5/Aug/2014 Vector Calculus & Index Notation

(Notes from Panton & Ganesh's class notes)

Two notations exist — Symbolic or Gibbs' notation; index or Cartesian notation.

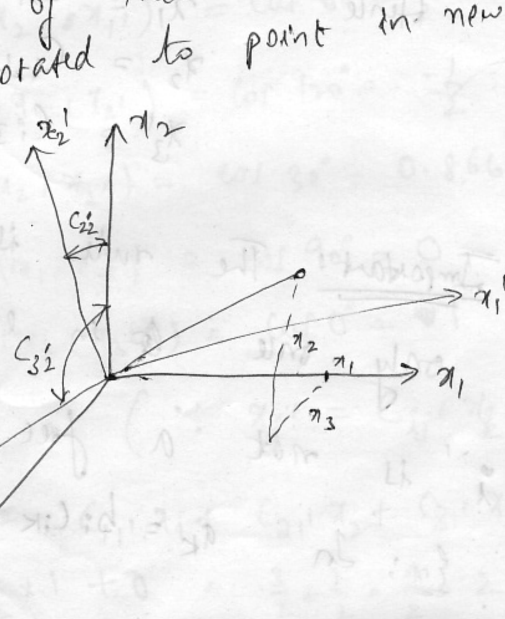
Gibbs notation: Scalars, vectors & tensors are viewed differently, operations like + & × have diff. meanings for each type of object. We therefore have to define specific operations for each new type of object.

Index notation:- Always deals with scalar variables. Hence no specific operations have to be defined, and all operations of simple algebra are directly applicable.

Index notation rules and co-ordinate notation:-

The key to classifying quantities as scalars, vectors, or tensors is how the values of these counterparts change if the coordinate axes are rotated to point in new directions.

Consider a right-handed co-ordinate system where P has co-ordinates x_1, x_2, x_3, i.e. co-ordinate of P is simply x_i, $i = 1, 2, 3$.



If the coordinate system is rotated, then the coordinates of P change. Let the new coordinates be x_j'.

Let $C_{ij'}$ be the cosine of the angles from x_i direction to x_j' direction.

Example, $C_{23'}$ is cosine between x_2 & x_3' -axes.

$$C_{ij'} = \cos(x_i, x_j') = \cos(x_j', x_i) = C_{j'i}$$

$$\Rightarrow C_{ij'} = C_{j'i} \Rightarrow \text{angle is not directed.}$$

From geometry, $\boxed{x_j' = C_{ij'} x_i}$, from $j' = 1', 2'$ or $3'$.
↳ index notation equation.

$j' \to$ free index $\Rightarrow$ Means we can write the above equation 3 times for $j' = 1'$ or $2'$ or $3'$.

Hence
$$x_1' = C_{i1'} x_i$$
$$x_2' = C_{i2'} x_i$$
$$x_3' = C_{i3'} x_i$$

__Important__: The rule is that a free index occurs only once in each term of the equation.

'i' is not a free index because it occurs twice.

__Ex__: In $a_k = b_i C_{ik} + d_{ijk} e_{ij}$; $k \to$ free index
$i, j \to$ repeated indices.

Free index can be changed to another letter if it does not repeat with an already existing ~~and~~ index.

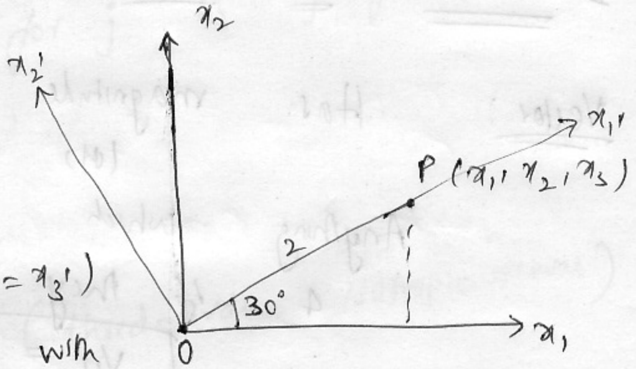
$\therefore$ Replacing $k \longrightarrow n$, we have $a_n = b_i c_{in} + d_{ijn} e_{ij}$

If an index appears twice, it is called dummy or Summation index.

Hence $x_1' = C_{i1'} x_i = \sum_{i=1}^{3} C_{i1'} x_i$

$= C_{11'} x_1 + C_{12'} x_2 + C_{13'} x_3$

Rotation about x_3-axis:-



Point P : $(\sqrt{3}, 1, 0)$

Rotating about x_3-axis ($x_3 = x_3'$) so that x_1'-axis is aligned with vector OP.

Direction cosines:

$C_{11'} = \cos(x_1, x_1') = \cos 30^\circ = \frac{\sqrt{3}}{2} = 0.866$

$C_{21'} = \cos(x_2, x_1') = \cos 60^\circ = \frac{1}{2} = 0.5$

$C_{12'} = \cos(x_1, x_2') = \cos 120^\circ = -\frac{1}{2}$

$C_{32'} = \cos(x_3, x_2') = \cos 90^\circ = 0$

$C_{22'} = \cos(x_2, x_2') = \cos 30^\circ = 0.866$

$C_{13'} = \cos(x_1, x_3') = \cos 90^\circ = 0$

$\cos_{23'} = \cos(x_2, x_3') = \cos 90^\circ = 0$

$C_{33'} = \cos(x_3, x_3') = \cos 0^\circ = 1$

$\therefore$ x_1' coordinate of OP is: $\left(\because x_j' = C_{ij'} x_i\right)$

$x_1' = C_{i1'} x_i = C_{11'} x_1 + C_{21'} x_2 + C_{31'} x_3$

$= \frac{\sqrt{3}}{2} \times \sqrt{3} + \frac{1}{2} \times 1 + 0 = \frac{3}{2} + \frac{1}{2} = \frac{4}{2} = 2$

Similarly, for $x_{2'}$, we have $\qquad (x_{j'} = C_{ij'} x_i)$

$$x_{2'} = C_{i2'} x_i = C_{12'} x_1 + C_{22'} x_2 + C_{32'} x_3$$

$$= -\frac{1}{2} \times \sqrt{3} + \frac{\sqrt{3}}{2} \times 1 + 0 \times 0 = 0$$

and $\quad x_{3'} = C_{i3'} x_i = C_{13'} x_1 + C_{23'} x_2 + C_{33'} x_3$

$$= 0 \times \sqrt{3} + 0 \times 1 + 1 \times 0 = 0$$

DEFINITION OF VECTORS AND TENSORS

Vector: Has magnitude and direction
(or)
Anything which has three scalar components
& if they transform according to the definition

$$\boxed{V_{j'} = C_{ij'} V_i} \leftarrow \text{rotation}$$

Tensor: A (rank 2) tensor is defined as a collection of 9 scalars that change under a rotation of axes as

$$\boxed{T_{ij'} = C_{ki'} C_{lj'} = T_{kl}}$$

k & l have to be summed.

