ME5310: Incompressible Fluid Flow
Solutions to Assignment - 1

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Problem 1
By manipulating the symbols, show that the product of ϵ_{ijk} and S_{ij} is zero, where S_{ij} is any symmetric tensor and ϵ is the usual alternating tensor.
Solution
The product of ϵ and S will be a vector in the kth direction. So we define a vector $\mathbf{w}$ such that

$$w_k = \epsilon_{ijk} S_{ij}.$$

Since S_{ij} is symmetric, we have $S_{ij} = S_{ji}$, where as $\epsilon_{ijk} = -\epsilon_{jik}$. Flipping i and j, we have

$$\begin{aligned} w_k &= \epsilon_{jik} S_{ji} \\ &= -\epsilon_{ijk} S_{ij}. \end{aligned}$$

Comparing the original definition of w_k and the above expression, we have $w_k = -w_k$. Hence $w_k = 0$.

Problem 2
You have learnt in class that any tensor $\mathbf{T}$ can be decomposed as follows:

$$T_{ij} = Q_{ij} + R_{ij}$$

where $\mathbf{Q}$ is a symmetry tensor and $\mathbf{R}$ is an antisymmetric tensor. We can also construct a vector $\mathbf{d}$ as a product of a two tensors. If we define $\mathbf{d}$ as

$$d_i = \epsilon_{ijk} T_{jk},$$

then show the following relation:

$$T_{ij} = Q_{ij} + \frac{1}{2} \epsilon_{ijk} d_k.$$

Solution

We first obtain the inverse relationship between d_i and T_{jk}. Let us consider the expression

$$\epsilon_{ilm} d_i$$

Using the definition of d_i, we get

$$\begin{aligned} \epsilon_{ilm} d_i &= \epsilon_{ijk} \epsilon_{ilm} T_{jk}, \\ &= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) T_{jk}, \\ &= T_{lm} - T_{ml}. \end{aligned}$$

But, we know that $R_{ij} = (T_{ij} - T_{ji})/2$. Hence interchanging $\{i, j\}$ with $\{l, m\}$, we have

$$R_{lm} = \frac{T_{lm} - T_{ml}}{2}.$$

Therefore

$$\epsilon_{ilm} d_i = 2R_{lm}, \implies \epsilon_{ijk} d_i = 2R_{ij}.$$

Hence,

$$T_{ij} = Q_{ij} + \frac{1}{2}\epsilon_{ijk} d_k.$$

Problem 3
If the second order tensor $\mathbf{T}$ is defined as

$$T_{ij} = v_k w_i S_{kj} + a\delta_{ij} + \epsilon_{ijk} w_k,$$

where a is a scalar, $\mathbf{v}$, $\mathbf{w}$ are vectors, $\mathbf{S}$ is an arbitrary second order tensor, ϵ and δ have their usual meaning.

The summation for the three indices $\{i, j, k\}$ goes from 1 to 3. Write down the expressions for the following tensor components:

(i) T_{11}

(ii) T_{12}

(iii) Contraction of T_{ij}: Make $i = j$ to obtain an expression for T_{ii}. Note that T_{ii} is equal to trace(T).

Solution

(i)

$$\begin{aligned} T_{11} &= \sum_k v_k w_1 S_{k1} + a\delta_{11} + \epsilon_{11k} w_k, \\ &= v_1 w_1 S_{11} + v_2 w_1 S_{21} + v_3 w_1 S_{31} + a. \end{aligned}$$

(ii)

$$\begin{aligned} T_{12} &= \sum_k v_k w_1 S_{k2} + a\delta_{12} + \epsilon_{12k} w_k, \\ &= v_1 w_1 S_{12} + v_2 w_1 S_{22} + v_3 w_1 S_{32} + \epsilon_{123} w_3. \end{aligned}$$

We have used $\delta_{12} = 0$ and only $k = 3$ gives a non-zero value for ϵ_{12k}.

