $B_m = B_{m+1}$, then by from Claim 1 above we obtain, $y' = y'' = y_m$. Thus, trivially, we have $x' \leq x'' \implies y' \leq y''$.

Case 2b: Let $B_m \neq B_{m+1}$. We will prove that $y' = \text{MOM}(B') \leq \text{MOM}(B'') = y''$.

Since, $x', x'' \in [x_m, x_{m+1}]$, for some $m \in \{1, 2, \dots, n-1\}$, from **Claim 1** we obtain, $y', y'' \in [y_m, y_{m+1}]$. Now, since A_m is strictly decreasing and A_{m+1} is strictly increasing on $[x_m, x_{m+1}]$, $x' \leq x''$ and $x', x'' \in [x_m, x_{m+1}]$ implies

$$\begin{aligned} A_m(x') &\geq A_m(x'') \text{ and } A_{m+1}(x') \leq A_{m+1}(x''), \\ \text{i.e., } s'_m &\geq s''_m \text{ and } s'_{m+1} \leq s''_{m+1}, \end{aligned} \quad (16)$$

where $s''_m = A_m(x'')$ and $s''_{m+1} = A_{m+1}(x'')$. Using (16), for any $y \in [y_m, y_{m+1}]$ we obtain the inequalities:

$$\begin{aligned} s'_m \rightarrow_{\mathbf{D1}} B_m(y) &\leq s''_m \rightarrow_{\mathbf{D1}} B_m(y), \\ &\implies B'_m(y) \leq B''_m(y), \text{ and} \\ s'_{m+1} \rightarrow_{\mathbf{D1}} B_{m+1}(y) &\geq s''_{m+1} \rightarrow_{\mathbf{D1}} B_{m+1}(y), \\ &\implies B'_{m+1}(y) \geq B''_{m+1}(y). \end{aligned}$$

Claim 3: $y' = \text{MOM}(B') \in \text{Supp}(B_m \cap B_{m+1})$ is the point of intersection of B'_m and B'_{m+1} .

On $[y_m, y_{m+1}]$, B'_m is strictly decreasing and B'_{m+1} is strictly increasing. Let B'_m and B'_{m+1} intersect at $y^0 \in [y_m, y_{m+1}]$, i.e.,

$$\begin{aligned} B'(y^0) &= \min\{B'_m(y^0), B'_{m+1}(y^0)\} \\ &= B'_m(y^0) = B'_{m+1}(y^0). \end{aligned} \quad (17)$$

Now, for $y \in [y_m, y^0]$, it holds that $B'_m(y) > B'_m(y^0) = B'_{m+1}(y^0) > B'_{m+1}(y)$. Since $B'_m(y) > B'_{m+1}(y)$ and B'_{m+1} is strictly increasing in $[y_m, y_{m+1}]$, using (17), we have

$$\begin{aligned} B'(y) &= \min(B'_m(y), B'_{m+1}(y)) = B'_{m+1}(y) \\ &< B'_{m+1}(y^0) = B'_m(y^0) = B'(y^0). \end{aligned} \quad (18)$$

Again for $y \in (y^0, y_{m+1}]$, it holds that, $B'_m(y) < B'_{m+1}(y)$ and, as above, we have the following inequality using the fact that B'_m is strictly decreasing in $[y_m, y_{m+1}]$:

$$\begin{aligned} B'(y) &= \min(B'_m(y), B'_{m+1}(y)) = B'_m(y) \\ &< B'_m(y^0) = B'_{m+1}(y^0) = B'(y^0). \end{aligned} \quad (19)$$

From (18) and (19), we have that $y' = \text{MOM}(B') = y^0$ and, in fact, is the point of intersection of B'_m and B'_{m+1} , thus proving **Claim 3**. $\square$

Since B'_m and B'_{m+1} are monotonic on $[y_m, y_{m+1}]$, we have that $y', y'' \in [y_m, y_{m+1}]$ are also the points which satisfy $B'_m(y') = B'_{m+1}(y')$ and $B''_m(y'') = B''_{m+1}(y'')$ i.e.,

$$s'_m \rightarrow_{\mathbf{D1}} B_m(y') = s'_{m+1} \rightarrow_{\mathbf{D1}} B_{m+1}(y'), \quad (20)$$

$$s''_m \rightarrow_{\mathbf{D1}} B_m(y'') = s''_{m+1} \rightarrow_{\mathbf{D1}} B_{m+1}(y''). \quad (21)$$

