Governing equations

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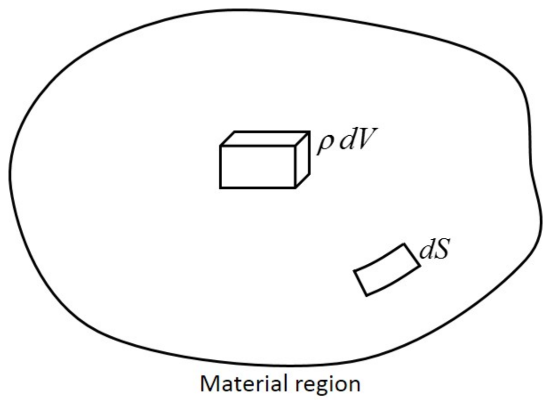
We will now derive the equations of motion for a fluid. This is essentially writing down the Newton's second law of motion for a material element of a fluid.



Consider a region with volume V and surface area A as shown in the figure. Within this region, consider a volume element dV_m and an area element dS. The subscript m denotes that this element is within the material region considered.

1 Mass conservation

The governing equation for mass conservation was derived in an earlier chapter (see kinematics-1). For the sake of completeness, we will state the governing equations again here. The mass of the small volume element is ρdV_m. The mass conservation equation then becomes:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0, \tag{1}$$

or

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \boldsymbol{\nabla} \cdot \mathbf{u} = 0, \tag{2}$$

or

$$\frac{D}{Dt}(\rho dV_m) = 0 \tag{3}$$

since mass associated with a material element is, by definition, conserved.

2 Equation of motion

The rate of change of momentum of a material element is given by $\dfrac{d}{dt}\left\{\int \rho \mathbf{u} dV_m\right\}$. This is the local momentum of the material element.

Since this is a material element, it is appropriate to regard $\rho \equiv \rho(\mathbf{x}(t), t)$ and $\mathbf{u} \equiv \mathbf{u}(\mathbf{x}(t), t)$ where $\mathbf{x}(t)$ is the location of the element at time t. This transferring the total derivation from outside the integral to the inside would lead to a material derivative, i.e. a derivative taken while the material element is moving in a fluid. Hence

$$\frac{d}{dt}\int \rho \mathbf{u} dV_m = \int \frac{D}{Dt}(\rho \mathbf{u} dV_m). \tag{4}$$

From the continuity equation, $\dfrac{D}{Dt}(\rho dV_m) = 0$.

$$\Longrightarrow \int \frac{D}{Dt}(\rho \mathbf{u} dV_m) = \int \rho dV_m \frac{D\mathbf{u}}{Dt}. \tag{5}$$

This material element is acted on by both surface forces and body forces. If $\mathbf{F}$ is the body force (say, gravity) per unit mass and $\boldsymbol{\sigma}$ is the stress tensor, then the two forces can be written as

$$\text{Body force} = \int \rho \mathbf{F} dV_m, \tag{6}$$

$$\begin{aligned}\text{Surface force} &= \int \boldsymbol{\sigma}\cdot \mathbf{n} dS,\\ &= \int \boldsymbol{\nabla}\cdot \boldsymbol{\sigma} dVm,\end{aligned} \tag{7}$$

The sum of these two forces gives the total force:

$$\int \rho \mathbf{F} dV_m + \int \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} dVm. \tag{8}$$

From Newton's second law, we have

$$\int \rho \frac{D\mathbf{u}}{Dt} dV_m = \int \left\{\rho \mathbf{F} + \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}\right\} dVm. \tag{9}$$

Since V_m is arbitrary, we must have

$$\boxed{\rho \frac{D\mathbf{u}}{Dt} = \rho \mathbf{F} + \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}.} \tag{10}$$

These are the *Cauchy's equations of motion*. Note that this is a vector equation and we have three different equations for the three scalar components of the velocity $\mathbf{u}$. The Cauchy's equations are applicable to any material, solid or liquid or gas, since we have not specified the nature of the response of the material to the applied stress, $\boldsymbol{\sigma}$.

