Vortices and the effects of atmospheric density stratification

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Abstract: Our goal is to study the effects of atmospheric density stratification on the formation and longevity of aircraft trailing vortices. Towards this end, we study a series of model two-dimensional problems, two of which are presented here. The studies are summarised here from [1, 2, 3]. The first is on the dynamics of a single vortex located at a density stratified layer. Due to the presence of the vortex, the density interface rolls up into a spiral. Centrifugal acceleration is shown to play a major role in reducing the life-span of the vortex. Due to the misalignment of the acceleration vector with the normal to the spiral interface, vorticity is created along the density interface, resulting in a spiral Kelvin-Helmholtz instability. At every interface where heavier fluid is closer to the vortex than lighter, the flow is also unstable due to a centrifugal Rayleigh-Taylor instability. In the second problem, centrifugal effects are neglected, i.e., the Boussinesq approximation is employed. Here gravity is shown to eat into the vortex and create small-scale structures, although the initial stratification is stable. In both cases, a turbulence-like state ensues. The relevance of the present work to aircraft trailing vortices is discussed in the introduction, and ongoing work addresses this problem more directly.

Key words: Brunt Vaisala frequency, instability

1. INTRODUCTION AND LITERATURE REVIEW

The wake behind an aircraft consists predominantly of pairs of counter-rotating vortices formed due to the finite span of the wing. This happens because a vortex sheet is shed behind the wing which subsequently rolls up at the wing tips. Significant progress on understanding trailing vortices began in the 1970's. The prediction of the structure and the evolution of trailing vortices is crucial to improving existing standards of air traffic control. In particular, an understanding of the life-span of the vortices may enable a reduction in the waiting time between successive take-offs and landings, resulting in higher airport efficiency. As mentioned in Spalart's *Annual Review* [4], a 100-ton airplane following a 300-ton airplane must be no closer than 5 nautical miles, and any reduction in this separation distance is welcome. The separation distance in turn depends in significant measure on the stability of trailing vortices. [5] showed the presence of a long-wavelength instability where the vortices are bent along their axes. Such was the importance of this discovery, that in 1971, a conference titled "Aircraft wake turbulence" was organized aimed at understanding the Crow instability. A few years later, it was shown that a shorter wavelength instability also exists in these vortex systems ([6]). This instabilily, also called the elliptic instability, was shown to be far more generic for a vortex in the presence of a strain field. This initial excitement could not be translated very easily to significant changes in air traffic control standards, and there is scope for work towards this end. The above instabilities are three-dimensional in nature. In this study, we are concerned with another set of instabilities, which are two-dimensional, and arise out of density stratification in the ambient atmosphere.

During the early stages of the evolution of the vortex sheet, the flow field can be treated as two-dimensional, especially for high aspect ratio wings. The vortex sheet is composed of a continuum of horse-shoe vortices in compliance with the Helmholtz theorem. This sheet rolls up into two spiral vortices near the wing tips, and was shown to obey a self-similar evolution by [7] in the absence of density variations. Near the centre of the spiral, viscous effects become important, resulting in the formation of a circular vortex core. [7] show that the flow field near the wing-tip consists of four regions. In the innermost region, viscous effects dominate resulting in a circular vortex core. Outside this region, spiral vortex sheets persists where the flow field is of an inviscid nature.

[8] (SD hereafter) were the first to address the problem

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of descent of a vortex pair in a stratified atmosphere, and their model predicted an accelerated descent of the vortex pair. As the vortices more downwards, baroclinic vorticity is generated along the streamline separating the two vortices from rest of the ambient fluid. Their model predicts that the vortex separation distance R decreases with time as

$$R = R_1 / \cosh(At), \tag{1}$$

and the descent velocity increases as

$$W = W_1 \cosh(At). \tag{2}$$

The exponential acceleration of the vortices was confirmed by [9] for weak stratifications. But the assumption of constant circulation has been questioned by [10] using laboratory experiments. [11] was also very critical of this assumption. Using a model where the distance betwen the vortices was kept constant and circulation was allowed to vary, Saffman arrived at a completely different prediction, that the descent height oscillates with a time period $2\pi/N$ as