Now, to prove monotonicity, we need to show that $y' \leq y''$. If possible, let $y' > y''$. Since B_m and B_{m+1} are, respectively, strictly decreasing and strictly increasing on $[y_m, y_{m+1}]$, $y', y'' \in [y_m, y_{m+1}]$ implies

$$B_m(y') < B_m(y'') \text{ and } B_{m+1}(y') > B_{m+1}(y''). \quad (22)$$

Use of strictness of $\rightarrow_{\mathbf{D1}}$, (22), $s'_{m+1} \leq s''_{m+1}$ from (16), (21), (22) and $s'_m \geq s''_m$ from (16) lead to the following thread of inequalities: $s'_{m+1} \rightarrow_{\mathbf{D1}} B_{m+1}(y') > s'_{m+1} \rightarrow_{\mathbf{D1}} B_{m+1}(y'') \geq s''_{m+1} \rightarrow_{\mathbf{D1}} B_{m+1}(y'') = s''_m \rightarrow_{\mathbf{D1}} B_m(y'') > s''_m \rightarrow_{\mathbf{D1}} B_m(y') \geq s'_m \rightarrow_{\mathbf{D1}} B_m(y')$ which results in $s'_{m+1} \rightarrow_{\mathbf{D1}} B_{m+1}(y') > s'_m \rightarrow_{\mathbf{D1}} B_m(y')$, a contradiction to (20). Thus, $x' \leq x'' \implies y' \leq y''$ and the system function g is monotonic. $\blacksquare$

Remark 7.2: For better readability the proof of Theorem 7.1 has been presented only for triangular fuzzy sets, whereas the proof is valid for any fuzzy sets which are normal, convex and strict on both sides of the ceiling. It should be noted that, the result remains unaffected, when we consider trapezoidal fuzzy sets instead of triangular fuzzy sets, since the only extra case that needs to be considered is when the input x' falls in the kernel of an antecedent fuzzy set A_m . However, in this case, due to the Ruspini partition of the antecedent fuzzy sets $\mathcal{P}_X$, it can be easily shown that the output $g(x')$ will fall within the kernel of the corresponding consequent fuzzy set B_m .

Theorem 7.3: Let us be given a fuzzy IF-THEN rule base $\mathcal{R}_M(A_i, B_i)$ as in (10) which is monotone and $A_i \in \mathcal{P}_X$, $i = 1, 2, \dots, n$, form a Ruspini partition on X and $B_i \in \mathcal{P}_Y$, $i = 1, 2, \dots, n$, form a Ruspini partition on Y , respectively. Further, let every element of $\mathcal{P}_X$ and $\mathcal{P}_Y$ be normal, convex and strictly monotone on both sides of the ceiling, i.e., $\mathcal{P}_X \subseteq \mathcal{F}^*(X)$ and $\mathcal{P}_Y \subseteq \mathcal{F}^*(Y)$. Then the system function g of the FRI with reducible composition $\mathbb{F}_{\rightarrow \mathbf{D1}^c} = (\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_{\rightarrow \mathbf{D1}^c}, \text{MOM})$ is monotonic, where $\rightarrow_{\mathbf{D1}^c} \in \mathbb{I}_{N_{\mathbf{D1}^c}}^{\text{st}}$.

Proof: Once again, while the proof is valid for any fuzzy sets which are normal, convex and strict on both sides of the ceiling, for better readability we prove this result only for triangular fuzzy sets.

For an input $x' \in X$ the fuzzy relational inference mechanism (FRI- R -Singleton) with $R = \hat{R}_{\rightarrow \mathbf{D1}^c}$ is of the form,

$$B'(y) = \hat{R}_{\rightarrow \mathbf{D1}^c}(x', y), \quad y \in Y. \quad (23)$$

Since A_i 's and B_i 's are convex and both form Ruspini partition (see Definition 2.5) on their corresponding domains, clearly, only adjacent A_i 's and B_i 's can overlap. Hence, w.l.o.g., let the supports of A_i, B_i be such that $\text{Supp}(A_i) = [x_{i-1}, x_{i+1}]$, $\text{Supp}(B_i) = [y_{i-1}, y_{i+1}]$, $i = 2, 3, \dots, n-1$, $\text{Supp}(A_1) = [x_1, x_2]$, $\text{Supp}(A_n) = [x_{n-1}, x_n]$, $\text{Supp}(B_1) = [y_1, y_2]$, $\text{Supp}(B_n) = [y_{n-1}, y_n]$. Further, let $A_i(x_i) = 1$ and $B_i(y_i) = 1$ for $i = 1, 2, \dots, n$.

Let $x' \in X$ be any given input. Clearly, $x' \in [x_m, x_{m+1}]$ for some $m \in \{1, 2, \dots, n-1\}$. Since $\{A_i\}_{i=1}^n$ are normal and form a Ruspini partition, $A_j(x') = 0$, for all $j \neq m, m+1$. From (23), $B'(y) = [A_m(x') \rightarrow_{\mathbf{D1}^c} B_m(y)] \wedge [A_{m+1}(x') \rightarrow_{\mathbf{D1}^c} B_{m+1}(y)]$. Once again, let $A_m(x') = s'_m$ and $A_{m+1}(x') = s'_{m+1}$ and hence, $s'_m + s'_{m+1} = 1$ and

$$B'(y) = [s'_m \rightarrow_{\mathbf{D1}^c} B_m(y)] \wedge [s'_{m+1} \rightarrow_{\mathbf{D1}^c} B_{m+1}(y)] \\ = B'_m(y) \wedge B'_{m+1}(y).$$

Clearly, since B_m, B_{m+1} are convex and normal, B'_m, B'_{m+1} are also convex and normal (see Proposition 6.4 of [18]). Hence $B' = B'_m \cap B'_{m+1}$ is also convex.