(Tensors $\longrightarrow$ capital letters.

Inverse relationships:- $\qquad V_j = C_{ij'} V_{i'} \quad$ or $\quad V_i = C_{j'i} V_{j'}$

ISOTROPIC TENSORS: Special tensors which will assist in mathematical operations or statements.

① Kronecker delta: δ_{ij} :-

Also known as substitution tensor or identity tensor.

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 \rightarrow & \text{if } i \neq j \end{cases}$$

This tensor is isotropic because the components are always the same no matter how the coordinates are rotated.

δ_{ij} substitutes i for j or j for i.

$$\delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

② Alternating unit tensor :- (third-order isotropic tensor)

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk = 123, 231, 312 \\ 0 & \text{any two indexes are alike} \\ -1 & \text{if } ijk = 321, 132, 213. \end{cases}$$

$$\begin{array}{ccc} 3 \leftarrow 2 & & 3 \rightarrow 2 \\ 1 & & 1 \\ \epsilon_{ijk} = +1 & & \epsilon_{ijk} = -1 \end{array}$$

$\boxed{\epsilon_{ijk} \text{ is used in cross-product.}}$

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$$

4. $\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj}$

Identity:

$$\epsilon_{ijk}\,\epsilon_{ilm} = \delta_{jl}\,\delta_{km} - \delta_{jm}\,\delta_{kl} \quad \text{— (1)}$$

Proof: Let $\epsilon_{ijk}\,\epsilon_{ilm} = C_1\,\delta_{jk}\,\delta_{lm} + C_2\,\delta_{jl}\,\delta_{km} + C_3\,\delta_{jm}\,\delta_{kl}$ — (2)

This term is symmetric $j \leftrightarrow k$ & $l \leftrightarrow m$

ie;

$$\epsilon_{ikj}\,\epsilon_{iml} = C_1\,\delta_{kj}\,\delta_{ml} + C_2\,\delta_{km}\,\delta_{jl} + C_3\,\delta_{kl}\,\delta_{jm} \quad \text{— (3)}$$

$$\Rightarrow (-1\times -1)\,\epsilon_{ijk}\,\epsilon_{ilm} = \text{"} \qquad \text{"}$$

Subtracting, we find $C_1 = 0$

$$\therefore \boxed{\epsilon_{ijk}\,\epsilon_{ilm} = C_2\,\delta_{jl}\,\delta_{km} + C_3\,\delta_{jm}\,\delta_{kl}} \quad \text{— (4)}$$

$j \leftrightarrow k$: $\epsilon_{ikj}\,\epsilon_{ilm} = C_2\,\delta_{kl}\,\delta_{jm} + C_3\,\delta_{km}\,\delta_{jl}$ — (5)

$$\Rightarrow -\epsilon_{ijk}\,\epsilon_{ilm} = C_2\,\delta_{jm}\,\delta_{kl} + C_3\,\delta_{jl}\,\delta_{km} \quad \text{— (6)}$$

Adding (5) & (6) :

$$0 = (C_2 + C_3)\left[\delta_{kl}\,\delta_{jm} + \delta_{km}\,\delta_{jl}\right]$$

$$\Rightarrow C_3 = -C_2$$

Let $C_2 = -C_3 = C$

$$\therefore \epsilon_{ijk}\,\epsilon_{ilm} = C\left[\delta_{jl}\,\delta_{km} - \delta_{jm}\,\delta_{kl}\right] \quad \text{— (7)}$$

Now, let $i=1,\ j=2,\ k=3$

& $l=2,\ m=3$

$$\Rightarrow \epsilon_{123}\,\epsilon_{123} = C\left[\delta_{22}\,\delta_{33} - \delta_{23}\,\delta_{32}\right]$$

$$\Rightarrow 1\times 1 = C\times(1\times 1 - 0\times 0) = C$$

$$\Rightarrow C = 1$$

$$\therefore \boxed{\epsilon_{ijk}\,\epsilon_{ilm} = \delta_{jl}\,\delta_{km} - \delta_{jm}\,\delta_{kl}}$$

How to remember: ~~$\epsilon_{ijk}\epsilon_{ilm} = \delta$~~

$$\epsilon_{ijk}\,\epsilon_{i\_\_} = \delta_{j\_}\,\delta_{k\_} - \delta_{j\_}\,\delta_{k\_}$$

$$\begin{array}{ccccccc} & l\ m & & l & m & m & l \end{array}$$

Using a similar argument, it can be shown that the only fourth-order isotropic tensor is related to δ_{ij} & ϵ_{ijk} in the form:

$$I_{ijkl} = a\,\delta_{ij}\,\delta_{kl} + b\left(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}\right) + c\left(\delta_{ik}\,\delta_{jl} - \delta_{il}\,\delta_{jk}\right)$$

NOTE: δ_{ij} & ϵ_{ijk} are the only isotropic tensors. Fundamental isotropic tensors. for their respective ranks.

DIRECTION COSINES:

Let $C_{1\alpha} = \cos(x_1, x_\alpha)$ where α represents some direction.

Rotation transformation:

$x_j = C_{i'j}\,x_{i'}$ Note that $C_{i'j} = C_{ji'}$ or simply $C_{ij} = C_{ji}$

Hence $x_j = C_{ji'}\,x_{i'}$ $\leftarrow$

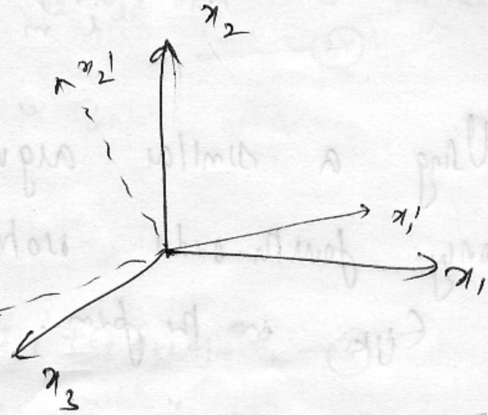
Replacing $j \to k'$ & $i' \to j$, we have

$$x_{k'} = C_{k'j}\,x_j$$
$$= C_{k'j}\,C_{ji'}\,x_{i'}$$

~~It is , $C_{k'j}\, C_{ji'} = \cos(x_{k'}, x_j)\cos(x_j, x_{i'})$~~,

~~with $k=1$, we have~~

~~$C_{1'j}\, C_{ji'}$~~



Since δ_{ij} is the substitution tensor,

we have

$$x_{k'} = \delta_{(\;)(\;)}\, x_{i'}$$

$$\downarrow$$

$$\delta_{k'i'}$$

Comparing, $\boxed{\delta_{k'i'} = C_{k'j}\, C_{ji'}}$

In general, $\boxed{\delta_{ij} = C_{ik}\, C_{kj}}$

ALGEBRA WITH VECTORS

We denote the cartesian unit vectors set as

$(\hat{e}_1, \hat{e}_2, \hat{e}_3) \longrightarrow$ form a right-handed orthogonal triplet.

unit vector in x-direction. y-direction z-direction.

