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Effect of density stratification on vortex merger

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We study the effect of density stratification in the plane on the merging of two equal vortices. Direct numerical simulations are performed for a wide range of parameters. Boussinesq and non-Boussinesq effects are considered separately. With the Boussinesq approximation, moderate to high Prandtl number and Froude number close to unity, there is a monotonic drifting away of the vortices from each other, and merger is completely prevented. Among non-Boussinesq effects, the inertial effects of density stratification are highlighted. These give rise to a breaking of symmetry, and consequently, the vorticity centroid is found to drift significantly from its initial position. Using an idealized model, we explore the role of baroclinic vorticity in determining these features of the merger process. © *2013 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4773445]

I. INTRODUCTION

One of the simplest forms of interaction between vortices is the merger of two co-rotating vortices as shown schematically in Figure 1. In the last few decades, vortex merger has received a great deal of attention, especially since it forms one of the basic interaction processes in a turbulent flow. This is more apparent in two-dimensional turbulence where smaller eddies "merge" to form larger eddies as the flow evolves, which is believed to be a fundamental mechanism for the transfer of energy to larger scales. Simultaneously, the merger of vortex cores is almost always associated with the formation of filamentary debris in the form of tight spirals with very thin cross sections. These filaments cause a cascade of enstrophy to smaller scales. The first detailed observations of vortex merging can be attributed to experiments on mixing layers. Freymuth[1] observed the coalescence of vortices in a separated laminar boundary layer. More detailed observations were made by Winand and Browand,[2] who attributed the growth of a turbulent mixing-layer to vortex merger.

Much of the early work involved studies in a purely inviscid context in an unstratified flow. Brandt and Iversen[3] and Rossow[4] studied the role of vortex merger in aircraft trailing vortices using point vortex methods. For finite but inviscid vortices of radii a and separation distance b, it was soon found that a critical ratio $(a/b)_{cr}$ exits below which a merger process is not initiated.[5–8] For $a/b < (a/b)_{cr}$ these inviscid vortices essentially rotate about their fixed centroid indefinitely with a constant angular velocity. During the initial stages, due to the straining field of its neighbor, each vortex supports Kelvin waves,[5] which amplify, eventually distorting the vortex to an equilibrium state. This *adaptation* process was studied by Le Dizès and Verga[9] using direct numerical simulations (DNS) with various vorticity profiles by monitoring the eccentricities of the vortex core. They showed that the eccentricity exhibits a damped oscillation and attributed this to the damped Kelvin modes (*quasi-modes*) of each vortex,[10] which results in an equilibrium state (in a rotating frame). Equilibrium states for multiple vortex configurations were first studied in the context of finding

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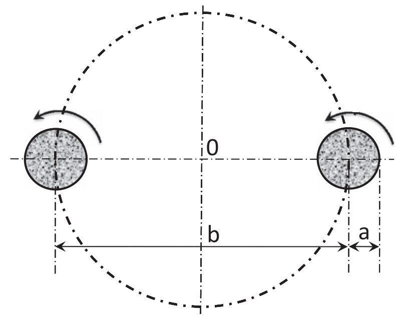

FIG. 1. Schematic of a two-dimensional initial vortex configuration, which will lead to merger. The arrows indicate the sense of vorticity, such that the two vortices initially describe an anti-clockwise revolution of period $\pi b_0^2/\Gamma$, where the subscript 0 stands for initial conditions, about the vorticity centroid "O." The $2\pi\Gamma$ is the circulation of each vortex.

stable non-axisymmetric solutions of the Euler's equations.[11] The stability of these states were then studied by Saffman and Szeto[5] and Overman and Zabusky[6] and it was found that these configurations become unstable if the vortices are too close to each other. This analysis was later extended[12] to many vortices rotating about a common centre. For symmetric systems, to arrive at an equilibrium configuration, it suffices to analyze the deformation of just one vortex. Equilibrium shapes with a non-uniform vorticity distribution were studied by Meunier *et al.*[8] to arrive at a more realistic critical separation distance, which could be compared with their experiments. They attribute the merger of the vortices to the conservation of angular momentum as the filaments are ejected outside the core. Meunier and Leweke[13] experimentally studied the three-dimensional instabilities in vortex merger. In the absence of viscosity, the vortices either reach an equilibrium state, or undergo a rapid merger depending on the initial a/b value. With viscosity, the vortex radius increases in time due to diffusion ensuring that the critical a/b value is always crossed. Hence with viscosity, vortex merger is inevitable. The viscous merger process for an unstratified flow has been well studied, and is described later in this section. We show below that density stratification can lead to surprising results, sometimes preventing merger completely.

Stratification effects on vortex dynamics has received a great deal of attention in the literature. In two dimensions, i.e., with the density gradient direction perpendicular to the vortex axes, the main focus has been on the dynamics of vortex dipoles, since this problem finds relevance in mitigating the threat posed by aircraft trailing vortices to follower aircraft.[14] Another geometry which has been studied well is that of a single asymmetric vortex with a radial density stratification. The stability of this flow was studied with broken-line profiles[15] and smooth profiles.[16,17] The effect of vertical density variation on a single vortex was studied by us[18] where it was shown that strong centrifugal effects could lead to instabilities inside the vortex core, resulting in a turbulence-like state. As for a vortex pair, the case of stratification along the axes of the vortices, being relevant in the geostrophic context, is studied well. The merger process in this case has been studied experimentally[7] in both a baroclinic and a barotropic setting. There have been far fewer studies on the effects of stratification perpendicular to the axes of the vortices on the vortex pair. Numerical simulations with the Boussinesq approximation were carried out by Brandt and Nomura[19] at low to moderate stratifications, which showed that merger always occurs, but the merger time depends both on Reynolds as well as Froude number. The present work extends the study of Brandt and Nomura[19] in three directions: we (i) examine the effect of strong stratification on vortex merger, (ii) study the role of diffusivity of density field, (iii) study the inertial effects of stratification (non-Boussinesq effects) on vortex merger. The results of the unstratified and Boussinesq cases are in good agreement with Brandt and Nomura,[19] and were used to validate the present numerical scheme. Before discussing the stratified case further, for the purpose of contrast, we briefly recapitulate the viscous merger process for an unstratified flow.

In an unstratified flow, when the two vortices are initially kept sufficiently far apart, the process of viscous merger can be divided into four stages. The first is the viscous diffusive stage, where the two vortices behave like point vortices rotating about a common axis. Viscous diffusion acts on the vortex cores increasing their size, thus increasing the value of *a*/*b* with time. The second, convective, stage begins when *a*/*b* crosses a critical value. Here, the vortices rapidly approach each other with the ejection of spirals of vorticity in the form of filaments. This stage is nearly independent of the Reynolds number, and can be explained by an inviscid mechanism. The third stage can be called the axisymmetrization stage where the vortex cores completely merge into each other, and the single final vortex relaxes towards an axisymmetric state. The final stage involves the viscous diffusion of the single vortex. The role of various regions in the flow on the merger process has been studied by Cerretelli and Williamson,[20] and Brandt and Nomura.[21] A detailed review of vortex merger in an unstratified flow can be found in Meunier *et al.*[22] and Josserand and Rossi.[23]

Atmospheric conditions such as stable density stratification can have an important role to play in the merger process. In the present work, we study the effect of stable density stratification on the interaction of two co-rotating vortices, with density gradient perpendicular to the vortex axes. In brief, we find that apart from the Froude number, *Fr*, which governs the strength of stratification, the Peclet number, *Pe*, also plays an important role. For flows with $Fr \sim 1$, merger never occurs and, remarkably, the vortices actually move away from each other. Two distinct mechanisms are found to operate depending on the value of Peclet number. For flows with high *Pe*, spirals of density field enter each vortex core resulting in the breakdown of the vortex cores.[18] Whereas for flows with low *Pe*, the density field diffuses rapidly near the vortex core and no such breakdown was observed.