Claim 4: If $x' \in [x_m, x_{m+1}]$, then $y' = \text{MOM}(B') \in [y_m, y_{m+1}]$ for $m \in \{1, 2, \dots, n-1\}$. Further, if $B_m = B_{m+1}$ then $y' = y_m \in [y_m, y_{m+1}]$.

The proof is by considering three different orderings between s'_m and s'_{m+1} .

Case-1: ($s'_m > s'_{m+1} \neq 0$): For a better understanding of the proof we refer to the Fig. 4 where the implication used is $I_{\mathbf{RC}}(x, y) = 1 - x + xy$, the Reichenbach implication. Note that $I_{\mathbf{RC}} \in \mathbb{I}_{\mathbf{D1}^c}^{\text{st}}$. Recall that $B'_m(y) = s'_m \rightarrow_{\mathbf{D1}^c} B_m(y)$

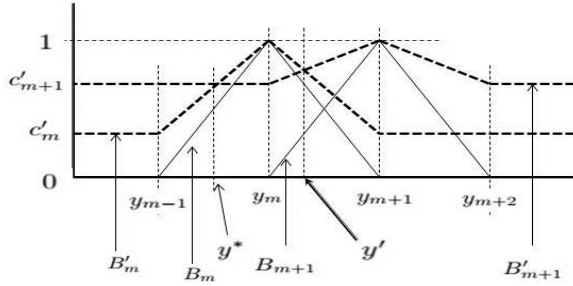


Fig. 4. The modified fuzzy sets using $I_{\mathbf{RC}}(x, y) = 1 - x + xy$, $0 < s'_{m+1} < s'_m$

and $B'_{m+1}(y) = s'_{m+1} \rightarrow_{\mathbf{D1}^c} B_{m+1}(y)$. We partition the space $Y = [y_1, y_n]$ into the following five sub-domains:

$$Y = \{y|y < y_{m-1}\} \cup [y_{m-1}, y_m] \cup [y_m, y_{m+1}] \\ \cup [y_{m+1}, y_{m+2}] \cup \{y|y > y_{m+2}\}, \quad (24)$$

and then discuss the behavior of B'_m and B'_{m+1} on these five sub-domains.

Behavior of B'_m and B'_{m+1} :

- **Over $\{y|y < y_{m-1}\}$:** For $y < y_{m-1}$, $B_m(y) = B_{m+1}(y) = 0$. Hence,

$$B'_m(y) = s'_m \rightarrow_{\mathbf{D1}^c} B_m(y) = s'_m \rightarrow_{\mathbf{D1}^c} 0 = c'_m \quad (25)$$

$$\text{and } B'_{m+1}(y) = s'_{m+1} \rightarrow_{\mathbf{D1}^c} B_{m+1}(y) \\ = s'_{m+1} \rightarrow_{\mathbf{D1}^c} 0 = c'_{m+1}. \quad (26)$$

Thus both B'_m and B'_{m+1} are a constant on this interval. Now using the strictness of $\rightarrow_{\mathbf{D1}^c}$, we have $B'_m(y) = c'_m = s'_m \rightarrow_{\mathbf{D1}^c} 0 < s'_{m+1} \rightarrow_{\mathbf{D1}^c} 0 = c'_{m+1} = B'_{m+1}(y)$. Thus B'_m and B'_{m+1} never intersect in $\{y|y < y_{m-1}\}$.

- **Over $[y_{m-1}, y_m]$:** On the interval $[y_{m-1}, y_m]$, $B_{m+1} \equiv 0$ while B_m is strictly increasing. Since $\rightarrow_{\mathbf{D1}^c}$ is strict,

$B'_m = s'_m \rightarrow_{\mathbf{D1}^c} B_m$ is strictly increasing. Hence, for any $y \in [y_{m-1}, y_m]$, $B'_{m+1}(y) = s'_{m+1} \rightarrow_{\mathbf{D1}^c} 0 = c'_{m+1}$, which is a constant value. Now, since B'_m is strictly increasing on this interval, $B'_m(y_{m-1}) = c'_m$, and $B'_m(y_m) = s'_m \rightarrow_{\mathbf{D1}^c} B_m(y_m) = s'_m \rightarrow_{\mathbf{D1}^c} 1 = 1$, the range of B'_m over $[y_{m-1}, y_m]$ is $[c'_m, 1]$. Once again, $c'_{m+1} = s'_{m+1} \rightarrow_{\mathbf{D1}^c} 0 > s'_m \rightarrow_{\mathbf{D1}^c} 0 = s'_{m+1} \rightarrow_{\mathbf{D1}^c} B_m(y_{m-1}) = B'_m(y_{m-1}) = c'_m$. Now, $c'_{m+1} > c'_m$ implies that $c'_{m+1} \in [c'_m, 1]$ and hence, clearly, B'_m and B'_{m+1} intersect at only one point in $[y_{m-1}, y_m]$. Let it be $y^* \in [y_{m-1}, y_m]$.