Since $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are orthogonal vectors, we have

$\hat{e}_1 \cdot \hat{e}_1 = \hat{e}_2 \cdot \hat{e}_2 = \hat{e}_3 \cdot \hat{e}_3 = 1$

and $\hat{e}_1 \cdot \hat{e}_2 = \hat{e}_1 \cdot \hat{e}_3 = \hat{e}_2 \cdot \hat{e}_3 = 0$

Comparing, we have

$$\boxed{\hat{e}_i \cdot \hat{e}_j = \delta_{ij}} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Vector addition: $\vec{w} = \vec{u} + \vec{v}$ is represented as

$w_i = u_i + v_i$

Scalar product:- $b = \vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$

(i is summation index)

$\Rightarrow b = u_i v_i$

In general, we have

$$b = \vec{u} \cdot \vec{v} = \sum_i u_i \hat{e}_i \cdot \sum_j v_j \hat{e}_j$$

$$= \sum_i \sum_j u_i v_j \, \hat{e}_i \cdot \hat{e}_j$$

$$= \sum_i \sum_j u_i v_j \delta_{ij} = \sum_i u_i v_i$$

$$= u_i v_i \quad \text{(ignoring the summation sign)}$$

Cross-Product:- $\hat{e}_1 \wedge \hat{e}_2 = \hat{e}_3$; $\hat{e}_2 \times \hat{e}_3 = \hat{e}_1$; $\hat{e}_3 \wedge \hat{e}_1 = e_2$

$\underline{e}_2 \wedge \underline{e}_1 = -\underline{e}_3$; $\underline{e}_1 \wedge \underline{e}_3 = -\underline{e}_2$; $\underline{e}_3 \wedge \underline{e}_2 = -\hat{e}_1$

Hence $\underline{e}_i \wedge \underline{e}_j = \sum_k \epsilon_{ijk} \underline{e}_k$

$\rightarrow$ Summation over k is redundant.

$\therefore \boxed{\hat{e}_i \times \hat{e}_j = \epsilon_{kij} \hat{e}_k}$

$\therefore \underline{a} \times \underline{b} = \underline{c}$ (say)

then $\underline{c} = \sum_i a_i \hat{e}_i \times \sum_j b_j \hat{e}_j$

$= \sum_i \sum_j a_i b_j (\hat{e}_i \times \hat{e}_j)$

$= \sum_i \sum_j \sum_k a_i b_j \sum_k \epsilon_{kij} \hat{e}_k$

$= \sum_k \sum_i \sum_j \epsilon_{kij} a_i b_j \hat{e}_k$

$= \sum_k c_k \hat{e}_k$

$\therefore \boxed{c_k = \sum_i \sum_j a_i b_j \epsilon_{kij}}$ $\quad$ i, j are again dummy indices.

$\Rightarrow c_k = \sum_l \sum_m \epsilon_{klm} a_l b_m$

We normally ignore the summation since it is cumbersome.

$\Rightarrow c_k = \epsilon_{kij} a_i b_j$ $\quad$ No summation involved. Just remember that the index which is not repeated gives the direction.

Check: $c_1 = \epsilon_{1ij} a_i b_j = \sum_j \{ \cancel{\epsilon_{11j} a_1 b_j} + \epsilon_{12j} a_2 b_j + \epsilon_{13j} a_3 b_j \}$

$= \epsilon_{123} a_2 b_3 + \epsilon_{132} a_3 b_2$

$= a_2 b_3 - a_3 b_2$

$$C_2 = \epsilon_{2jk}\, a_j b_k = \epsilon_{213}\, a_1 b_3 + \epsilon_{231}\, a_3 b_1$$
$$= -a_1 b_3 + a_3 b_1 = (a_3 b_1 - a_1 b_3)$$

$$C_3 = \epsilon_{3jk}\, a_j b_k = \epsilon_{312}\, a_1 b_2 + \epsilon_{321}\, a_2 b_1$$
$$= a_1 b_2 - a_2 b_1$$

$$\therefore \underline{c} = (a_2 b_3 - a_3 b_2)\,\hat{e}_1 + (a_3 b_1 - a_1 b_3)\,\hat{e}_2 + (a_1 b_2 - a_2 b_1)\,\hat{e}_3$$

$$= \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \longrightarrow \text{OK.}$$

Symmetric and Antisymmetric tensors:-

Tensor; $T_{ij} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$

Transpose of T_{ij} is T_{ji}.

$$T_{ji} = \begin{bmatrix} T_{11} & T_{21} & T_{31} \\ T_{12} & T_{22} & T_{32} \\ T_{13} & T_{23} & T_{33} \end{bmatrix} = T^t$$

Ex: $\begin{bmatrix} 3 & 4 & 1 \\ 4 & 5 & -2 \\ 1 & -2 & 2 \end{bmatrix}$

Symmetric Tensor: $Q_{ij} = Q_{ji}$

Ex: $\begin{bmatrix} 0 & 3 & 1 \\ -3 & 0 & -5 \\ -1 & 5 & 0 \end{bmatrix}$

Antisymmetric Tensor: $R_{ij} = -R_{ji}$

Six independent entries

only three independent entries.

Theorem:
Any arbitrary tensor can be impressed as a sum of a symmetric tensor & an antisymmetric tensor.