$$H = H_0 \sin(Nt), \tag{3}$$

where N is the Brunt-Vaisala frequency. [12] attemped to model entrainment of the ambient field and showed that trailing vortices first decelerate and come to rest, and then are accelerated upwards. Accurate experiments were carried out by [13] to test the above theoretical predictions, but his experiments had the drawback that they were carried out with low aspect ratio wings making the flow three dimensional. Oscillations of wake height as predicted by Saffman were not seen. The wake velocity was shown to decrease with time, in variance with the predictions of SD or Crow. The experimental results are similar to the predictions of [12], but that comparison was surprisingly not made by Sarpkaya.

We skip detailed discussion on the early numerical calculations of Hill (1974) and Hecht *et al.* (1981) and focus on the detailed two-dimensional calculations of [14]. As in the theoretical analysis, he considers a pair of compact vortices in a linear density stratification. He finds good agreement with Sarpkaya's experiments at short time for the descent height of the vortices. In other words, he does not find good agreement with either SD/Crow or Saffman's theories. It appears that to improve comparison with experiments or numerical simulations, one needs to consider the effects of varying circulation and vortex separation distance.

We have broken our efforts to study this problem into a series of two-dimensional model problems. We fist consider a single compact vortex placed at a single straight density interface ([1]). We study this dynamics using a combination of stability theory and direct numerical simulations. In order to simplify the analysis, we neglect effects of gravity and consider only centrifugal forces. As the density sheet evolves into a spiral, vorticity is created along the spiral since the normal to the spiral and the acceleration vectors do not coincide. This leads to a spiral Kelvin-Helmholtz instability. A combined centrifugal Rayleigh-Taylor instability and spiral Kelvin-Helmholtz instability can be seen in the direct numerical simulations of figure 4. A key feature seen in fig. 4 is the occurrence of significant growth of the instability at a distinct radial location, along with a specific wavenumber. In the analysis below (summarised from [1], we show that the instabilility mechanism is inherently inviscid, and can be demonstrated by a point-vortex vortex at a density interface, whereas the quantitative features of the instability require a vortex with a finite size, and viscosity.

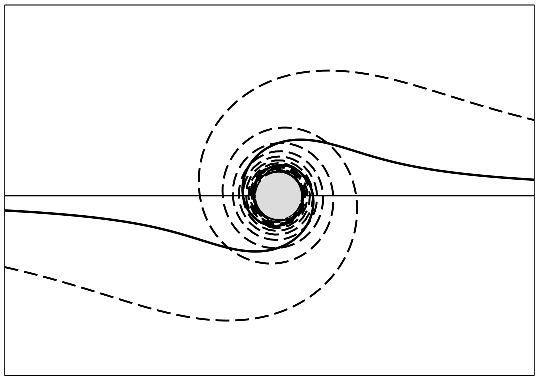


Figure 1: Roll up of a density interface in the neighbourhood of the vortex

2. POINT VORTEX MODEL

A point vortex is placed at a density interface resulting in a spiral roll-up as mentioned earlier, shown schematically in fig. 1.

Let the circulation of the vortex be $2\pi\Gamma$ and the initial density jump across the flat interface be $\Delta\rho$. The spiral interface is nearly circular in the neighbourhood of the vortex and the spacing between sucessive turns increases radially outwards. A simple scaling argument shows that the spacing scales with radius as

$$\lambda \sim \frac{r^3}{\Gamma t}. \tag{4}$$

Very close to the centre, successive turns of the interface are very close to each other, so diffusivity causes the density field to homogenise within a radius

$$\frac{r_h}{l_d} \sim Pe^{1/3}. \tag{5}$$

The spiral interface however, exists up to a radius

$$r_s \sim l_d Pe^{1/2}. \tag{6}$$

Here l_d is the diffusive length scale, and Pe is the Peclet number. For a very large Peclet number, a large number of spiral density jumps exists between r_h and r_s.