- **Over $[y_m, y_{m+1}]$:** On the interval $[y_m, y_{m+1}]$, B_m is strictly decreasing while B_{m+1} is strictly increasing. Since $\rightarrow_{\mathbf{D1}^c}$ is strict, B'_m is strictly decreasing, B'_{m+1} is strictly increasing and thus they intersect exactly at one point. Let it be $y' \in [y_m, y_{m+1}]$.
- **Over $[y_{m+1}, y_{m+2}]$:** For any $y \in [y_{m+1}, y_{m+2}]$, $B_m(y) = 0$, while on this interval B_{m+1} is strictly decreasing. Hence, $B'_m(y) = s'_m \rightarrow_{\mathbf{D1}^c} 0 = c'_m$, which is a constant value and by the strictness of $\rightarrow_{\mathbf{D1}^c}$, $B'_{m+1} = s'_{m+1} \rightarrow_{\mathbf{D1}^c} B_{m+1}$ is strictly decreasing. Now, since

$$B'_{m+1}(y_{m+1}) = s'_{m+1} \rightarrow_{\mathbf{D1}^c} B_{m+1}(y_{m+1}) \\ = s'_{m+1} \rightarrow_{\mathbf{D1}^c} 1 = 1, \\ B'_{m+1}(y_{m+2}) = s'_{m+1} \rightarrow_{\mathbf{D1}^c} 0 = c'_{m+1},$$

and B'_{m+1} is strictly decreasing on this interval, the range of B'_{m+1} over the interval $[y_{m+1}, y_{m+2}]$ is $[c'_{m+1}, 1]$. Again, since $c'_{m+1} > c'_m$ and $c'_m \notin [c'_{m+1}, 1]$ clearly, B'_m and B'_{m+1} do not intersect in $[y_{m-1}, y_m]$.

- **Over $\{y|y > y_{m+2}\}$:** For $y > y_{m+2}$, $B_m(y) = B_{m+1}(y) = 0$. Thus both B'_m and B'_{m+1} are a constant on this interval as given in (25) and (26) above. Now using the strictness of $\rightarrow_{\mathbf{D1}^c}$, we have $B'_m(y) = c'_m = s'_m \rightarrow_{\mathbf{D1}^c} 0 < s'_{m+1} \rightarrow_{\mathbf{D1}^c} 0 = c'_{m+1} = B'_{m+1}(y)$, and hence B'_m and B'_{m+1} never intersect on $\{y|y > y_{m+2}\}$.

The behavior of both B'_m and B'_{m+1} on the partition of Y as given in (24) is summarized in the following Table II, where by $\nearrow$ and $\searrow$, we mean strictly increasing and strictly decreasing, respectively. Hence the only points of intersection between B'_m and B'_{m+1} are $y^* \in [y_{m-1}, y_m]$ and $y' \in [y_m, y_{m+1}]$. Once again, due to the strictness of $\rightarrow_{\mathbf{D1}^c}$, it can be shown that $y' = \text{MOM}(B')$ is a point of intersection of B'_m and B'_{m+1} , similar to **Claim 3** of Theorem 7.1. Thus, we have

$$y_0 = g(x') = \text{MOM}(B') = \text{MOM}(B'_m \cap B'_{m+1}) \\ = \text{Mean}(\{y \in Y | B'_m(y) = B'_{m+1}(y) \text{ and } y \in \text{Ceil}(B')\}) \\ = \begin{cases} y^*, & \text{if } B'(y^*) > B'(y') \\ y', & \text{if } B'(y') > B'(y^*) \end{cases}$$

where, $\text{Mean}(S)$ = Average of the values of the elements in the set S . Since B'_{m+1} is constant on $[y^*, y_m]$ and increasing on $[y_m, y_{m+1}]$, $B'_{m+1}(y') > B'_{m+1}(y^*)$. Again, we have, $B'_m(y^*) = c'_{m+1} = B'_{m+1}(y^*) < B'_{m+1}(y') = B'_m(y')$. Now,

	$y < y_{m-1}$	$[y_{m-1}, y_m]$	$[y_m, y_{m+1}]$	$[y_{m+1}, y_{m+2}]$	$y > y_{m+2}$
B'_m	Const (c'_m)	$\nearrow$	$\searrow$	Const (c'_m)	Const (c'_m)
B'_{m+1}	Const (c'_{m+1})	Const (c'_{m+1})	$\nearrow$	$\searrow$	Const (c'_{m+1})
Points of Intersection	None	y^*	y'	None	None

TABLE II
BEHAVIOR OF B'_m AND B'_{m+1} ON THE OUTPUT SPACE Y , WHEN $s'_m > s'_{m+1}$

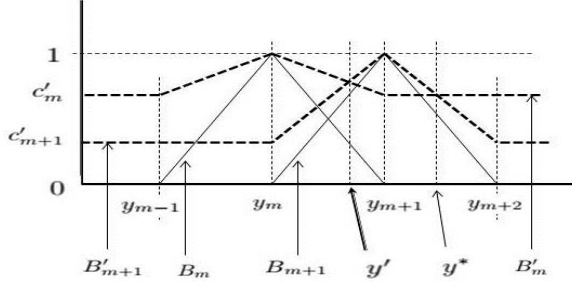


Fig. 5. The modified fuzzy sets using $I_{RC}(x, y) = 1 - x + xy$, $s'_{m+1} > s'_m > 0$