Proof: Let $T_{ij} = \frac{1}{2} T_{ij} + \frac{1}{2} T_{ji} + \frac{1}{2} T_{ij} - \frac{1}{2} T_{ji}$

$= \frac{1}{2}(T_{ij} + T_{ji}) + \frac{1}{2}(T_{ij} - T_{ji})$

$= \underset{(\text{Symmetric})}{Q_{ij}} + \underset{(\text{Antisymmetric})}{R_{ij}}$

$$\frac{1}{2}(T_{ij} + T_{ji}) = \frac{1}{2}\begin{bmatrix} 2T_{11} & T_{12}+T_{21} & T_{13}+T_{31} \\ T_{21}+T_{12} & 2T_{22} & T_{23}+T_{32} \\ T_{31}+T_{13} & T_{32}+T_{23} & 2T_{33} \end{bmatrix} = Q_{ij}$$

$$\frac{1}{2}(T_{ij} - T_{ji}) = \frac{1}{2}\begin{bmatrix} 0 & T_{12}-T_{21} & T_{13}-T_{31} \\ T_{21}-T_{12} & 0 & T_{23}-T_{32} \\ T_{31}-T_{13} & T_{32}-T_{23} & 0 \end{bmatrix} = R_{ij}$$

ALGEBRA WITH TENSORS:-

Inner product of two tensors:- Double summation on inner and outer indices:

$$a = T_{ij} \, S_{ji} = \sum_{j=1}^{3} \sum_{i=1}^{3} T_{ij} \, T_{ji} = \underset{(\text{Scalar})}{T : S}$$

If T_{ij} is symmetric & S_{ji} is antisymmetric, the product is zero.

Also $T_{ij} S_{ij} = T : (S)^t$ (differ from $T : S$)

<u>Dual vector</u>: Constructing a vector using two tensors. (we use contraction)

Ex: $d_i = \epsilon_{ijk} T_{jk}$

~~Proof that this is a vector:~~ $R_{jk} =$

$$\therefore d_i = \epsilon_{ijk} (Q_{jk} + R_{jk})$$

$$= \underbrace{\epsilon_{ijk}}_{\text{Antisymmetric}} \underbrace{Q_{jk}}_{\text{Symmetric}} + \epsilon_{ijk} R_{jk}$$

$$= 0 + \epsilon_{ijk} R_{jk}$$

Consider $\epsilon_{ilm} d_i = \epsilon_{ilm} \epsilon_{ijk} T_{jk}$

$$= (\delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj}) T_{jk}$$

$$= T_{lm} - T_{ml} = 2R_{lm}$$

$$\therefore R_{lm} = \frac{1}{2}\epsilon_{ilm} d_i$$

Hence $\boxed{T_{ij} = Q_{ij} + \frac{1}{2}\epsilon_{ijk} d_k}$

Other multiplication types:-

→ $\underset{=}{S} \cdot \underset{=}{T} = \underset{=}{R} \Rightarrow S_{ij} T_{jk} = R_{ik}$

(Tensor product of two tensors)

→ ~~Vector~~ product of a vector & a tensor:-

$$u_j = v_i T_{ij} = T_{ij} v_i$$

or $\vec{u} = \vec{v} \cdot \vec{\vec{T}}$ (Note $\vec{u} \neq T \cdot \vec{v}$ ✗)

(first index is repeated)

→ $\vec{\vec{T}} \cdot \vec{v} = T_{ij} v_j$ (second index is repeated)

$$\Rightarrow w_i = T_{ij} v_j$$

note $w_i \neq u_i$ in general.

→ Dyadic product of two tensors:-

$$T_{ij} = u_i v_j = v_j u_i$$ (Reynolds stress)

$$\Rightarrow \underset{=}{T} = \underset{-}{u}\,\underset{-}{v}$$ (order is important in symbolic notation).

Transpose of T: $\underset{=}{Q} = (\underset{=}{T})^t = \underset{-}{v}\,\underset{-}{u}$

$$Q_{ij} = T_{ji} = u_j v_i = v_i u_j$$

Here $u_j v_i$ is the transpose of $u_i v_j$.

Ex: Consider $T_{ij} = v_K w_i \delta_{Kj} + a\delta_{ij} + \epsilon_{ijK} w_K$

(a) What is T_{11}?

$$T_{11} = v_K w_1 \delta_{K1} + a \underset{1}{\underset{\shortparallel}{\delta_{11}}} + \underset{0}{\underset{\shortparallel}{\epsilon_{11K}}} w_K \quad \text{(repeated index)}$$

$$= w_1 v_K \delta_{K1} + a$$

$$= w_1 v_1 \delta_{11} + w_1 v_2 \delta_{21} + w_1 v_3 \delta_{31} + a$$

(a) What is T_{12}?

$$T_{12} = v_K w_1 \delta_{K2} + a \overset{0}{\cancel{\delta_{12}}} + \epsilon_{12K} w_K$$

$$= v_1 w_1 \delta_{12} + v_2 w_1 \delta_{22} + v_3 w_1 \delta_{32} + \epsilon_{123} w_3$$

$$= v_1 w_1 \delta_{12} + v_2 w_1 \delta_{22} + v_3 w_1 \delta_{32} + w_3$$

(a) Contraction of T_{ij} :- Make $i = j$:

$$T_{ii} = v_K w_i \delta_{Ki} + a\delta_{ii} + \epsilon_{iiK} w_K$$

$$= v_K w_i \delta_{Ki} + 3a = \mathrm{Trace}(T)$$

$$T_{ii} = \mathrm{tr}(\underline{T})$$

Vector Cross Product :-

$\underline{w} = \underline{u} \times \underline{v}$

$\Rightarrow w_i = u_j v_k \epsilon_{ijk}$

$\Rightarrow w_i = \epsilon_{ijk} u_j v_k$

The cross-product produces a vector that is $\perp^{ar}$ to the plane of the two vectors.

Proof: To ~~prove~~ show that $\underline{u} \cdot \underline{w} = 0$
& $\underline{v} \cdot \underline{w} = 0$

$\underline{v} \cdot \underline{w} = v_i w_i = v_i \epsilon_{ijk} u_j v_k$
$= \epsilon_{ijk} v_i v_k u_j$

Consider $S_{ik} = v_i v_k$

$S^t = S_{ik}^t = S_{ki} = v_k v_i$

$\Rightarrow S_{ik} = v_i v_k$ is symmetric

$\Rightarrow \epsilon_{ijk} v_i v_k = 0$

$\Rightarrow \underline{v} \cdot \underline{w} = 0$

Product of Symmetric & anti-symmetric tensor :-

Let α be the symmetric tensor &

β be the antisymmetric tensor,

Inner product :-

$$\underline{\underline{\alpha}} : \underline{\underline{\beta}} = \alpha_{ij} \beta_{ji} \quad \text{— (a)}$$

$\alpha_{ij} = \alpha_{ji}$

& $\beta_{ij} = -\beta_{ji}$

$$\Rightarrow \boxed{\underline{\alpha} : \underline{\beta} = -\alpha_{ij} \beta_{ij}} \quad \text{— (b)}$$