In summary, the aims of this paper are two-fold. First, we extend the Boussinesq simulations of Brandt and Nomura[19] to higher stratifications, where it will be shown that strong baroclinic vorticity can completely prevent vortex merger, and to lower diffusivity. For higher stratifications, inertial effects of stratification cannot always be ignored. We show that certain symmetries satisfied by the Boussinesq system are now broken, which can lead to drift in the centroid of vorticity.

The paper is organized as follows. In Sec. II, the governing equations and the numerical method is discussed. In Sec. III, we consider vortex merger under the Boussinesq approximation. The effect of Prandtl number is also treated in this section. In Sec. IV, we consider non-Boussinesq effects on the merger process. To highlight the importance of inertia, in this section we either neglect gravity altogether, or stay at large Froude numbers. In Sec. V, we summarize our main results. The physics of the merger process is also discussed qualitatively using an idealized model.

II. GOVERNING EQUATIONS AND NUMERICAL METHOD

Direct numerical simulations in two-dimensions are performed with a co-rotating Gaussian vortex pair in a stably stratified fluid. The initial background density stratification is taken to be linear. We further split the total density field, $\tilde{\rho}(x, y, t)$ in the form (see Turner[24])

$$\tilde{\rho}(x, y, t) = \rho_0 + \bar{\rho}(y) + \rho'(x, y, t), \tag{1}$$

where x and y are the horizontal and vertical directions, respectively. Note that the subscript 0, when used with ρ denotes a constant reference value, and not the initial conditions. $\bar{\rho}(y)$ is the vertical variation of the mean density about ρ_0. We prescribe linear stratification, i.e., $\bar{\rho}(y)$ has a linear dependence on the y coordinate. And ρ' is a time dependent perturbation density field generated due to the motion of vortices. The governing non-Boussinesq equations in the velocity-pressure formulation can be written as

$$\nabla \cdot \mathbf{u} = 0, \tag{2}$$

$$\tilde{\rho}\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right) = -\nabla P + \rho' \mathbf{g} + \tilde{\rho} \nu \nabla^2 \mathbf{u}, \tag{3}$$

$$\frac{\partial \tilde{\rho}}{\partial t} + \mathbf{u} \cdot \nabla \tilde{\rho} = \kappa \nabla^2 \tilde{\rho}. \tag{4}$$

In the above equations, ν is the kinematic viscosity, κ is the thermal or mass diffusivity, and these quantities are taken to be constant for simplicity. Gravity $\mathbf{g} = g\mathbf{e_y}$ where $\mathbf{e}_y$ is the unit vector in the vertical direction and P is the pressure. The fluid is incompressible, which allows us to define a streamfunction to solve the momentum equation in the vorticity-streamfunction formulation. Notice that the total density $\tilde{\rho}$ appears in the inertial acceleration terms. At low to moderate density variations, it is common to employ the Boussinesq approximation. In the limit of

$$\rho' \ll \bar{\rho} \ll \rho_0, \tag{5}$$

rewriting total density as

$$\tilde{\rho} = \rho_0 \left(1 + \frac{\rho'}{\rho_0} + \frac{\bar{\rho}(y)}{\rho_0}\right), \tag{6}$$

and retaining only the lowest order inertial terms, we get the Boussinesq approximation, under which, the momentum equations take the form

$$\rho_0 \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right) = -\nabla P + \rho' \mathbf{g} + \rho_0 \nu \nabla^2 \mathbf{u}. \tag{7}$$

In many physical systems, such as in the oceans, $\rho_0 + \bar{\rho}(y) \approx \rho_0$ on an average, so the inequality (5) is valid, whereas in atmospheric flows, this approximation can be restrictive. Even when (5) is a fair approximation, whenever centrifugal accelerations are comparable with or larger than gravity, such as in the immediate vicinity of a vortex, inertial effects of density stratification cannot be ignored. This is especially true when the transition from heavy to lighter fluid occurs across a thin layer, since significant amounts of baroclinic torque can then be generated from inertial effects. In our recent papers,[17,18] we have studied the inertial effects of density stratification on the stability of single vortex. The evolution equation for the vorticity $Z(x, y, t) \equiv \boldsymbol{\nabla} \times \mathbf{u}$ for a non-Boussinesq system is obtained by taking the curl of Eq. (3), as

$$\frac{\partial Z}{\partial t} + \mathbf{u} \cdot \nabla Z = \frac{\nabla \tilde{\rho} \times \nabla P}{\tilde{\rho}^2} + \frac{g}{\rho_0} \frac{\partial \rho'}{\partial x} + \nu \nabla^2 Z, \tag{8}$$

where the streamfunction in the fixed frame is related to vorticity in the standard way,

$$\nabla^2 \psi = -Z. \tag{9}$$

The initial condition consists of two Gaussian (Lamb-Oseen) vortices with centres at $(x_1, 0)$ and $(x_2, 0)$ with a radius a_0 (see Figure 1),

$$Z(x, y, 0) = Z_0 \left[\exp\left(\frac{-((x - x_1)^2 + y^2)}{a_0^2}\right) + \exp\left(\frac{-((x - x_2)^2 + y^2)}{a_0^2}\right)\right]. \tag{10}$$

The peak vorticity Z_0 is related to the circulation Γ' and the initial vortex radius a_0 as $Z_0 = 2\Gamma/a_0^2$, where $\Gamma \equiv \Gamma'/2\pi$. The initial separation between the two vortices is given by $b_0 = |x_1 - x_2|$. This initial condition is consistent with the experiments of Meunier and Leweke.[13] The Reynolds number and the Prandtl number are defined by

$$Re = \frac{\Gamma'}{\nu}, \qquad Pr = \frac{\nu}{\kappa}. \tag{11}$$

The Peclet number *Pe* is the product of the two. We report all our results in terms of *Pr* rather than in terms of *Pe*. We do not distinguish here between the Schmidt number (for mass diffusivity) and the Prandtl number, so this value can be quite high. The Froude number is defined as

$$Fr = \frac{\Gamma}{b_0^2 N}, \tag{12}$$

where N is the Brunt-Väisälä frequency given by

$$N^2 = -\frac{g}{\rho_0} \frac{d\bar{\rho}}{dy}. \tag{13}$$

The strength of stratification can be conveniently defined by a non-dimensional density gradient or the Atwood number, as

$$At = -\frac{b_0}{\rho_0} \frac{d\bar{\rho}}{dy}. \tag{14}$$

If the vortices are oriented vertically with their centers at $b_0/2$ and $-b_0/2$, then the relevant Atwood number is given by $At = (\rho_2 - \rho_1)/\rho_0$ where ρ_1 and ρ_2 are the density values at $\pm b_0/2$. Note that Froude number in (12) in terms of At can be written as $Fr^2 = \Gamma^2/g b_0^3 At$. In the absence of gravity, centrifugal acceleration plays the role of gravity and this scales as Γ^2/b_0^3 (see Dixit and Govindarajan[18] for more details). In such a case, replacing gravity by this quantity, we see that $At^{1/2}$ is just the inverse of Froude number.