To proceed further, we require how stress affects the deformation. Such a relationship is called the *constitutive relation.*

Examples:

- If stress $\propto$ strain $\rightarrow$ Hookean solid
- If stress $\propto$ rate-of-strain $\rightarrow$ Newtonian liquid

3 Generalized constitutive relationship for a fluid

Before we begin a detailed investigation of the nature of the constitutive relationship between the stress and rate-of-strain, we first recall the structure of the stress tensor for a fluid at rest. As seen in an earlier chapter, the stress tensor for a fluid at rest (in equilibrium), is isotropic and characterized by a single scalar, the pressure, i.e,

$$\sigma_{ij} = -p\delta_{ij}. \tag{11}$$

The pressure p is called the *hydrostatic pressure*. A fluid can only sustain tangential stresses if in motion. The constitutive relation therefore relates the tangential stresses to motion (deformation). The first departure from isotropy leads to the Newtonian fluid approximation. It is reasonable to assume this departure is linear in the applied forcing. The response of a fluid to external stresses is characterized by the velocity gradient tensor. Therefore the departure from isotropy can be written as $A_{ijkl}(\boldsymbol{\nabla}\mathbf{u})_{lk}$ where A_{ijkl} is a fourth-order tensor. Since the $\boldsymbol{\sigma}$ and $\boldsymbol{\nabla}\mathbf{u}$ are both second order tensors, the most generation relationship between the two is via a fourth order tensor.
A rigid body pressure leads to a hydrostatic pressure in a rotating frame and is again characterized by an isotropic pressure field[1]. This is because, a solid-body rotation is like a fluid at rest when viewed from a rotating frame of reference.
Since solid-body rotation arises from the anti-symmetric part of the velocity gradient tensor, ignoring the possibility of tangential stresses arising from rigid-body rotation is equivalent to completely ignoring the anti-symmetric part of velocity gradient tensor from the constitutive relationship. The most general for the constitutive relationship with an isotropic part and a deviation from isotropy can then be written as

$$\boxed{\sigma_{ij} = -p\delta_{ij} + A_{ijkl}S_{kl}} \tag{12}$$

where S_{kl}, the rate-of-strain tensor, is the symmetric part of the velocity gradient tensor $(\boldsymbol{\nabla}\mathbf{u})_{kl}$.

3.1 On the structure of A_{ijkl}

We impose certain constraints on the structure of $\mathbf{A}$ which will help us at a later stage when we derive specific constitutive model relations between $\boldsymbol{\sigma}$ and $\mathbf{S}$.

Constraint - 1:
We have already seen that $\mathbf{S}$ is a symmetric tensor. Since the product of a symmetric and anti-symmetric tensor is zero, we require A_{ijkl} to be symmetric as well, i.e.

$$A_{ijkl} = A_{jikl} = A_{jilk}. \tag{13}$$

Such a constraint on $\mathbf{A}$ ensures that any double contraction of $\mathbf{A}$ with a symmetric second order tensor does not vanish.

Constraint - 2:
We also make the fourth-order tensor $\mathbf{A}$ traceless. This will enable use to separate the compressive/expansion part of the deformation from the incompressible part. Using this constraint and applying it to eq. (12), we get

$$A_{iikl} = 0 \quad \Longrightarrow \quad \underbrace{p = -\frac{\sigma_{ii}}{3}}_{\text{Definition of mechanical pressure}} \quad . \tag{14}$$

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[1] If $\dfrac{\partial p}{\partial x_i} = \rho F_i = -\rho\dfrac{\partial \Psi}{\partial x_i}$ with Ψ as the potential, then for

- gravity $\rightarrow \Psi = \mathbf{g}\cdot\mathbf{x}$,
- centrifugal force $\rightarrow \Psi = -\Omega^2\left[r^2 - (\mathbf{r}-\mathbf{e}_\Omega)^2\right]$.

This is a generalization of the static pressure seen earlier and is the average of the three components of the normal stresses. In a moving fluid, components of the normal stress are no longer equal to each other as in a static fluid.
The enforcement that p is the mechanical pressure even for fluids in motion leads to a subtlety in that the mechanical and thermodynamic pressure, p_e, are no longer the same. The difference arises because a fluid in motion is no longer in thermodynamic equilibrium. Whereas in a static fluid, we need not make such a distinction.
We state here without proof that the departure of pressure from the thermodynamic pressure is proportional to the change of volume of a material element. Hence

$$\begin{aligned} p - p_e &\propto \boldsymbol{\nabla} \cdot \mathbf{u}, & (15) \\ &= -\mu_b \boldsymbol{\nabla} \cdot \mathbf{u}, & (16) \end{aligned}$$

where μ_b is a scalar material constant called the bulk viscosity.