Replacing i & j in the original expression :

$$\underline{\alpha} : \underline{\beta} = \alpha_{ji} \beta_{ij}$$

$$= \alpha_{ij} \beta_{ij} \quad \text{— (c)}$$

$\downarrow$ using $\alpha_{ij} = \alpha_{ji}$

Comparing (b) & (c) : $\alpha_{ij} \beta_{ij} = -\alpha_{ij} \beta_{ij}$

$$\Rightarrow \alpha_{ij} \beta_{ij} = 0$$

What about $\epsilon : \delta$:-

$\epsilon_{ijk} = -\epsilon_{jik}$

Let $Q_k = \epsilon_{ijk} \delta_{ij}$ — (a)

$\delta_{ij} = \delta_{ji}$

$i \Leftrightarrow j$, we have $Q_k = \epsilon_{jik} \delta_{ji} = -\epsilon_{ijk} \delta_{ij}$ — (b) from (a) & (b) : $Q_k = 0$

Brute force approach:

$$\alpha : \beta = \alpha_{ij}\beta_{ji}$$

$$= \alpha_{1j}\beta_{j1} + \alpha_{2j}\beta_{j2} + \alpha_{3j}\beta_{j3}$$

$$= (\cancel{\alpha_{11}}\beta_{11} + \alpha_{12}\beta_{21} + \alpha_{13}\beta_{31})$$

$$+ (\alpha_{21}\beta_{12} + \cancel{\alpha_{22}}\beta_{22} + \alpha_{23}\beta_{22})$$

$$+ (\alpha_{31}\beta_{13} + \alpha_{32}\beta_{23} + \cancel{\alpha_{33}}\beta_{33})$$

Since $\alpha_{ij} = \alpha_{ji}$, $\alpha_{12} = \alpha_{21}$, $\alpha_{13} = \alpha_{31}$ & so on

& $\beta_{ij} = -\beta_{ji}$, $\beta_{ii} = 0$ & $\beta_{12} = -\beta_{21}$, ...

$\Rightarrow \alpha : \beta = 0$

DERIVATIVE OPERATIONS

Gradient operator: $\nabla_{\underline{x}} = \frac{\partial}{\partial \underline{x}} \equiv \sum_i e_i \frac{\partial}{\partial x_i}$

Thus, if the gradient (vector) operates on a scalar, we could still have a vector.

If f is a scalar, then $\nabla_{\underline{x}} f = \sum_i e_i \frac{\partial}{\partial x_i}(f) = \sum_i e_i \frac{\partial f}{\partial x_i}$

$$= \frac{\partial f}{\partial x_i}$$ (ignoring e_i & summation & remembering that i gives the direction).

Ex: Conductive heat flux:

$$\underline{q} = -K \nabla_{\underline{x}} T \Rightarrow q_i = -K \frac{\partial T}{\partial x_i}$$

(3D heat conduction equation)

Multiple vector products:-

① $$\underline{u} \cdot (\underline{v} \times \underline{w}) = \sum_i u_i e_i \cdot \sum_j (\underline{v} \times \underline{w})_j e_j$$

$$= \sum_i \sum_j u_i (v \times w)_j \; (e_i \cdot e_j)$$

$$= \sum_i \sum_j u_i \, (v \times w)_j \; \delta_{ij}$$

$$= \sum_i u_i \, (v \times w)_i$$

$$= \sum_i u_i \sum_j \sum_k \epsilon_{ijk} v_j w_k$$

$$= \sum_i \sum_j \sum_k \epsilon_{ijk} \, u_i v_j w_k$$

$$\underline{u} \cdot (\underline{v} \times \underline{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

: Volume of a parallelopiped defined by vectors $\underline{u}, \underline{v}, \underline{w}$

② Position vector :-

$$\vec{r} = \sum_i \hat{e}_i x_i = \hat{e}_1 x_1 + \hat{e}_2 x_2 + \hat{e}_3 x_3$$

$$|r| = \sqrt{\vec{r}\cdot\vec{r}} = \sqrt{\sum_i \hat{e}_i x_i \cdot \sum_j \hat{e}_j x_j}$$

$$= \sqrt{\sum_i \sum_j (\hat{e}_i \cdot \hat{e}_j)\, x_i x_j} = \sqrt{\sum_i \sum_j \delta_{ij}\, x_i x_j}$$

$$= \sqrt{\sum_i x_i x_i} = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

③ Proof of a vector identity :-

Prove that

$$\underline{u} \times (\underline{v} \times \underline{w}) = \underline{v}(\underline{u}\cdot\underline{w}) - \underline{w}(\underline{u}\cdot\underline{v})$$

The ith component of the expression on LHS is :

$$[\underline{u} \times (\underline{v} \times \underline{w})]_i = \sum_j \sum_k \epsilon_{ijk}\, u_j\, (\underline{v} \times \underline{w})_k$$

$$= \sum_j \sum_k \epsilon_{ijk}\, u_j \left\{ \sum_l \sum_m \epsilon_{klm}\, v_l\, w_m \right\}$$

$$= \sum_j \sum_k \sum_l \sum_m \epsilon_{ijk}\, \epsilon_{klm}\, u_j\, v_l\, w_m$$

$$= \sum_j \sum_l \sum_m \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \right) u_j\, v_l\, w_m$$

$$= \sum_j \sum_l \sum_m \delta_{il}\, \delta_{jm}\, u_j\, v_l\, w_m - \sum_j \sum_l \sum_m \delta_{im}\, \delta_{jl}\, u_j\, v_l\, w_m$$

$$= v_i \sum_j \sum_m \delta_{jm} u_j w_m - w_i \sum_j \sum_l \delta_{jl} u_j v_l$$

$$= v_i \sum_j u_j w_j - w_i \sum_j u_j v_j$$

$$= v_i (\underline{u} \cdot \underline{w}) - w_i (\underline{u} \cdot \underline{v})$$

$$\Rightarrow \boxed{\underline{u} \times (\underline{v} \times \underline{w}) = \underline{v} (\underline{u} \cdot \underline{w}) - \underline{w} (\underline{u} \cdot \underline{v})}$$

Other vector identities:-

→ $\underline{u} \cdot (\underline{v} \times \underline{w}) = \underline{v} \cdot (\underline{w} \times \underline{u})$

→ $(\underline{u} \times \underline{v}) \cdot (\underline{w} \times \underline{z}) = (\underline{u} \cdot \underline{w})(\underline{v} \cdot \underline{z}) - (\underline{u} \cdot \underline{z})(\underline{v} \cdot \underline{w})$

→ $(\underline{u} \times \underline{v}) \times (\underline{w} \times \underline{z}) = [(\underline{u} \times \underline{v}) \cdot \underline{z}]\underline{w} - [(\underline{u} \times \underline{v}) \cdot \underline{w}]\underline{z}$