To allow for a simplified stability analysis, we treat the spiral interface as being circular. For an incompressible flow in the inviscid limit, the dispersion relation with a single circular density jump and a point vortex can be written as

$$\omega = \frac{m\Gamma}{r_1^2} \pm \sqrt{-\frac{m\Gamma^2}{r_1^4}\frac{(\rho_0 - \rho_1)}{(\rho_0 + \rho_1)}}. \tag{7}$$

When heavier fluid is inside lighter, we thus have a centrifugal Rayleigh-Taylor instability, as expected.

We next relax the imposition of a perfectly circular interface and account for the creation of vorticity along it. For this, the shape of the interface needs to be obtained, for which it is simplest to approximate the interface as winding up passively. The interface then takes the form of a Lituus spiral. For small Atwood numbers, we expect this shape of the spiral to be unchanged. The Lituus spiral in polar coordinates can be written as $\theta_s = \frac{\Gamma t}{r^2}$, and the tangent of the angle α made by the normal to the spiral interface with acceleration vecto is given by $\tan\alpha = \frac{r^2}{2\Gamma t}$. Treating the density gradient at the interface as a *delta*-function, the vorticity created along thi interface can be given by the following equation:

$$\Omega(r,t) = \mp\mathcal{A}U \log(\theta_s + (1+\theta_s^2)^{1/2})\delta(r\theta - r\theta_s) \equiv \Delta U_\theta \delta(r\theta-$$

This predicts a velocity jump at the interface equal to

$$\Delta U_\theta \simeq \mp\mathcal{A}U \log(\frac{2\Gamma t}{r^2}). \qquad (9$$

This is a Kelvin-Helmholtz instability created at a density interface. For more details, see [1]. Since the velocity jump varies logarithmically, the growth of the instability is slightly faster than an exponential growth, taking the form, $u_r \sim At^{Bt}$

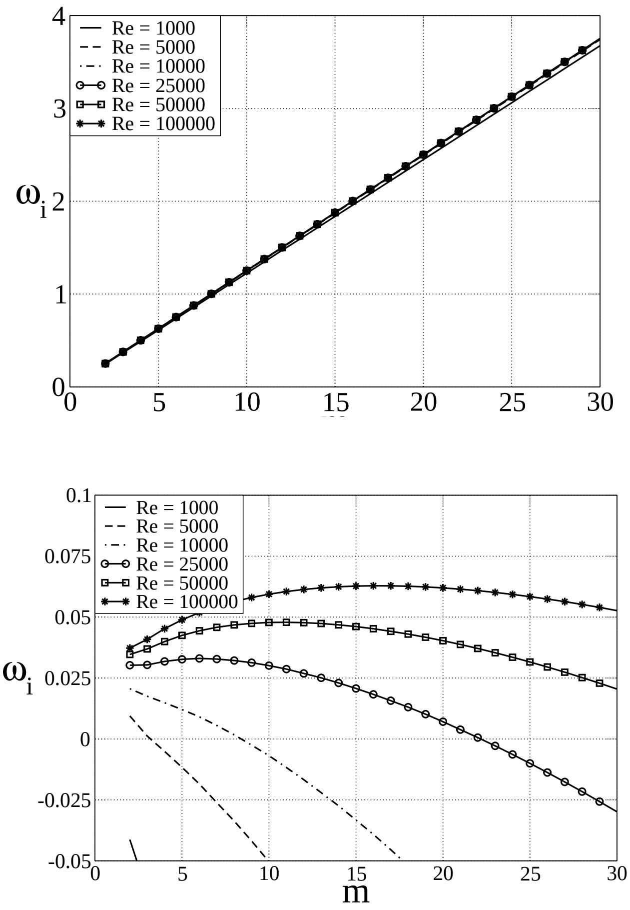
3. STABILITY ANALYSIS OF FINITE-CORES:

We now consider a vortex with a finite core with circular density jumps outside the core. Consistent with the predictions of equn.(9), velocity jumps of appropriate size are placed at these density jumps. Both a viscous and an inviscid stability analysis have been carried out and are presented below. For the governing equations, refer to equations (3.1-3.4) of [1]. The equations were solved using the standard Chebyshev collocation method. For a single circular density jump with out any velocity jump, viscous effects dampen the large m modes as shown in fig. 2. For a better comparison with the numerical simulations to be presented later, we have to include the effects of a vortex sheet at the density interface.