4 Newtonian fluid

In spite of the constraints, the structure of $\mathbf{A}$ is still now known. If we assume that the fluid in consideration is isotropic, then A_{ijkl} is required to be an isotropic tensor. The general form of a fourth order isotropic tensor is given by

$$A_{ijkl} = a\delta_{ij}\delta_{kl} + b(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \tag{17}$$

Constraint - 1:
(i) $A_{ijkl} = A_{jikl}$: Interchanging i and j in eq. (17), we have

$$A_{jikl} = a\delta_{ij}\delta_{kl} + b(\delta_{jk}\delta_{il} + \delta_{jl}\delta_{ik}) + c(\delta_{jk}\delta_{il} - \delta_{jl}\delta_{ik}). \tag{18}$$

Subtracting (17) from (18), we get $c = 0$.

(ii) $A_{jikl} = A_{jilk}$: It is easy to verify that this constraint is identically satisfied. We now have only two constants a and b.

(iii) $A_{jikl} = 0$:

$$\begin{aligned} A_{iikl} &= a\delta_{ii}\delta_{kl} + b(\delta_{ik}\delta_{il} + \delta_{il}\delta_{ik}) = 0, & (19) \\ &= 3a\delta_{kl} + b(\delta_{kl} + \delta_{kl}) = 0, \\ &\implies a = -\frac{2b}{3}. & (20) \end{aligned}$$

We are now reduced to just one constant, b. Following convention, we replace b with μ. The general form of $\mathbf{A}$ for an isotropic fluid then becomes

$$A_{ijkl} = -\frac{2\mu}{3}\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{21}$$

The deviation of stress tensor from isotropy then becomes

$$\begin{aligned} \tau_{ij} &= \left[-\frac{2\mu}{3}\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right] S_{kl}, \\ &= 2\mu \underbrace{\left(S_{ij} - \frac{S_{kk}}{3}\delta_{ij}\right)}_{\text{traceless}}. & (22) \end{aligned}$$

τ_{ij} is the called the deviatoric part of the stress tensor and it describes the response of a Newtonian fluid to volume-preserving shape deformations as determined by the traceless part of the rate-of-strain tensor. Also note that τ_{ij} is characterized by a single scalar material constant, μ called the *shear viscosity* or *dynamic viscosity*. We will use the latter name for μ since the deviatoric stress can arise even in fluids without any shear.

$$\sigma_{ij} = -p\delta_{ij} + 2\mu\left(S_{ij} - \frac{S_{kk}}{3}\delta_{ij}\right). \tag{23}$$

or

$$\sigma_{ij} = -(p_e - \mu_b S_{kk})\delta_{ij} + 2\mu\left(S_{ij} - \frac{S_{kk}}{3}\delta_{ij}\right). \tag{24}$$

In the second form, we have written the pressure term in terms of p_e. This is a linear relationship between $\boldsymbol{\sigma}$ and $\mathbf{S}$. This is the Newton's constitutive relation, or simply the Newton's law of viscosity, albeit, expressed in a generalized fashion. Thus, the response of a Newtonian fluid to deformation of all types, including those that change the volume, is characterized by two material constants, a dynamic (μ) and a bulk (μ_b) viscosities.

Substituting the general form of σ_{ij} into the Cauchy's equations of motion, we have

$$\rho\frac{Du_i}{Dt} = \rho F_i + \frac{\partial}{\partial x_j}\left\{-p\delta_{ij} + 2\mu\left(S_{ij} - \frac{S_{kk}}{3}\delta_{ij}\right)\right\} \tag{25}$$

or in term of p_e, we have

$$\rho\frac{Du_i}{Dt} = \rho F_i + \frac{\partial}{\partial x_j}\left\{-(p_e - \mu_b S_{kk})\delta_{ij} + 2\mu\left(S_{ij} - \frac{S_{kk}}{3}\delta_{ij}\right)\right\}. \tag{26}$$

Simplifying further, we get

$$\boxed{\rho\frac{Du_i}{Dt} = \rho F_i - \frac{\partial p_e}{\partial x_j} + \frac{\partial}{\partial x_j}\left\{\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right\} + \frac{\partial}{\partial x_i}\left\{\left(\mu_b - \frac{2\mu}{3}\right)\frac{\partial u_k}{\partial x_k}\right\}} \tag{27}$$

This is the generalized form of the **Navier-Stokes equation** with variable transport coefficients.