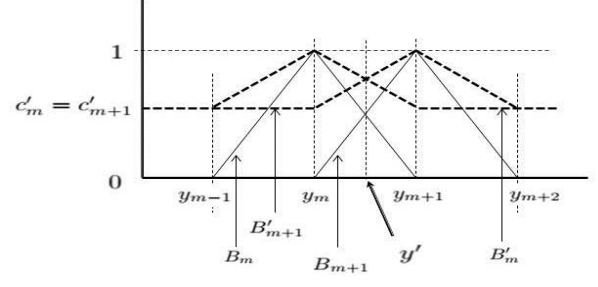


Fig. 6. The modified fuzzy sets using $I_{RC}(x, y) = 1 - x + xy$, $s'_{m+1} = s'_m = \frac{1}{2}$

$B'_{m+1}(y^*) < B'_{m+1}(y')$ and $B'_m(y^*) < B'_m(y')$ implies
 $\min\{B'_m(y^*), B'_{m+1}(y^*)\} < \min\{B'_m(y'), B'_{m+1}(y')\}$
 $\Rightarrow B'(y^*) < B'(y') .$

Hence, $y_0 = y' \in [y_m, y_{m+1}]$. Hence the **Claim 4**. $\square$

Case-2: ($s'_{m+1} > s'_m \neq 0$): Along similar lines as argued in **Case-1**, **Claim 4** can be proven in this case too.

For a better understanding we refer to Fig. 5 where the implication used is the same Reichenbach implication.

The behavior of both B'_m and B'_{m+1} on the partition of Y as given in (24) when $s'_{m+1} > s'_m > 0$ is summarized in the following Table III, where, once again, by $\nearrow$ and $\searrow$, we mean strictly increasing and strictly decreasing, respectively.

Case-3: ($s'_m = s'_{m+1} = \frac{1}{2}$): Since A_i 's form a Ruspini partition, $s'_m + s'_{m+1} = 1$ implies that if $s'_m = s'_{m+1}$ then their common value is $\frac{1}{2}$.

Similarly, as in **Case-1**, we partition the space Y as given in (24) and discuss the behavior of B'_m and B'_{m+1} over these five sub-domains.

For a better understanding we refer to the Fig. 6 where the implication used is once again the same Reichenbach implication. The behavior of both B'_m and B'_{m+1} on the partition of Y as given in (24) when $s'_m = s'_{m+1} = \frac{1}{2}$ is summarized in the following Table IV.

Once again, due to the strictness of $\rightarrow_{D1^c}$, it can be shown that $y' = \text{MOM}(B')$ is a point of intersection of B'_m and B'_{m+1} , similar to **Claim 3** of Theorem 7.1 and the set of points over which B'_m and B'_{m+1} intersect can be summarised as follows:

$$\{y \in Y | B'_{m+1}(y) = B'_m(y)\} \\ = \{y | y < y_{m-1}\} \cup \{y'\} \cup \{y | y > y_{m+2}\} .$$

Firstly, note that $B'_{m+1}(y) = B'_m(y) = c'_m$ on the sub-domains $\{y | y < y_{m-1}\}$ and $\{y | y > y_{m+2}\}$. Secondly, since $y' \in [y_m, y_{m+1}]$ $B'_m(y') = s'_m \rightarrow_{D1^c} B'_m(y') >$

$s'_m \rightarrow_{D1^c} B'_m(y_{m+1}) = \frac{1}{2} \rightarrow_{D1^c} 0 = c'_m$, while $B'_{m+1}(y') = s'_{m+1} \rightarrow_{D1^c} B'_{m+1}(y') > s'_{m+1} \rightarrow_{D1^c} B'_{m+1}(y_m) = \frac{1}{2} \rightarrow_{D1^c} 0 = c'_m$. Thus, we have

$$g(x') = \text{MOM}(B') = \text{MOM}(B'_m \cap B'_{m+1}) \\ = \text{Mean}(\{y \in Y | B'_m(y) = B'_{m+1}(y) \text{ and } y \in \text{Ceil}(B')\}) \\ = y' .$$

Thus we have proven that if $x' \in [x_m, x_{m+1}]$, then $y' \in [y_m, y_{m+1}]$ for $m \in \{1, 2, \dots, n-1\}$.

Once again, the fact that when $B_m = B_{m+1}$, $y' = y_m \in [y_m, y_{m+1}]$ follows along similar lines of the corresponding case in Theorem 7.1 above.

Claim 5: The system function g is monotonic, i.e., if $x' \leq x''$ then $g(x') = y' \leq y'' = g(x'')$.

Case 1: Let $x' \in [x_m, x_{m+1}]$ and $x'' \in [x_{m+p}, x_{m+p+1}]$, where $p \geq 1$. By the **Claim 4** above, irrespective of the orderings between B_m, B_{m+1} and B_{m+p}, B_{m+p+1} , we have that $y' \in [y_m, y_{m+1}]$ and $y'' \in [y_{m+p}, y_{m+p+1}]$ and hence $y' \leq y''$.

Case 2: Let $x', x'' \in [x_m, x_{m+1}]$, i.e., $p = 0$ in the above **Case 1**.