Results are presented in terms of the non-dimensional time $t^* = t/T_{ref}$, where the reference time $T_{ref} \equiv \pi b_0^2/\Gamma$ is the time period of rotation of two point vortices of the same circulation. The buoyancy force is related to the deviations in the instantaneous density from a mean (time invariant) density profile. Though simulations were carried out for a slightly wider range of non-dimensional numbers, we present results here for $1000 \leq Re \leq 10000$, $1 \leq Fr \leq \infty$ and $0.005 \leq Pr \leq 10$. Some of these values, especially very low Prandtl numbers, may not appear commonly in physical situations, but are useful to highlight the physics. It is useful to remember a few points regarding the effect of Pr and Fr. At large Pr, momentum diffuses much more rapidly than density, whereas at small Pr, the reverse is true. Therefore, for a low Pr fluid, the rapid homogenization of density gradients ensures that stratification does not survive for too long. At $Fr = 1$, the time scale associated with rotation of the vortices and the time scale associated with density oscillations, i.e., $1/N$, are comparable. Since Fr is inversely related to stratification, its effects will be felt strongly for $Fr \sim \mathcal{O}(1)$.

The symmetries in a Boussinesq system are such that the equations of motion are invariant under the following transformations for Z and ρ, i.e.,

$$Z^*(x, y) \rightarrow Z(-x, -y), \tag{15}$$

$$\rho^*(x, y) \rightarrow -\rho(-x, -y), \tag{16}$$

$$\rho'^*(x, y) \rightarrow -\rho'(-x, -y). \tag{17}$$

Because of this, so long as the initial configuration is symmetric with respect to gravity, the centroid of vorticity is preserved in a Boussinesq system. As will be shown later, symmetry is broken when employing the non-Boussinesq equations, so the vortex system migrates as the merger process continues.

The numerical simulations are carried out in Cartesian coordinates in a doubly periodic domain using the Fourier pseudospectral method with a second-order Adams-Bashforth time stepping. Due to the incompressibility assumption, we can solve in the vorticity-streamfunction formulation. In all the simulations, the width of the domain was at least 10 times the separation distance. The number of collocation points varied depending on the Re and Fr used, but we always had more than 30 points across each vortex. The present simulations were validated against many test cases including the results of Brandt and Nomura.[19] Due to the splitting of the density field as shown in equation (1), ρ' can be taken to be a spatially periodic function. Domain dependence and grid dependence tests were conducted to arrive at the parameters used for the simulations. The code was also used in our earlier published work.[17,18] To measure the separation distance to sub-grid accuracy, we track the location of the vortex maximum as follows. We first isolate the grid box in which the vorticity maximum lies, and then use the values of the gradient at the four corners of the grid box to determine the sub-grid location where the gradient of vorticity vanishes. For this, we use an iterative technique on a finer spatial mesh of 10×10. Even without this, our results are accurate owing to the fine grids used in most simulations.

III. EFFECT OF STRATIFICATION UNDER THE BOUSSINESQ APPROXIMATION

Under the Boussinesq approximation, the vorticity Eq. (8) reduces to

$$\frac{\partial Z}{\partial t} + (\mathbf{u} \cdot \nabla Z) = \nu \nabla^2 Z - \frac{g}{\rho_0} \frac{\partial \rho'}{\partial x}. \tag{18}$$

Note that horizontal gradients of density contribute to vorticity generation. As the two vortices rotate, they advect the entire density field around them such that horizontal density gradients are

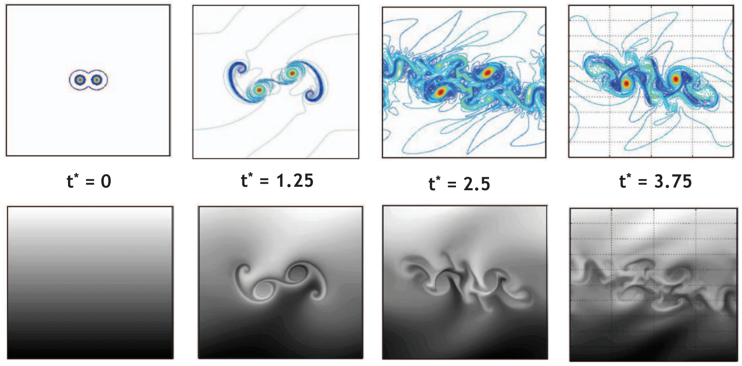

FIG. 2. Time evolution of vorticity (upper panel) and density (lower panel) contours for merger at higher levels of stratification than in Figure 4 with $Re = 5000$, $Fr = 1$, and $Pr = 1$. Significant generation of small scales can be observed. This figure is not to the same scale as the previous figure.

created. This in turn influences the evolution of the vortices themselves. It can be easily seen that both positive and negative vorticity will be created due to density effects. Simulation results at $Fr = 2$ and $Pr = 1$ were practically identical to those presented in Brandt and Nomura[19] and this served to validate our approach. It is relevant to mention that at $Fr = 2$ two dominant filaments of baroclinic vorticity appear. The position, orientation, and sign of this additional vorticity relative to the primary vortices can either accelerate, or decelerate the merger process. When stratification increases further to $Fr = 1$, a qualitative change is evident. The negative signed baroclinic vorticity due to stratification is comparable to the strength of the vortex, as shown in Figure 2. When each vortex approximately completes one rotation, i.e., the vortex rotation time scale equals the buoyancy time scale, these baroclinically generated filaments were observed to roll-up into two smaller vortices as can be seen at $t^* \approx 1.25$. In this case, the result of stratification is to cause the primary vortices to move away from each other, preventing merger. Another noticeable feature is the strong mixing that is generated in this flow. Local regions of unstable stratification further enhance the mixing, causing the breakdown of the flow into a turbulence-like state. Internal gravity waves appear to radiate out from the mixing zone. A kinematic view of the flow field and the role of various time scales in the flow will be examined later in Sec. V and this will help us understand the reasons behind merger/non-merger.

A series of numerical simulations were carried out for different Fr and different Pr with Re ranging from 1000 to 10 000. We validated our numerical method by comparing the our results with that of Brandt and Nomura.[19,21] For all the simulations presented in this paper, we use a smaller a_0/b_0 ratio than that used in Brandt and Nomura.[19,21] In agreement with their results, we find that up to a Froude number of 2, stratification slows down merger at low Reynolds number, and speeds up merger at high Re, with a cross-over in the range $1500 < Re < 2000$. But our values of merger time at the cross-over are slightly lower than those of Brandt and Nomura,[19] which is to be expected from a smaller a_0/b_0.