For incompressible flows, $D\rho Dt = 0$, hence $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$. Absence of volume changes implies that the difference between p and p_e is irrelevant.
The Navier-Stokes equation[2] for an incompressible fluid becomes

$$\begin{aligned}\rho\frac{Du_i}{Dt} &= \rho F_i - \frac{\partial p}{\partial x_j} + \frac{\partial}{\partial x_j}\left\{\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right\}, \\ &= \rho F_i - \frac{\partial p}{\partial x_j} + \mu\frac{\partial^2 u_i}{\partial x_j^2} + \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial \mu}{\partial x_j}\right).\end{aligned} \tag{28}$$

For a constant bulk viscosity, we get

$$\rho\frac{Du_i}{Dt} = \rho F_i - \frac{\partial p}{\partial x_j} + \mu\frac{\partial^2 u_i}{\partial x_j^2} \quad \text{with} \quad \mu = \text{constant}. \tag{29}$$

[2] The Navier-Stokes equation was first proposed by Navier, a French physicist, in 1822. Subsequently, Stokes (a British physicist) and St. Venant (French mechanician and mathematician) derived the same equation independently in 1843. Stokes in particular was trying to study the 'flow' of ether around Earth when it travels around the Sun to understand how light travels through a medium. We now know that there is no evidence of the existence of ether and that electromagnetic waves don't require a medium for their travel.

The last term of RHS in the above equation is the viscous stress. Viscosity play's the role of friction and diffuses momentum from one location to another. We will examine the physical meaning of this term in the next section. The presence of this term can have dramatic effects on the motion of fluids near solid objects. We will return to this point later in the course.

If the viscous term is absent, we obtain the Euler's equation named after the great mathematician Leonard Euler.

$$\rho \frac{Du_i}{Dt} = \rho F_i - \frac{\partial p}{\partial x_j}. \tag{30}$$

In summary, we have for an incompressible fluid

$$\rho \frac{Du_i}{Dt} = \rho F_i - \frac{\partial \sigma_{ij}}{\partial x_j}, \tag{31}$$

where $\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$ and the deviatoric or viscous stress tensor is given by

$$\tau_{ij} = 2\mu S_{ij}.$$

This is the Newton's law of viscosity.

5 Molecular theory of viscosity

We discussed earlier the the stress tensor, $\boldsymbol{\sigma}$, is a consequence of the surface force. Surface forces have a microscopic origin. The subject of viscous stress is really coming from a microscopic perspective. To understand momentum transport due to viscosity, we examine the viscous transport from an elementary kinetic theory perspective.

We first consider a pure gas consistent of rigid, non-attracting spherical molecules of diameter, b, and mass, m. Let the number density (number of molecules per unit volume) is taken to be n. We assume that the average distance between the molecules, $O(n^{-1/3}) \gg d$.

The average molecular velocity is

$$c = \sqrt{\frac{8kT}{\pi m}}. \tag{32}$$

The frequency of molecular bombardment per unit area on one side of a stationary surface exposed to the gas is