Figure 3 shows that combined effects of a density and a velocity jump for various interface thicknesses. The inviscid analysis predicts a wavenumber of 10 for an interface thickness of $d = 0.03r_c$ which is comparable to the thicknesses found in the numerical simulations. Numerical simulations with were also carried out for three different Reynolds numbers, and a viscous stability analysis for these Reynolds numbers predicted a wavenumber between 3 – 5.

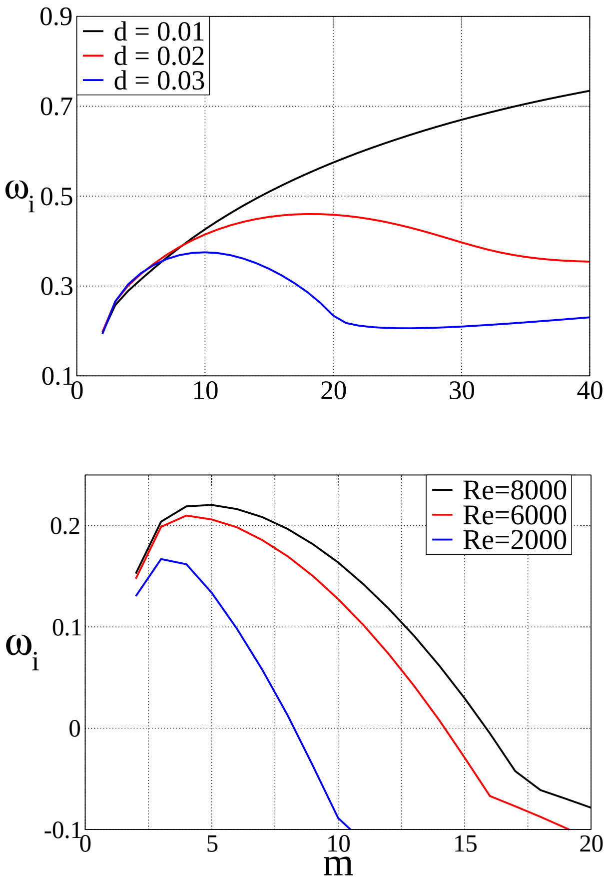
We now present results from direct numerical simulations. These simulations have enabled us to check the predictions of the linear stability analysis and further probe the nonlinear behaviour.

The growth rate of this instability shown in fig.5, which scales with the inertial scales in the flow, can be seen to be in very good agreement with the stability analysis shown in fig. 3(a). In all of the above analysis, effect of gravity was completely neglected. Inclusion of gravity [2] was found to



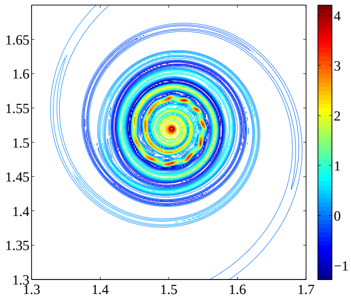
(b)

Figure 2: Frequency and growth rate of the most unstable mode for rankine vortex with a single circular density jump at $r_j = 2r_c$ whose thickness is $d = 0.02$.

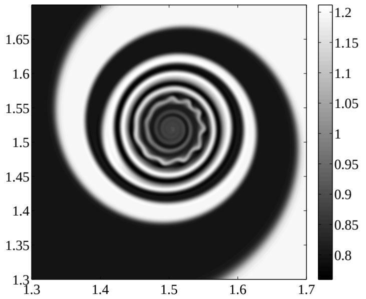


(b)

Figure 3: Growth rate for a smooth vortex with two circular density and velocity jumps of same size at $r_1 = 1.3r_c$ and $r_2 = 1.6r_c$, (a) for varying thickness with inviscid analysis, $\mathcal{A} = 0.2$, (b) Viscous analysis at various Reynolds numbers for $\mathcal{A} = 0.1, d = 0.05$. The dashed line is for $Re = 2000$ but with $\mathcal{A} = 0.05, d = 0.05$. The black filled circle shows the highest growth rate for $Re = 2000, \mathcal{A} = 0.1, d = 0.2, r_2 = 1.4$.