A new finding in the present simulations is that merger was always prevented at all Reynolds number when Froude number approached unity, except for the cases when Pr was small. We examine the role of Pr separately in Sec. III A. Therefore, merger time for high Re cases is a non-monotonic function of Fr, i.e., merger time reduces with decrease in Fr till about $Fr \approx 2$ and rapidly increases and diverges at $Fr = 1$. The reasons for this non-monotonic variation are discussed later, for which it will help to briefly discuss the energetics of the system, as we do now. As the vortices rotate, the density field is advected, creating a large-scale overturning. This process has a time-scale dependent on the vortex time scale and the distance from the origin, but small perturbations in the density field would oscillate with the Brunt-Väisälä frequency. At $Fr = 3$ for example, the vortices would complete three rotations for one cycle of oscillation of the density. During one half-cycle of density oscillation, heavier fluid rises upwards and lighter fluid sinks downwards. This causes a local increase in potential energy (P.E.) at the expense of the kinetic energy (K.E.) of the system. In a neutral oscillation, the

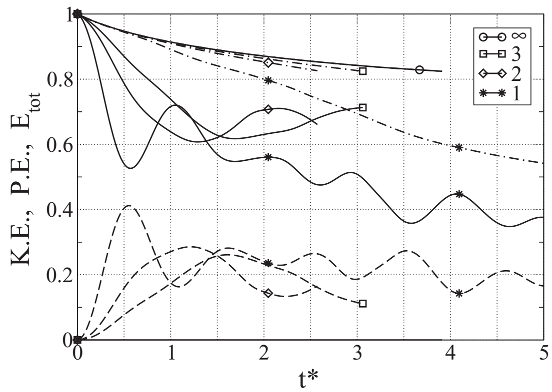

FIG. 3. Evolution of kinetic (*K.E.* - solid), potential (*P.E.* - dashed), and total (E_{tot} - dashed-dotted lines) for $Re = 5000$ and $Pr = 1$ for four different Froude numbers indicated with different colors. The kinetic and potential energies are always in anti-phase with each other.

balance would have been restored during the second half of oscillation. Indeed, this is seen to happen in our simulations as shown in Figure 3, though small differences exist in the timescales observed. As expected from the explanation above, the kinetic and potential energies are in anti-phase with each other. Simultaneously, owing to viscous effects, the total energy decreases in all cases, but a much larger dissipation is observed with $Fr = 1$. This is due to the significant generation of small scales in the flow at this Froude number, as was seen in Figure 2, which makes viscosity more effective. On the other hand, for higher *Fr*, the final state is a single large vortex, with the only small scales being due to weak filaments of vorticity.

In the above simulations, *Pr* was held fixed at unity, but for very small *Pr*, things can be different, and therefore the role of *Pr* is discussed separately below.

A. Effect of diffusivity and of low Froude number

The Prandtl number is a measure of the rate at which the density field diffuses in relation to diffusion of momentum. For a low *Pr* fluid, density inhomogeneities are rapidly erased, thus removing the source of baroclinic torque. However, for a high *Pr* fluid, there is a prolonged effect of density stratification due to the slow diffusion of density gradients. We first show in Figure 4 the time evolution of vorticity and density contours with $Re = 5000$ and $Fr = 2$ at $Pr = 10$. The vorticity and density contours differ in detail from those of $Pr = 1$, as shown in Brandt and Nomura,[19] but the basic structure of these fields remain unaltered.

It is instructive to study the effect of varying Prandtl and Froude numbers on the time evolution of the separation distance between the two vortices. In Figure 5, we plot the variation of separation distance for two different *Fr* by varying *Pr* in each case separately. By comparing merger time for the same *Pr* in the two cases, it is clear that vortex merger is delayed, and even prevented, with increasing stratification. The effect of *Pr* is marginal at high Froude number, but at low *Fr*, unity in this case, diffusivity can become important due to the generation of small scale structures in the flow as shown in Figure 2. Clearly at $Fr = 1$, vortex merger is delayed or completely prevented if the *Pr* is not too low. Conversely, as *Pr* decreases below unity, the density field is rapidly homogenized, and little baroclinic torque is generated, so the merger process would be progressively closer to that in a constant density fluid. In the small *Pr* limit, merger was always found to occur, even for very large stratifications, as can be seen in Figure 5(b) for $Fr = 1$.

In this section, Boussinesq equations were employed in all the simulations. In spite of the complexity of the flow field, the centroid of vorticity was always invariant as can be seen in Figures 2 and 4. This is due to the symmetries in a Boussinesq system as discussed in Sec. II. In Sec. IV, we consider inertial effects of stratification, and no such symmetries exist.

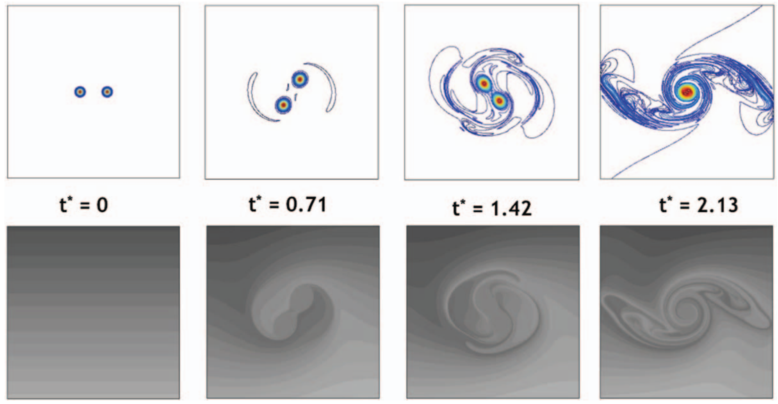

FIG. 4. Time evolution of vorticity (color) and density (gray scale) contours for merger in the presence of density stratification with $Re = 5000$, $Fr = 2$, and $Pr = 10$.

IV. NON-BOUSSINESQ EQUATIONS

We now study the effect of inertial (non-Boussinesq) terms on the merger process. As discussed in Sec. II, inertial effects of density stratification can lead to symmetry breaking. Therefore, the main goal of this section is to study this symmetry breaking process inherent in non-Boussinesq equations. To demonstrate this effectively, we deliberately employ large density stratifications, which enhance the impact of inertial terms. In many flows, it is possible that large density gradients arise not from large density contrasts, but from rapid variations in density across narrow regions in the flow. In such scenarios, it was shown in an earlier paper[18] that significant baroclinic vorticity can be generated.