$$Z = \frac{1}{4} nc. \tag{33}$$

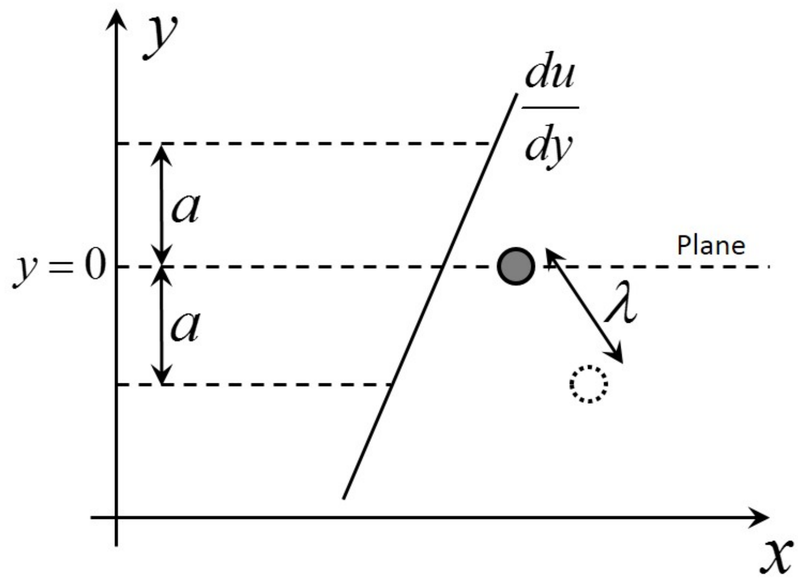
The goal of this approach is to understand the molecular mechanism of momentum transport that leads to a tangential stress in the equations of motion. We hope to achieve this by determining the total force (tangential) on an imaginary plane aligned in a flow due to molecular transport as shown in the figure.

The average distance travelled by a molecule between successive collisions is the "mean free path", i.e.,

$$\lambda = \frac{1}{\sqrt{2}\pi d^2 n}. \tag{34}$$

From kinetic theory, we can show that on an average, molecules reaching a plane will have experienced their last collision at a distance of a from the plane, where

$$a = \frac{2}{3}\lambda. \tag{35}$$



Let us consider a monoatomic gas flowing in the x-direction with a velocity $u(y)$ alone. The velocity gradient is therefore $\frac{\partial u}{\partial y}$. The viscous stress tensor, $\tau_{xy} = \tau_{yx} = \mu \frac{du}{dy}$.

We will now derive the expression for τ_{yx} using kinetic theory. The flux of x-momentum across any plane y=constant is found by summing the total x-momentum of molecules that cross the positive direction, and subtracting the total x-momentum of those that cross in the opposite direction.

$$\tau_{yx} = Zm\, u|_{y=a} - Zm\, u|_{y=a} \tag{36}$$

We have assumed that all molecules below and above the $y = 0$ plane have velocities which are in equilibrium with the local flow, i.e.

$$u|_{y=\pm a} = u|_{y=0} \pm a \left.\frac{du}{dy}\right|_{y=0}. \tag{37}$$

Therefore

$$\begin{aligned} \tau_{yx} &= Zm \times 2a \left.\frac{du}{dy}\right|_{y=0}, &(38)\\ &= \frac{1}{3} nmc\lambda \frac{du}{dy} = \frac{1}{3}\rho c\lambda \frac{du}{dy}, &(39) \end{aligned}$$

where $\rho = mn$ is the density of the gas. Comparing this with the continuum description, we have

$$\mu \frac{du}{dy} = \frac{1}{3} nmc\lambda \frac{du}{dy}. \tag{40}$$

Hence

$$\mu = \frac{1}{3} nmc\lambda = \frac{1}{3}\rho c\lambda. \tag{41}$$

Writing the above expression in terms of fundamental parameters, we have

$$\mu = \frac{2}{3\pi} \frac{\sqrt{\pi m k T}}{\pi b^2}. \tag{42}$$

Therefore, for a dilute gas $\mu \sim T^{1/2}$ and $\mu \sim 1/d^2$. Hence increasing temperature of a gas decreases its viscosity, heavier molecules also increase viscosity and larger molecules decrease viscosity.

For liquids, a molecular theory is significantly more involved. The above description does not hold in the case of liquids since the molecules in a liquid, like in a solid, are more closely spaced and we cannot ignore the intermolecular forces that exist between molecules of liquid. A liquid can be described as matter where the molecules are perpetually involved in cage-breaking and cage-formation events. If ΔG is the typical activation energy of a cage-breaking event, then the viscosity of a liquid can be written as

$$\mu \approx \frac{N_a h}{V} \exp\left(\frac{\Delta G}{RT}\right), \tag{43}$$

where N_a and h is Avogadro and Planck's constant respectively and $R = N_a k$ is the gas constant.
For a derivation of the above law, the reader is referred to Transport Phenomenon by Bird, Lightfoot & Stewart.

Clearly, in the case of a liquid, increasing temperature reduces the viscosity. This is in complete contrast to the way viscosity of gases varies. But such differences only arise with regards to variations of the transport constants and do not in any way affect the validity of the Navier-Stokes equations.