(a)



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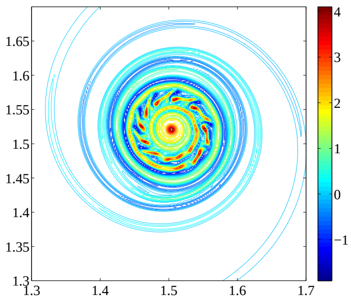
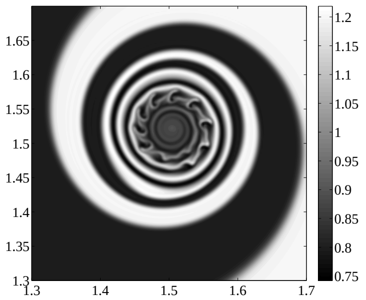


(c)



(d)

Figure 4: Evolution of the vorticity (a, c) and density (b, d) fields in the inviscid simulations. The time t, non-dimensionalized with respect to the period of rotation of the vortex core r_c^2/Γ, is (a,b) 45, (c,d) 50. The Atwood number is 0.2.

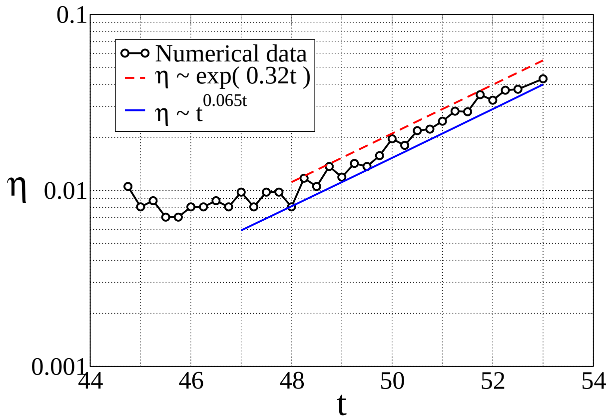


Figure 5: Growth rate from the numerical simulation of fig.4. The straight line is an exponential fit. The stability analysis of fig.3(a) predicted a growth as $\exp(0.37t)$.

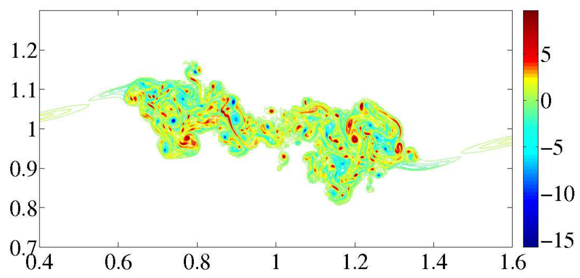


Figure 6: Vorticity field in the presence of gravity with no centrifugal effects at $t = 95$ and Atwood number of 0.2.

result in a catastrophic breakdown of the vortex as shown in fig. 6. Further details of this part of the work will be discussed at the conference.

4. FUTURE WORK:

In the present work, we considered the evolution of an initially flat density interface in the presence of a vortex. The instability found in the present work was found to scale with the inertial time scale of the system. However, in the case of an aircraft trailing vortex, a vortex sheet peels off the wing which subsequently rolls up along with a density interface. Some questions that remain unanswered are:
1. What is the structure of the trailing vortex when the density field evolves along with the vortex sheet during the roll-up process?
2. In the presence of density stratification, how do the centrifugal Rayleigh-Taylor and the spiral Kelvin-Helmholtz instabilities manifest themselves, if at all, during this evolution?
3. For a trailing vortex pair descending in a stratified medium, the results of various groups are not in agreement. Does studying the effects of the transient roll-up process lead to a better understanding of their evolution in a strafied medium?
4. The viscous cases present here are for a Prandtl number of 10. What would happen to them in the case of a Prandtl number of 1?
Ongoing work ([3]) is aimed at addressing these.

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