For simplicity, all transport coefficients are held constant. Two cases can be separately studied: (1) where purely inertial effects of density variation are considered with gravity being absent, and (2) where combined inertial-gravity effects are present in the system. To evaluate the relative contribution of Boussinesq and non-Boussinesq effects on separation distance, we carried out a sample simulation at $Re = 5000$ with $Fr = 3$. To make a fair comparison, we fix a mean density profile and adjust gravity such that a desired Froude number is attained. In Figure 6, we compare the variation of separation distance with time for unstratified flow, a Boussinesq fluid with $Fr = 3$, a non-Boussinesq fluid with $Fr = 3$ and a purely inertial case where baroclinic vorticity is generated from inertial effects of density stratification and gravity is set to zero. Clearly, purely inertial effects do not affect the separation distance when compared with an unstratified case. It is confirmed in

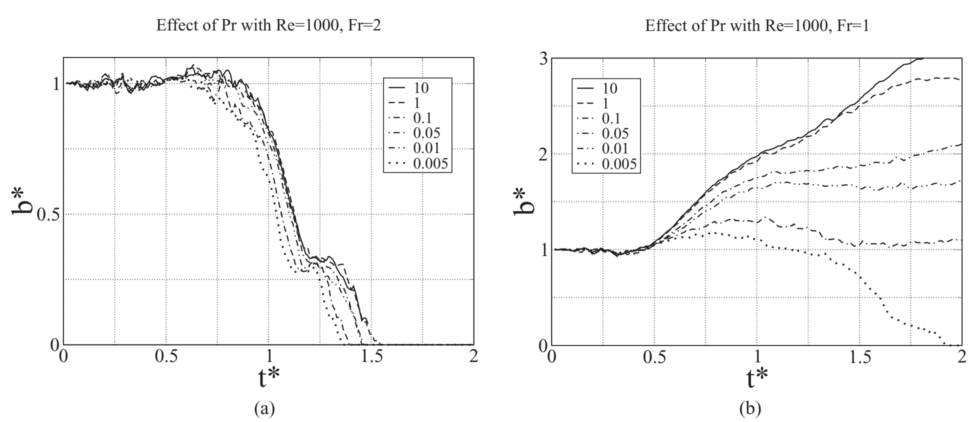

FIG. 5. Effect of Prandtl number, shown in the legend, on the separation distance for a fixed $Re = 1000$ and different Froude numbers, (a) $Fr = 2$ and (b) $Fr = 1$.

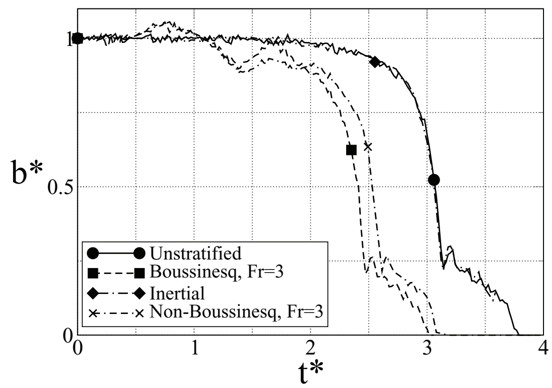

FIG. 6. Variation of separation distance with time for Re = 5000 with a mean non-dimensional density gradient of At = 0.0954. For the purely Boussinesq flow, gravity is chosen to give Fr = 3. In the purely inertial case, gravity is neglected and baroclinic torque is generated only due to the nonlinear terms in the governing equation.

Figure 7 (where we plot the separation distance as a function of time for different Atwood numbers at Re = 3000) that inertial effects have a negligible effect on the separation distance, a quantity, which changed significantly due to other parameters, as discussed earlier.

In spite of negligible effect on separation distance, there is an unexpected and significant effect of inertia, which we focus on in this section. This effect becomes apparent when we look at the trajectory of the two vortices. This is first shown by plotting the vorticity contours at four different times for Re = 3000. The vortex merger process is similar to the unstratified case except for the drift the vorticity centroid as shown in Figure 8. To highlight the effect of symmetry breaking due to inertia, we first consider a higher Reynolds number, viz., Re = 10 000. At this Re and the prescribed initial conditions, the two vortices complete approximately 5 rotations before convective merger begins. This provides ample time for baroclinic torque to act on the vortices and symmetry breaking to become noticeable. In Figure 9, the trajectory of one of the vortices is plotted as a function of time. It can be seen that for a system where inertial effects of density stratification are present, the centroid of vorticity is no longer an invariant. For this case, the equivalent centrifugal Froude number is approximately 3.24. Though the separation distance does not differ significantly from the unstratified case, the vortices can be seen to exhibit a pronounced drift leftwards. The exact direction and rate of drift depends both on Re and the density stratification employed. The same conclusion can be arrived at by monitoring the x-coordinate of the two vortices as shown in Figure 10. These results show that for large density stratifications, where inertial effects of density field become important, symmetry breaking can lead to the large-scale meandering of vortices.

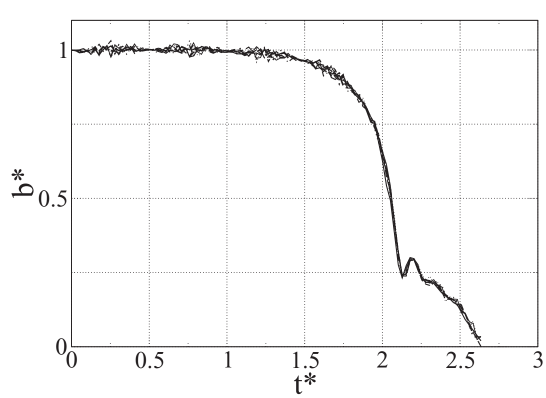

FIG. 7. Variation of separation distance with time for Re = 3000 by varying the Atwood number from 0.0106 to 0.0954 in nine equal steps of 0.0106. All the nine curves are clearly identical to each other.

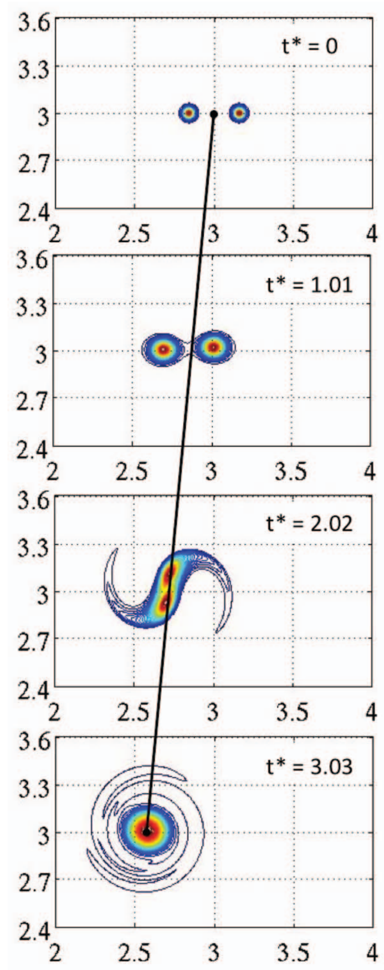

FIG. 8. Non-Boussinesq effects on the time evolution of vorticity contours for merger in the presence of density stratification with $Re = 5000$, $At = 0.0954$, and $Pr = 1$. The black line shows the drift of the vorticity centroid.