Case 2a: If $B_m = B_{m+1}$, then by the above Claim 4 we obtain, $y' = y'' = y_m$. Thus, trivially, we have $x' \leq x'' \Rightarrow y' \leq y''$.

Case 2b: Let $B_m \neq B_{m+1}$. The argument to show that $y' \leq y''$ proceeds along similar lines of the corresponding case as given in Theorem 7.1 above and this proves the Claim 5 and completes the proof. $\blacksquare$

VIII. ILLUSTRATIVE EXAMPLES WITH YAGER'S CLASS OF FUZZY IMPLICATIONS

In this section we begin by illustrating the results of the previous section through some examples. We have chosen two fuzzy implications, one each from the classes of $\mathbb{I}_{N_{D1}^c}^{\text{st}}$

	$y < y_{m-1}$	$[y_{m-1}, y_m]$	$[y_m, y_{m+1}]$	$[y_{m+1}, y_{m+2}]$	$y > y_{m+2}$
B'_m	Const (c'_m)	$\nearrow$	$\swarrow$	Const (c'_m)	Const (c'_m)
B'_{m+1}	Const (c'_{m+1})	Const (c'_{m+1})	$\nearrow$	$\swarrow$	Const (c'_{m+1})
Points of Intersection	None	None	y'	y^*	None

TABLE III
BEHAVIOR OF B'_m AND B'_{m+1} ON THE OUTPUT SPACE Y , WHEN $s'_m < s'_{m+1}$

	$y < y_{m-1}$	$[y_{m-1}, y_m]$	$[y_m, y_{m+1}]$	$[y_{m+1}, y_{m+2}]$	$y > y_{m+2}$
B'_m	Const (c'_m)	$\nearrow$	$\swarrow$	Const (c'_m)	Const (c'_m)
B'_{m+1}	Const (c'_{m+1})	Const (c'_{m+1})	$\nearrow$	$\swarrow$	Const (c'_{m+1})
Points of Intersection	$y < y_{m-1}$	y_{m-1}	y'	y_{m+2}	$y > y_{m+2}$

TABLE IV
BEHAVIOR OF B'_m AND B'_{m+1} ON THE OUTPUT SPACE Y , WHEN $s'_m = s'_{m+1} = \frac{1}{2}$

and $\mathbb{I}_{N_{D1}}^{\text{st}}$. Further, these fuzzy implications also belong to the class of Yager's implications, see Section VI-E. In fact, as a corollary of the results in Section VII, in Section VIII-B, we show the monotonicity of BKS- $\mathcal{Y}$ inference mechanisms that use the Yager's family of fuzzy implications to model the rule base. While we have considered only the MOM defuzzification so far, in Section VIII-C, we consider other defuzzification methods which allow us to make some interesting observations.

A. Some Illustrative Examples of the monotonicity of $\mathbb{F}_{\rightarrow D1^c}$ and $\mathbb{F}_{\rightarrow D1}$

Let us consider the rule base as given in Example 5.2 and the FRIs:

- (i) $\mathbb{F}_{\rightarrow D1} = (\{A_i\}_{i=1}^3, \{B_i\}_{i=1}^3, \hat{R}_{\rightarrow D1}, \text{MOM})$ and
- (ii) $\mathbb{F}_{\rightarrow D1^c} = (\{A_i\}_{i=1}^3, \{B_i\}_{i=1}^3, \hat{R}_{\rightarrow D1^c}, \text{MOM})$.

In the examples, we have investigated the behaviour of the system function for monotonicity.

Example 8.1: Let us consider the fuzzy system $\mathbb{F}_{\rightarrow D1^c}$ with the rule base (11) given in Example 5.2, and let the implication operator employed in the relation $\hat{R}_{\rightarrow D1^c}$ be the Reichenbach implication, which is strict but $N_{I_{RC}} \neq N_{D1}$, i.e., $I_{RC} \in \mathbb{I}_{N_{D1}}^{\text{st}}$ (see Table I). Let us denote this FRI by $\mathbb{F}_{RC}$ for brevity. The system function g with MOM defuzzification is given in Fig. 7, which is clearly monotonic as claimed in Theorem 7.3.

Example 8.2: Let us consider the fuzzy system $\mathbb{F}_{\rightarrow D1}$ with the rule base (11) given in Example 5.2, and let the implication operator employed in the relation $\hat{R}_{\rightarrow D1}$ be the Yager's implication, which is strict and $N_{I_{YG}} = N_{D1}$, i.e., $I_{YG} \in \mathbb{I}_{N_{D1}}^{\text{st}}$ (see Table I). Let us denote this FRI by $\mathbb{F}_{YG}$ for brevity. The system function g with MOM defuzzification is given in Fig. 7, which is clearly monotonic as claimed in Theorem 7.1.

B. Monotonicity of BKS- $\mathcal{Y}$ Inference Mechanisms

In our previous works, [16], [18], we have seen that BKS with Yager's families of fuzzy implications $\mathbb{F}_{\rightarrow \mathcal{Y}}$, where $\rightarrow_{\mathcal{Y}}$ stands for any of the Yager's families of fuzzy implications, possess the following desirable properties, namely, interpolativity, continuity, robustness and universal approximation capability.