→ Identity of Lagrange:

$$\left[(\underline{v} \times \underline{w}) \cdot (\underline{v} \times \underline{w})\right] + (\underline{v} \cdot \underline{w})^2 = v^2 w^2$$

$v^2 \to |v|^2$, $w^2 \to |w|^2$

TENSOR OPERATIONS IN TERMS OF COMPONENTS

In previous section, we wrote a vector in terms of unit vector

$$\underline{v} = \sum_i \hat{e}_i v_i, \text{ i.e. } \hat{e}_i \text{ \& Component } v_i$$

we will now write a tensor $\underline{\underline{T}}$ in terms of e_i & e_j as:

$$\underline{\underline{T}} = \sum_i \sum_j e_i e_j T_{ij}$$

The Unit Dyads:-

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$$
$$\hat{e}_i \times \hat{e}_j = \epsilon_{kij} \hat{e}_k$$

Dyadic product: $\underbrace{\hat{e}_i \hat{e}_j}_{\text{unit dyads.}} \longrightarrow$ this is a tensor of second order.

Since each unit vector represents a single co-ordinate direction, unit dyads represent ordered pairs of coordinate directions.

Ex: Such unit dyads are helpful in dealing with quantities which simultaneous require two directions.

Flux of x-mom. across a unit area of surface perpendicular to the y-direction is a quantity of this type.

In general, this flux is not the same as $\Rightarrow$ y-momentum flux $\perp$ar to x-direction.

We must also agree on the order in which these directions are given.

~~Relation~~

Relations:

① $\left[\hat{e}_i \hat{e}_j : \hat{e}_k \hat{e}_l\right] = (\hat{e}_j \cdot \hat{e}_k)(\hat{e}_i \cdot \hat{e}_l) = \delta_{jk}\,\delta_{il}$

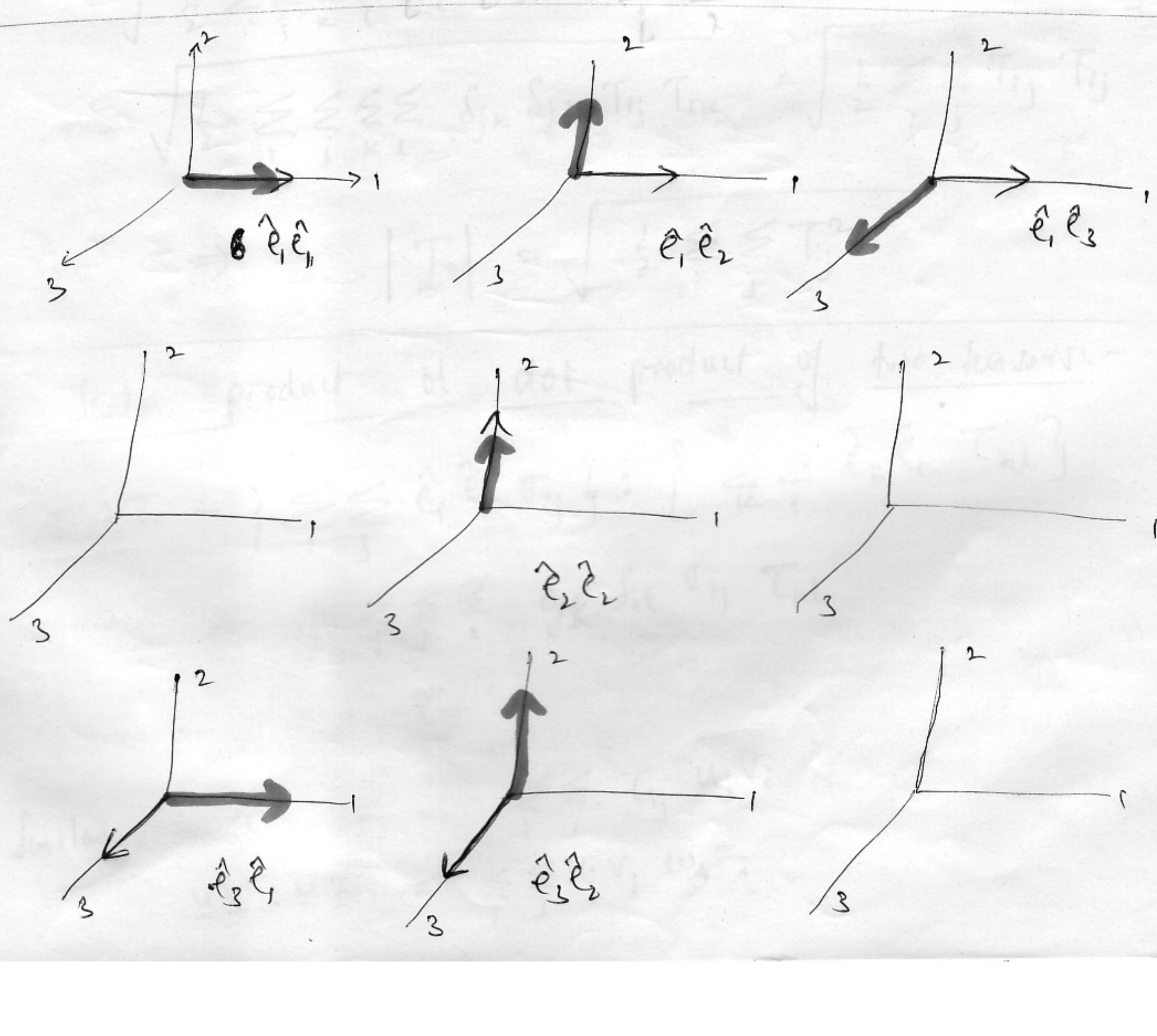
② $\left[\hat{e}_i \hat{e}_j \cdot \hat{e}_k\right] = \hat{e}_i (\hat{e}_j \cdot \hat{e}_k) = \hat{e}_i\, \delta_{jk}$

③ $\left[\hat{e}_i \cdot \hat{e}_j \hat{e}_k\right] = (\hat{e}_i \cdot \hat{e}_j)\, \hat{e}_k = \delta_{ij}\, \hat{e}_k$

④ $\left[\hat{e}_i \hat{e}_j \cdot \hat{e}_k \hat{e}_l\right] = \hat{e}_i (\hat{e}_j \cdot \hat{e}_k)\, \hat{e}_l = \delta_{jk}\, \hat{e}_i\, \hat{e}_l$

⑤ $\left[\hat{e}_i \hat{e}_j \times \hat{e}_k\right] = \hat{e}_i (\hat{e}_j \times \hat{e}_k) = \sum_{l=1}^{3} \epsilon_{ljk}\, \hat{e}_l\, \hat{e}_i$