To quantify the symmetry breaking process, we measure (i) the magnitude of drift quantified by the distance the centroid travels in one rotation cycle and (ii) *initial* drift velocity as a function of Atwood and Reynolds numbers. The magnitude of drift is defined as

$$\Delta_{drift} = |\mathbf{r}(T_{ref}) - \mathbf{r}(0)|, \tag{19}$$

and the drift velocity is defined as

$$V_{drift} = \frac{1}{T_{ref}} \int_0^{T_{ref}} \frac{|\mathbf{r}(t + \Delta t) - \mathbf{r}(t)|}{\Delta t} dt, \tag{20}$$

where $\mathbf{r}(t)$ is the position vector of the centroid at time t and Δt is a chosen time interval. Note that T_{ref} is the time taken for one complete rotation in the unstratified case. In most of our results presented above, Δt was approximately 200 times the timestep used in the simulation. The drift velocity is averaged over the time the vortices take to complete one rotation. This averaging process was necessary since the drift velocity was found to have a small oscillation about a mean value. It has to be mentioned here that the average drift velocity is not dependent on the value of the upper limit used in the above integral. Even after the merger of the two vortices, the merged vortex was found to drift in the same direction with the same drift velocity. We carry out a series of simulations for different Reynolds and Atwood numbers. The magnitude of drift after one rotation cycle is shown in Figure 11 for four different Reynolds numbers by sequentially varying the Atwood

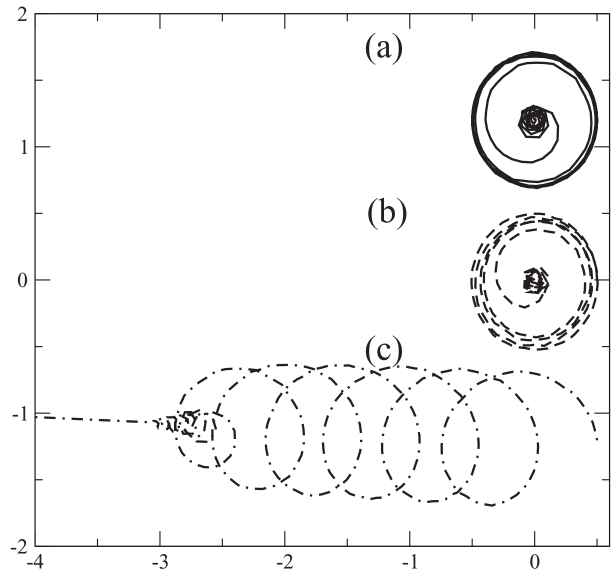

FIG. 9. Trajectory of a single vortex with $Re = 10\,000$ (a) no stratification, (b) with Boussinesq approximation at $Fr = 3$, (c) with only inertial effects of stratification at $At = 0.0954$. For visual clarity, the curves are shifted vertically from each other.

number. It is quite remarkable that the magnitude of drift is independent of Reynolds number. Moreover, the universality of the curve suggests that a scaling relationship exists between Δ_{drift} and At. A exponential fit for the data revealed that $\Delta_{drift} \sim e^{0.636At}$. More simulations are required to understand this scaling relationship, which is left for future study. The drift velocity was found to have a weak dependence on the Reynolds number as shown in Figure 12. It is clearly evident that V_{drift} again follows a universal curve as At increases, independent of Reynolds number. The weak dependence of Δ_{drift} and V_{drift} on Re suggests that drifting of the vortices is due to an inviscid mechanism and can be modelled using point vortices in a stratified fluid.

Drift of a single vortex due to inertial effects was observed by us in an earlier work[18] where stratification was in the form of a density jump at the location of the vortex. The results above show that a smooth density gradient, such as the one used here, can lead to significant drift of the vortices and could have important implications for other stratified turbulent flows.

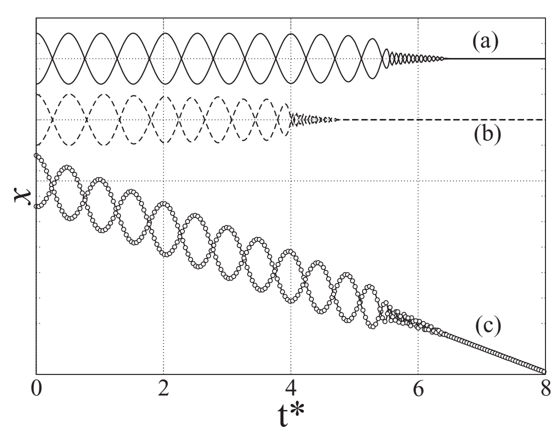

FIG. 10. Same as Figure 9, but showing the x-coordinate of both the vortices as a function of time. (a) Unstratified fluid, (b) Boussinesq fluid, (c) non-Boussinesq fluid.

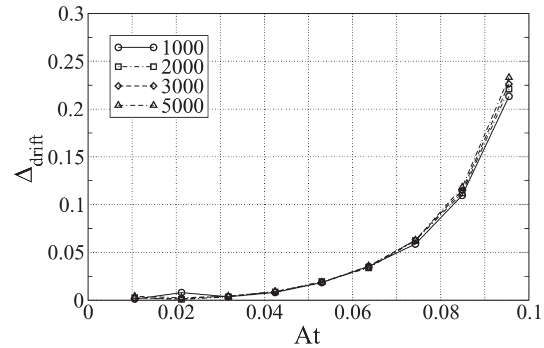

FIG. 11. Drift of the vortex centroid as a function of Atwood number for four different Reynolds numbers with $Pr = 1$.

V. MERGER MECHANISM IN A STRATIFIED FLUID

We briefly examine some aspects of the physical mechanism in vortex merger when the density in the vicinity is stratified. We have seen that two kinds of baroclinic torque are created, as a result of misalignment of the density gradient vector with gravity and with the centrifugal acceleration. The second would be missing in a Boussinesq calculation.

To understand the conditions under which merger will be accelerated or decelerated, we make a simple model, of two point vortices, placed on an initially horizontal line, which defines a density jump. The density takes on different constant values on either side of the horizontal line. We evolve this system inviscidly, and move in the frame of reference of the centroid of the system. In this idealized system, baroclinic torque can only be produced at the density interface, at a rate proportional to the cross product of the density jump and the local body force. There are two contributions to this force: gravitational, acting vertically, and centrifugal, acting radially. Since we can analytically write down the shape of the interface[18] at a given time, this model serves to explain the mechanism.

Figure 13 shows the right and left density interfaces at a particular instant later in time, in black and red, respectively. At an arbitrarily chosen point on the interface, the normal to the density interface is seen neither to be aligned with gravity, nor with the centrifugal acceleration. Thus, both gravity and centrifugal forces will induce vorticity at the interface, and the productions of both are positive at the point and time instant chosen. The sign of vorticity production at other places on the two interfaces are also shown. The magnitude of the baroclinic vorticity Z_g created by gravity increases and decreases alternately as the interface is advected. The centrifugal acceleration, on the other hand, always produces vorticity Z_c of one sign on a given interface (left or right).