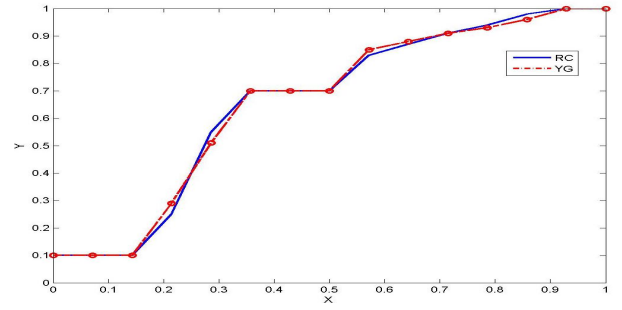


Fig. 7. System function with MOM defuzzification of the FRI (a) $\mathbb{F}_{RC}$ given in Example 8.1 (b) $\mathbb{F}_{YG}$ given in Example 8.2 with $\rightarrow_{D1} = I_{YG}$, the Yager's implication.

Based on the results in Section VII, we show that BKS with Yager's families of fuzzy implications are also monotonic, i.e., the corresponding system functions are monotonic. The results are, in fact, some special cases of the above Theorem 7.1 and Theorem 7.3.

Corollary 8.3: Let us be given a fuzzy IF-THEN rule base $\mathcal{R}_M(A_i, B_i)$ as in (10) which is monotone and $A_i \in \mathcal{P}_X$, $i = 1, 2, \dots, n$, form a Ruspini partition on X and $B_i \in \mathcal{P}_Y$, $i = 1, 2, \dots, n$, form a Ruspini partition on Y , respectively. Further, let every element of $\mathcal{P}_X$ and $\mathcal{P}_Y$ be normal, convex and strictly monotone on both sides of the ceiling, i.e., $\mathcal{P}_X \subseteq \mathcal{F}^*(X)$ and $\mathcal{P}_Y \subseteq \mathcal{F}^*(Y)$. Then the system function g of the FRI $\mathbb{F}_{\rightarrow f} = (\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_f, \text{MOM})$ is monotonic, where $\hat{R}_f$ is defined as in (27) and $\rightarrow_f \in \mathbb{I}_{\mathbb{F}} = \mathbb{I}_{\mathbb{F}, \infty} \cup \mathbb{I}_{\mathbb{F}, 1}$:

$$\hat{R}_f(x, y) = \hat{R}_{\rightarrow f}(x, y) = \bigwedge_{i=1}^n (A_i(x) \rightarrow_f B_i(y)), \quad (27)$$

Proof: Every $\rightarrow_f \in \mathbb{I}_{\mathbb{F}}$ is strict. Thus $\rightarrow_f \in \mathbb{I}_{\mathbb{F}} \subsetneq \mathbb{I}^{\text{st}}$ and the result follows from Theorem 7.1 and Theorem 7.3. ■

Corollary 8.4: Let us be given a fuzzy IF-THEN rule base $\mathcal{R}_M(A_i, B_i)$ as in (10) which is monotone and $A_i \in \mathcal{P}_X$, $i = 1, 2, \dots, n$, form a Ruspini partition on X and $B_i \in \mathcal{P}_Y$, $i = 1, 2, \dots, n$, form a Ruspini partition on Y , respectively. Further, let every element of $\mathcal{P}_X$ and $\mathcal{P}_Y$ be normal, convex and strictly monotone on both sides of the ceiling, i.e., $\mathcal{P}_X \subseteq \mathcal{F}^*(X)$ and $\mathcal{P}_Y \subseteq \mathcal{F}^*(Y)$. Then the system function g of the

FRI $\mathbb{F}_{\rightarrow_g} = (\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_g, \text{MOM})$ is monotonic, where $\hat{R}_g$ is defined as in (28) and $\rightarrow_g \in \mathbb{I}_{\mathbb{G}}$.

$$\hat{R}_g(x, y) = \hat{R}_{\rightarrow_g}(x, y) = \bigwedge_{i=1}^n (A_i(x) \rightarrow_g B_i(y)), \quad (28)$$

Proof: Every $\rightarrow_g \in \mathbb{I}_{\mathbb{G}}$ is strict and its natural negation is the Gödel negation. Thus $\rightarrow_g \in \mathbb{I}_{\mathbb{G}} \subsetneq \mathbb{I}_{N_{D1}}^{\text{st}} \subsetneq \mathbb{I}^{\text{st}}$ and the result follows from Theorem 7.1. ■

C. Monotonicity under Different Defuzzification Methods

In Section VII, we have proven our results, viz., Theorems 7.1 and 7.3 for the FRIs $\mathbb{F}_{\rightarrow_{D1}}$ and $\mathbb{F}_{\rightarrow_{D1^c}}$ with the MOM defuzzification. The examples in Section VIII-A above illustrate these results, albeit by considering some specific fuzzy implications from each of the classes of $\mathbb{I}_{N_{D1}}^{\text{st}}$ and $\mathbb{I}_{N_{D1^c}}^{\text{st}}$, but here again we have used only the MOM defuzzification.