⑥ $\left[\hat{e}_i \times \hat{e}_j \hat{e}_k\right] = (\hat{e}_i \times \hat{e}_j)\, \hat{e}_k = \sum_{l} \epsilon_{lij}\, \hat{e}_l\, \hat{e}_k$



Expansion of a tensor in terms of its components:-

Tensor: a quantity that associates a scalar with each ordered pair of coordinate directions in the following sense:-

$$\underline{T} = \hat{e}_1 \hat{e}_1 T_{11} + e_1 e_2 T_{12} + e_1 e_3 T_{13}$$
$$+ \hat{e}_2 \hat{e}_1 T_{21} + e_2 e_2 T_{22} + e_2 e_3 T_{23}$$
$$+ \hat{e}_3 \hat{e}_1 T_{31} + e_3 e_2 T_{32} + e_3 e_3 T_{33}$$
$$= \sum_i \sum_j \hat{e}_i \hat{e}_j T_{ij}$$

Based on the unit dyadic product rules, we can now define various kinds of Tensor products.

① Magnitude of a tensor:-

$$|\underline{\underline{T}}| = \sqrt{\tfrac{1}{2}(\underline{\underline{T}} : \underline{\underline{T}}^t)} = \sqrt{\tfrac{1}{2}\left\{\sum_i\sum_j \hat{e}_i \hat{e}_j T_{ij}\right\} : \left\{\sum_k\sum_l \hat{e}_k \hat{e}_l T_{lk}\right\}}$$

$$= \sqrt{\tfrac{1}{2} \sum_i\sum_j\sum_k\sum_l (\hat{e}_i \hat{e}_j : \hat{e}_k \hat{e}_l)\, T_{ij} T_{lk}}$$

$$= \sqrt{\tfrac{1}{2} \sum_i\sum_j\sum_k\sum_l \hat{e}_i (\hat{e}_j \cdot \hat{e}_k) \hat{e}_l\, T_{ij} T_{lk}}$$

$$= \sqrt{\tfrac{1}{2} \sum_i\sum_j\sum_k\sum_l (\hat{e}_j \cdot \hat{e}_k)(\hat{e}_i \cdot \hat{e}_l)\, T_{ij} T_{lk}}$$

$$= \sqrt{\tfrac{1}{2} \sum_i\sum_j\sum_k\sum_l \delta_{jk}\delta_{il}\, T_{ij} T_{lk}} = \sqrt{\tfrac{1}{2}\sum_i\sum_j T_{ij} T_{ij}}$$

$$\Rightarrow \quad |\underline{\underline{T}}| = \sqrt{\tfrac{1}{2}\sum_i\sum_j T_{ij}^2}$$

② Scalar product or dot product of two tensors:-

$$\sigma : \tau = \left\{\sum_i\sum_j \hat{e}_i \hat{e}_j \sigma_{ij}\right\} : \left\{\sum_k\sum_l \hat{e}_k \hat{e}_l \tau_{kl}\right\}$$

$$= \sum_i\sum_j\sum_k\sum_l \delta_{jk}\delta_{il}\, \sigma_{ij}\tau_{kl}$$

$$= \sum_i\sum_j \sigma_{ij}\tau_{ji}$$

Similarly, $\underline{\underline{\tau}} : \underline{u}\underline{v} = \sum_i\sum_j \tau_{ij} u_j v_i$

$$\underline{u}\underline{v} : \underline{w}\underline{z} = \sum_i\sum_j u_i v_j w_j z_i$$

(13) Tensor product of two tensors:-

$$\underline{\underline{\sigma}} \cdot \underline{\underline{\tau}} = \left\{ \sum_i \sum_j \hat{e}_i \hat{e}_j \sigma_{ij} \right\} \cdot \left\{ \sum_k \sum_l \hat{e}_k \hat{e}_l \tau_{kl} \right\}$$

$$= \sum_i \sum_j \sum_k \sum_l \hat{e}_i \delta_{jk} \hat{e}_l \sigma_{ij} \tau_{kl}$$

$$= \sum_i \sum_j \sum_l \hat{e}_i \hat{e}_l \sigma_{ij} \tau_{jl}$$

$$= \sum_i \sum_l \hat{e}_i \hat{e}_l \left\{ \sum_j \sigma_{ij} \tau_{jl} \right\}$$

The il component of $(\underline{\underline{\sigma}} \cdot \underline{\underline{\tau}})$ is $\left\{ \sum_j \sigma_{ij} \tau_{jl} \right\}$

---

(14) Vector Product (or Dot Product) of a Tensor with a vector:-

$$(\underline{\underline{\tau}} \cdot \underline{v}) = \left\{ \sum_i \sum_j \hat{e}_i \hat{e}_j \tau_{ij} \right\} \cdot \left\{ \sum_k \hat{e}_k v_k \right\}$$

$$= \sum_i \sum_j \hat{e}_i \delta_{jk} \tau_{ij} v_k = \sum_i \hat{e}_i \left(\sum_j \tau_{ij} v_j \right)$$

$\Rightarrow$ the i^{th} component of $(\underline{\underline{\tau}} \cdot \underline{v})$ is

$$\sum_j \tau_{ij} v_j$$

Similarly, the ith component of $\underline{v} \cdot \underline{\underline{\tau}}$ is $\sum_j v_j \tau_{ji}$

Clearly $(\underline{\underline{\tau}} \cdot \underline{v}) \neq (\underline{v} \cdot \underline{\underline{\tau}})$ unless $\underline{\underline{\tau}}$ is symmetric.

Vector & Tensor Differential Operators:-

$$\nabla_x = \hat{e}_1 \frac{\partial}{\partial x_1} + \hat{e}_2 \frac{\partial}{\partial x_2} + \hat{e}_3 \frac{\partial}{\partial x_3} = \sum_i \hat{e}_i \frac{\partial}{\partial x_i}$$

Gradient of a scalar:- $\nabla f = \sum_i \hat{e}_i \frac{\partial f}{\partial x_i}$

Ex: Heat conduction: $\underline{q} = -k \nabla_{\underline{x}} T \Rightarrow q_i = -k \frac{\partial T}{\partial x_i}$

Divergence of a vector field:- (gives a scalar)

$$\nabla \cdot \underline{v} = \left\{ \sum_i \hat{e}_i \frac{\partial}{\partial x_i} \right\} \cdot \left\{ \sum_j \hat{e}_j v_j \right\}$$

$$= \sum_i \sum_j \underbrace{(\hat{e}_i \cdot \hat{e}_j)}_{\delta_{ij}} \frac{\partial v_j}{\partial x_i} = \sum_i \frac{\partial v_i}{\partial x_i}$$

or simply $\frac{\partial v_i}{\partial x_i}$

Curl of a vector:- (gives a vector)

$$\nabla \times \underline{v} = \left\{ \sum_j \hat{e}_j \frac{\partial}{\partial x_j} \right\} \times \left\{ \sum_k \hat{e}_k v_k \right\}$$

$$= \sum_j \sum_k (\hat{e}_j \times \hat{e}_k) \frac{\partial v_k}{\partial x_j}$$

$$= \sum_i \sum_j \sum_k \epsilon_{ijk} \hat{e}_i \frac{\partial v_k}{\partial x_j}$$ → vector in the $\hat{e}_i$ th direction.