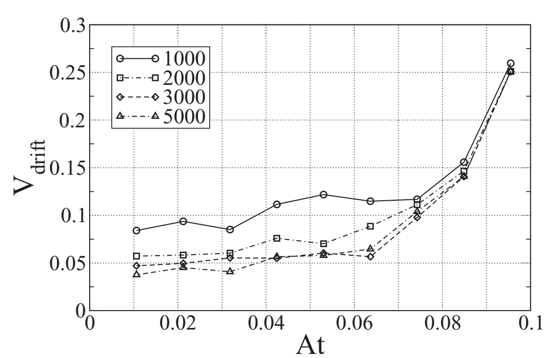

FIG. 12. Variation of drift velocity as a function of Atwood number for four different Reynolds numbers with $Pr = 1$.

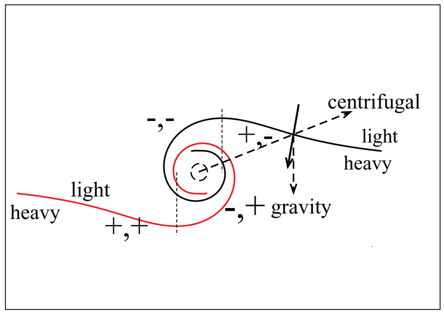

FIG. 13. Baroclinic torque produced due to centrifugal effects and gravity. Shown here is the locus at a given time of an initially horizontal interface separating light and heavy fluid. The solid arrow represents the normal to this line at one point. The effect of gravity (centrifugal acceleration) produces the same (opposite) sign of vorticity along diametrically opposite points as shown in the figure. The sign of the vorticity produced depends on the orientation of the normal vector to the interface with respect to gravity and to the radial vector. The first in the pair of signs shown indicates the sense of torque produced by gravity effects, while the second shows the sign of torque production due to centrifugal effects. The small dashed circle shows the locus of the two point vortices.

The relative strengths of Z_g and Z_c can be estimated far away from the primary vortices, where the density field responds as to a single point vortex of twice the circulation Γ'. It is straightforward to show[18] that a step density interface which is initially horizontal will evolve into an ever-tightening Lituus spiral. In the inviscid case, considering gravity alone and neglecting non-Boussinesq effects, Eq. (3) may be simplified to give

$$\dot{Z}_g = \frac{\Delta\rho}{\rho_{ave}} g \cos\left\{\theta_s + \tan^{-1}\left[\frac{1}{2\theta_s}\right]\right\} \delta(r\theta - r\theta_s). \tag{21}$$

On the other hand, neglecting gravity and taking into consideration the non-Boussinesq (centrifugal) term, we would have[18]

$$\dot{Z}_c = \frac{\Delta\rho}{\rho_{ave}} \frac{4\Gamma^2}{r^3} \frac{1}{\left(1+4\theta_s^2\right)^{1/2}} \delta(r\theta - r\theta_s). \tag{22}$$

The net baroclinic torque $Z = Z_g + Z_c$. Here, $\theta_s(t)$ is the total angle traversed up to time t by a point on the spiral density interface at a distance r from the centroid. At large r, $\theta_s(t)$ may be approximated as

$$\theta_s = \frac{2\Gamma t}{2\pi r^2}. \tag{23}$$

For the portion of the interface initially to the right of the two vortices, it is evident that the centrifugally created vorticity is always positive, whereas the gravity-created vorticity as the point on the interface traverses one quadrant (approximately) is more or less canceled out during its travel through the next.

The acceleration or deceleration of vortex merger may be explained by gravity effects alone. We simplify the argument by clubbing the net vorticity created by gravity into two small vortices, as shown in Figure 14. It is clear that the phase between the baroclinic vortices "B" and the primary ones "P" will determine whether the P vortices proceed towards or away from merger. Being at large r, the location of the B vortices is a slowly changing function of time, whereas the P vortices rotate much faster. Depicted in the figure is a time instant where the vortices are at an angle $\theta < \pi/2$ to the horizontal. At this instant, the gravity-created baroclinic torque serves to push the primary vortices apart. On the other hand, if the vortices had executed a larger angle by this time, say π, the baroclinic vorticity would push them together. Remembering that the strength of B increases with time, we see that if the stratification is weak, the pushing-apart in the initial quarter cycle will be overcome by the pulling together in the next quarter cycle, and we could have earlier merger than in an unstratified case. Figure 6 shows such an initial increase in separation followed by a quicker merger. On the other hand, if the stratification is large, the strength of B will be high enough to

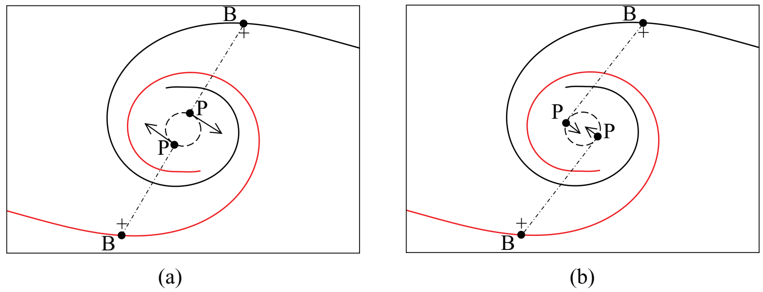

FIG. 14. A schematic of the dominant vorticity due to gravity alone, based on Figure 13. The two primary vortices are shown in solid circles, and marked "P," and the "clubbed" baroclinic vortices are marked "B" and shown by open circles (not to any scale). In (a), the primary vortices are at such a phase with respect to the baroclinic vortices causing them to move away from each other. In (b), the net effect on the primary vortices is to push them towards each other leading to accelerated merger.