An interesting question that crops up now is this: *What if we use an alternate defuzzification method? Does the monotonicity of the system function still hold?*

In the examples, we investigate the monotonicity of the system function by considering the same FRIs with reducible composition, $\mathbb{F}_{RC}$ and $\mathbb{F}_{YG}$, as in Section VIII-A,

- (i) $\mathbb{F}_{YG} = (\{A_i\}_{i=1}^3, \{B_i\}_{i=1}^3, \hat{R}_{\rightarrow_{D1}}, d), \rightarrow_{D1} = I_{YG}$ and
- (ii) $\mathbb{F}_{RC} = (\{A_i\}_{i=1}^3, \{B_i\}_{i=1}^3, \hat{R}_{\rightarrow_{D1^c}}, d), \rightarrow_{D1^c} = I_{RC}$,

but where the defuzzification method d is one of the following (see Section II-B): (i) LOM, (ii) SOM, (iii) COG and (iv) BIS. The system functions corresponding to the FRIs $\mathbb{F}_{RC}$ and $\mathbb{F}_{YG}$, with different types of defuzzification methods, are shown in Fig. 8 and Fig. 9, respectively.

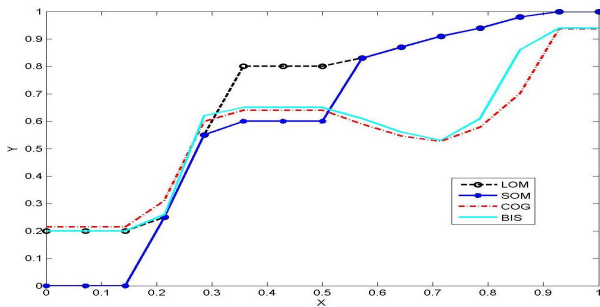


Fig. 8. System function of the FRI $\mathbb{F}_{RC}$ given in Example 8.1 with LOM, SOM, COG and BIS defuzzifier.

Note that, among the defuzzification methods considered, the MOM, SOM and LOM methods can be seen as *Ceiling-based* methods, since the defuzzified value depends only on the ceiling of the fuzzy set under consideration. In the case of convex fuzzy sets, these three methods are such that the defuzzified output falls within the ceiling of the fuzzy set, while the same is not true, in general, for COG and BIS methods.

However, it is interesting to note the following. On the one hand, for the FRI $\mathbb{F}_{RC}$, the corresponding system function is monotonic with MOM defuzzification as well as with other ceiling-based defuzzification methods (e.g., SOM and LOM),

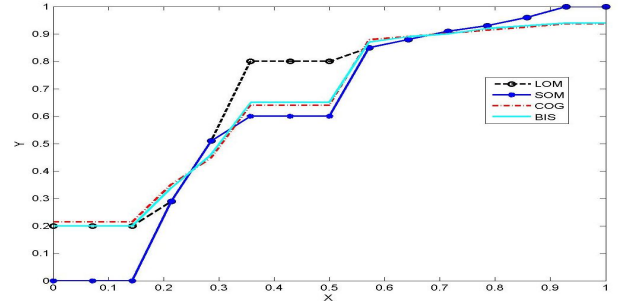


Fig. 9. System function of the FRI $\mathbb{F}_{YG}$ given in Example 8.2 with LOM, SOM, COG and BIS defuzzifier.

see Fig. 8(—●—●—) and Fig. 8(—○—○—), whereas it is not monotonic while considering the defuzzification methods COG and BIS, see Fig. 8(—·—·—·—) and Fig. 8(—■—■—). On the other hand, for the FRI $\mathbb{F}_{YG}$, the corresponding system function is monotonic with all the defuzzification methods considered here, see Fig. 9

This seems to point to the fact that Theorems 7.1 and 7.3 may be valid even when d is taken to be any ceiling based defuzzification method instead of the MOM defuzzifier.

IX. CONCLUDING REMARKS

In this work, we have investigated the monotonicity of a single input single output (SISO) FRI with fuzzy implications under suitable choice of operations for the other components of the fuzzy system. The highlights of this work are two fold:

- (i) This is the first work that has illustrated that monotonicity for FRIs can be ensured without modifying the given rule base, as is common in the literature.
- (ii) The class of fuzzy implications considered do not come from a residuated setting, which is once again the common setting in all the earlier works.

In fact, our results are valid for a large class of fuzzy implications, viz., $\mathbb{I}^{\text{st}}$ (see Section VI-D) and the proofs presented are general enough not to depend on their form or representation. Further, from the discussions in Section VIII-C, it appears that our results could also be generalised for any ceiling-based defuzzification methods.

There exist many families of fuzzy implications, other than those obtained as residuals of generalised conjunctions. So far, however, their employability in applications has not received much attention. The first such work dealing with FRIs that employ the Yager's families of fuzzy implications and their suitability appeared recently in this very journal [16] and further studies on it appeared in [18]. Our results in Section VIII-B have taken this to the next logical step by showing that these FRIs can also be employed without compromising on monotonicity.

Thus, it is clear from these works, that other well known families of fuzzy implications should not be treated as just objects of mathematical curiosity but as those with the potential to be used in an applicational setting.

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