Gradient of a vector :- (gives a tensor) :

$$\nabla_x \underline{v} = \left\{ \sum_i \hat{e}_i \frac{\partial}{\partial x_i} \right\} \left\{ \sum_j \hat{e}_j v_j \right\}$$

$$= \sum_i \sum_j \hat{e}_i \hat{e}_j \frac{\partial v_j}{\partial x_i}$$

$\Rightarrow$ the ij^{th} component of $\nabla \underline{v}$ is $\dfrac{\partial v_j}{\partial x_i}$

$$(\nabla \underline{v})^t = \sum_i \sum_j \hat{e}_i \hat{e}_j \frac{\partial v_i}{\partial x_j}$$

The ij^{th} component of $(\nabla \underline{v})^t$ is $\dfrac{\partial v_i}{\partial x_j}$

Divergence of a tensor :-

$$(\underline{\nabla} \cdot \underline{\underline{\tau}}) = \left\{ \sum_i \hat{e}_i \frac{\partial}{\partial x_i} \right\} \cdot \left\{ \sum_j \sum_k \hat{e}_j \hat{e}_k \tau_{jk} \right\}$$

$$= \sum_i \sum_j \sum_k \hat{e}_k \delta_{ij} \frac{\partial}{\partial x_i} \tau_{jk}$$

$$= \sum_k \hat{e}_k \left\{ \sum_i \frac{\partial \tau_{ik}}{\partial x_i} \right\}$$

The k^{th} component of $\underline{\nabla} \cdot \underline{\underline{\tau}}$ is $\displaystyle\sum_i \frac{\partial \tau_{ik}}{\partial x_i}$

In general, we can write $\underline{\nabla} \cdot \underline{\underline{\tau}} = \dfrac{\partial \tau_{ik}}{\partial x_i}$

If $\underline{\underline{\tau}} = s\,\underline{u}\,\underline{v}$, then $\underline{\nabla} \cdot \underline{\underline{\tau}} = \sum_k \hat{e}_k \left(\sum_i \frac{\partial}{\partial x_i} (s u_i v_k) \right)$

Laplacian of a scalar:-

$$\nabla \cdot \nabla s = \left(\sum_i \hat{e}_i \frac{\partial}{\partial x_i} \right) \cdot \left(\sum_j \hat{e}_j \frac{\partial s}{\partial x_j} \right)$$

$$= \sum_i \underbrace{\left(\frac{\partial^2 s}{\partial x_i^2} \right)}_{\text{scalar..}}$$

Note that, we can write this in other forms:

$$\nabla \cdot \nabla s = (\nabla \cdot \nabla) s = \nabla^2 s = \Delta s$$

Laplacian of a vector:-

$$\nabla \cdot \nabla v = \sum_k \hat{e}_k \left(\sum_i \frac{\partial^2}{\partial x_i^2} v_k \right)$$

Some vector identities:-

$$(\underline{\nabla} \cdot s \underline{v}) = \underline{\nabla} s \cdot \underline{v} + s (\underline{\nabla} \cdot \underline{v})$$

$$(\underline{\nabla} \cdot \underline{\nabla}\, \underline{v}) = \underline{\nabla} (\underline{\nabla} \cdot \underline{v}) - \underline{\nabla} \times (\underline{\nabla} \times \underline{v})$$

$$\underline{u} \cdot \nabla \underline{u} = \frac{1}{2} \nabla (\underline{u} \cdot \underline{u}) - \underline{u} \times (\underline{\nabla} \times \underline{u})$$

Vector and Tensor Integral Theorems:-

Gauss-Ostrogradskii Divergence Theorem

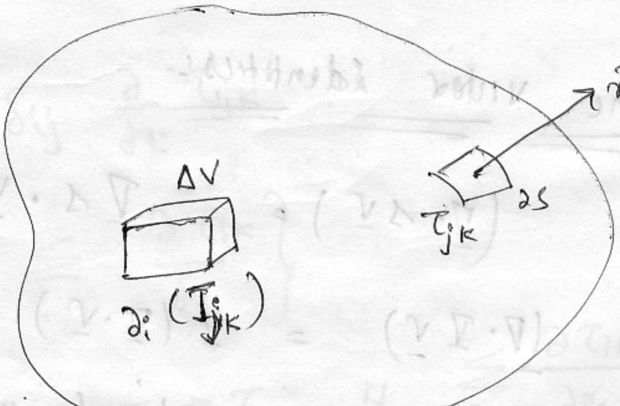
If V is a closed region in space enclosed by a surface S, then

$$\int_V (\nabla \cdot \underline{u})\, dV = \int_S (\underline{n} \cdot \underline{u})\, dS$$

This relates the derivative of a volume integral to an integral over a surface.

In index notation:

$$\int_V \frac{\partial u_i}{\partial x_i}\, dV = \int_S n_i u_i\, dS$$



For a scalar ϕ, we have

$$\int_V \frac{\partial \phi}{\partial x_i}\, dV = \int_S \underbrace{n_i \phi}_{\text{vector}}\, dS$$

(In vector notation, we have

$$\int_V \nabla \phi\, dV = \int_S \hat{n}\, \phi\, dS$$

Divergence theorem for tensors:-

$$\int_V (\underline{\nabla} \cdot \underline{\underline{\tau}})\, dV = \int_S (\hat{n} \cdot \underline{\underline{\tau}})\, dS$$

or

$$\int_V \frac{\partial (\tau_{jk})}{\partial x_i}\, dV = \int_S n_i \, \tau_{jk}\, dS$$

Higher order tensors:-

$$\int_V \frac{\partial}{\partial x_i} (\tau_{jkl\ldots})\, dV = \int_S n_i (\tau_{jkl\ldots})\, dS$$

Navier-Stokes equation:-

$$\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} = -\underline{\nabla} p + \frac{1}{Re} \nabla^2 \underline{u}$$

In index notation, we have

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{1}{Re} \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$