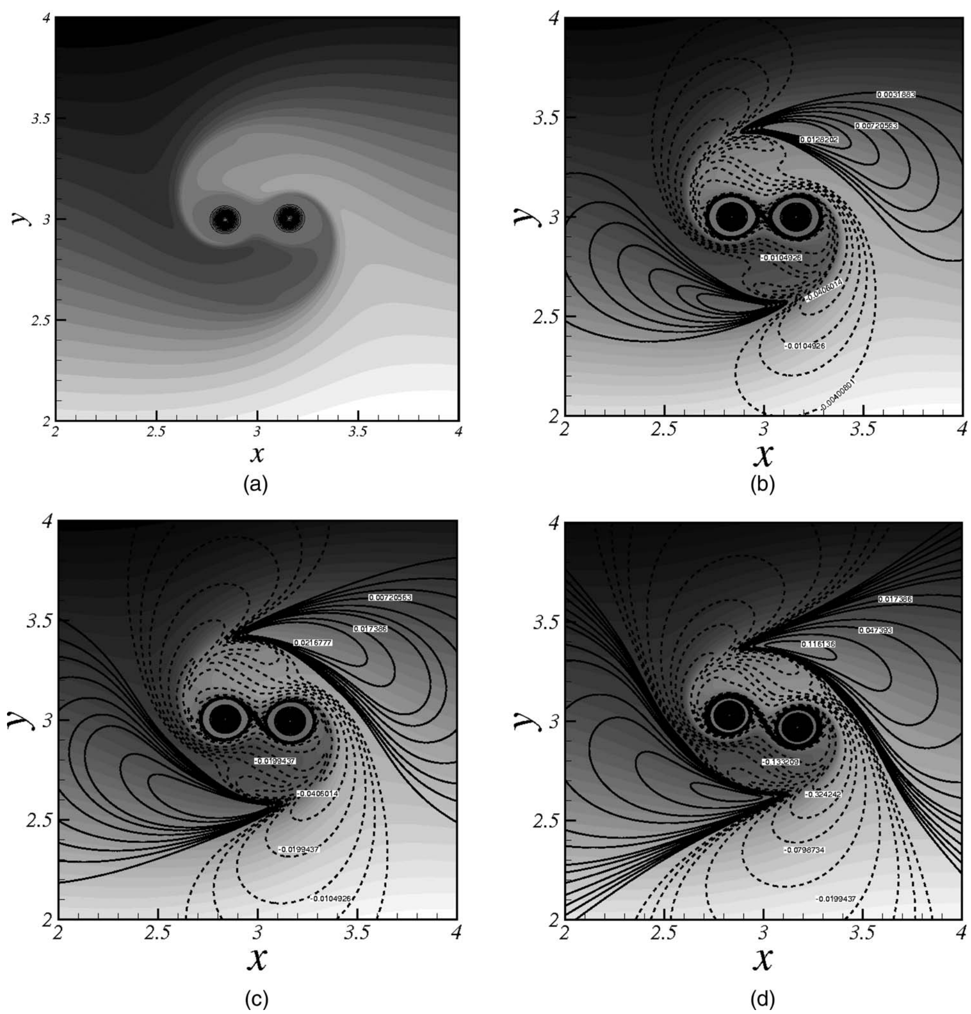

FIG. 15. Vorticity (lines) and density (gray scale) contours for $Re = 5000$, $Pr = 1$ at time $t^* = 0.5$ for various Fr: (a) $Fr = \infty$, (b) $Fr = 3$, (c) $Fr = 2$, (d) $Fr = 1$. Solid and dashed lines represent positive and negative vorticity levels. Note that in (a), the density field is a passive scalar.

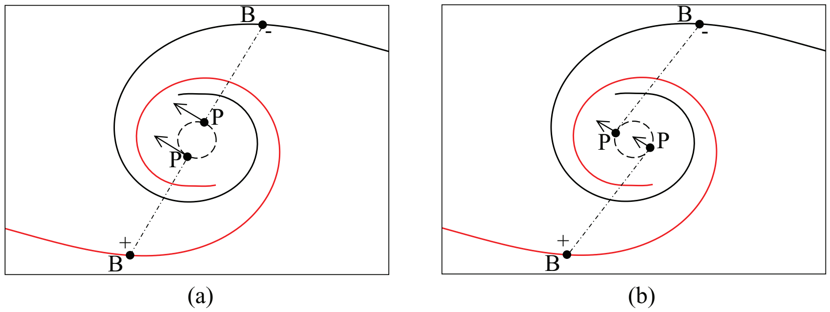

FIG. 16. Same as Figure 14 with "B" now representing baroclinic vorticity generated from inertial effects of stratification. The baroclinic vortices act like a dipole imparting a push on the primary vortices. In both (a) and (b), the net effect on the primary vortices is to push them leftward.

influence P before they execute a large angle. The vortices are then pushed apart in the beginning of the process at a rate too large for the rest of the cycle to proceed undisturbed. The complicated structure of the vorticity field now generated has the effect of a breakdown into decaying turbulence. From scaling arguments, it is easy to see that a Froude number of about unity would constitute a large stratification. Figure 15 shows the vorticity and density field at $t^* = 0.5$ from a Boussinesq calculation at various levels of stratification. For comparison, an unstratified case is also presented. The figure demonstrates that at this short time, the vorticity and density fields are similar in structure in all the cases, but the magnitude of the baroclinic vorticity scales with the inverse square of the Froude number. For example, the maximum baroclinic vorticity at this time for $Fr = 3, 2, 1$ is approximately 0.013, 0.029, and 0.117, respectively, which closely follows the required scaling. The primary vortices are not too affected by stratification at this time, so the assumption in the model is a reasonable one.

Now considering centrifugal effects alone, Eq. (22) shows that the baroclinic vorticity above the primary vortices would be negative, while that below would be positive. This leads to the formation of a dipole around the vortices, which would impart a net linear momentum to the primary pair P as shown in Figure 16. The direction and rate of advection would depend on the ratio of buoyancy timescale to the characteristic rotation timescale of the primary vortices, i.e., on a centrifugal Froude number as suitably defined in Eq. (12). Though highly simplistic this model explains the basic effect of density stratification in the merging process.

VI. SUMMARY AND DISCUSSION

In this paper, we study the effect of ambient density stratification on the merger of two like-signed vortices, employing a wide range of stratification levels. We also analyze the effect of Prandtl number. Apart from the Boussinesq equations commonly employed, we also study the inertial effects of stratification on the merger process.

The two key results of this paper are: (i) a demonstration and a mechanism, for the initial moving apart of the vortices at intermediate levels of stratification, and prevention of merger at high stratification levels ($Fr \approx 1$ or lower), for all Re when $Pr \sim \mathcal{O}(1)$, and (ii) drift of the vortical system due to inertial effects of stratification. The first result can be summarized as follows: when the Prandtl number is $\mathcal{O}(1)$ or higher, the diffusion of the density interface is slow enough for baroclinic vorticity to be produced, and for a sufficient amount to accumulate in spiral filaments around the vortices. At moderate stratification, this vorticity leads to an initially oscillatory response of the separation distance, and at higher stratifications, causes a monotonically increasing separation, completely preventing merger. Our low to moderate stratification cases are in complete agreement with Brandt and Nomura.[19] Since flow fields governed by the Boussinesq equations are constrained by certain symmetries, the location of the centroid is an invariant quantity. When this approximation is relaxed, we observe a significant drift of the vortical system. This drift may be attributed to baroclinic vorticity generated by the centrifugal acceleration. The direction and rate of drift may be modeled as well. We construct a simple point vortex model with a single density interface to understand the essential features of the flow kinematically. Owing to the simplicity of the model, quantitative comparisons with the DNS results were not possible, but the model reveals key features of accelerated/ decelerated merger, and drift in the "inertial" case.

Finally, we briefly examine the relevance of the present work in a broader context. A key application of this work lies in geophysical flows. Merger events have been observed on Jupiter, and also in Earth's atmosphere. Both these systems tend to be strongly stratified in the meridional (latitudinal) direction. For large vortices such as a tropical cyclone or a polar vortex, strong meridional stratifications could be an important factor affecting the lifetime and trajectory of the vortices. In a simplified 2D analysis for atmospheric flows, stratification is present only in the meridional direction and inertial effects of stratification could become important. This can lead to significant drift of the vortex trajectories as shown in this work. Another important application is the merger events of Kelvin-Helmholtz billows. Often, the rate of growth of a mixing layer is attribute to vortex merger. The stratification of the billows is in the same sense as studied in this paper and in this case, both gravity and inertial effects are important. Quicker merger of vortices due to stratification should therefore lead to a faster growth of the mixing layer.

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