(iii) With $i = j$, we have

$$\begin{aligned} T_{ii} &= \sum_i \sum_k v_k w_i S_{ki} + a\delta_{ii} + \epsilon_{iik} w_k, \\ &= \sum_k v_k w_1 S_{k1} + v_k w_2 S_{k2} + v_k w_3 S_{k3} + a(\delta_{11} + \delta_{22} + \delta_{33}), \\ &= \sum_k v_k w_1 S_{k1} + v_k w_2 S_{k2} + v_k w_3 S_{k3} + 3a, \\ &= \text{trace}(\mathbf{T}). \end{aligned}$$

We have used $\delta_{12} = 0$ and only $k = 3$ gives a non-zero value for ϵ_{12k}.

Problem 4
Using index notation, prove the following vector algebra identities between the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}$:

(i) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u})$.

(ii) $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{z}) = (\mathbf{u} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{z}) - (\mathbf{u} \cdot \mathbf{z})(\mathbf{v} \cdot \mathbf{w})$: Binet-Cauchy identity.

(iii) $(\mathbf{u} \times \mathbf{v}) \times (\mathbf{w} \times \mathbf{z}) = [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{z}]\mathbf{w} - [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}]\mathbf{z}$

(iv) $[(\mathbf{v} \times \mathbf{w}) \cdot (\mathbf{v} \times \mathbf{w})] + (\mathbf{v} \cdot \mathbf{w})^2 = v^2 w^2$, where v and w are the magnitudes of $\mathbf{v}$ and $\mathbf{w}$ respectively.

Solution

I will ignore the $\sum$ symbol in all my solutions below for simplicity.

(i)

$$\begin{aligned}
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{v}) &= u_i (\mathbf{v} \times \mathbf{w})_i, \\
&= u_i \epsilon_{ijk} v_j w_k, \\
&= v_j \epsilon_{ijk} w_k u_i, \\
&= v_j \epsilon_{jki} w_k u_i, \\
&= v_j (\mathbf{w} \times \mathbf{u})_j, \\
&= \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \text{RHS}.
\end{aligned}$$

(ii)

$$\begin{aligned}
(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{z}) &= (\mathbf{u} \times \mathbf{v})_i (\mathbf{w} \times \mathbf{z})_i, \\
&= \epsilon_{ijk} u_j v_k \epsilon_{ilm} w_l z_m, \\
&= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) u_j v_k w_l z_m, \\
&= \delta_{jl}\delta_{km} u_j v_k w_l z_m - \delta_{jm}\delta_{kl} u_j v_k w_l z_m, \\
&= u_j v_k w_j z_k - u_j v_k w_k z_j, \\
&= (\mathbf{u} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{z}) - (\mathbf{u} \cdot \mathbf{z})(\mathbf{v} \cdot \mathbf{w}) = \text{RHS}.
\end{aligned}$$

(iii) Let us consider the i^{th} component of LHS:

$$\begin{aligned}
[(\mathbf{u} \times \mathbf{v}) \times (\mathbf{w} \times \mathbf{z})]_i &= \epsilon_{ijk} (\mathbf{u} \times \mathbf{v})_j (\mathbf{w} \times \mathbf{z})_k, \\
&= \epsilon_{ijk} (\epsilon_{jlm} u_l v_m)(\epsilon_{knp} w_n z_p), \\
&= (\epsilon_{kij} \epsilon_{knp}) \epsilon_{jlm} u_l v_m w_n z_p, \\
&= (\delta_{in}\delta_{jp} - \delta_{ip}\delta_{jn}) \epsilon_{jlm} u_l v_m w_n z_p, \\
&= \epsilon_{jlm} u_l v_m w_i z_j - \epsilon_{jlm} u_l v_m w_j z_i, \\
&= (\mathbf{u} \times \mathbf{v})_j z_j w_i - (\mathbf{u} \times \mathbf{v})_j w_j z_i, \\
&= [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{z}] w_i - [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}] z_i
\end{aligned}$$

(iv) Consider the first term in the LHS:

$$\begin{aligned}
(\mathbf{v}\times\mathbf{w})\cdot(\mathbf{v}\times\mathbf{w}) &= (\mathbf{v}\times\mathbf{w})_i(\mathbf{v}\times\mathbf{w})_i, \\
&= \epsilon_{ijk}v_jw_k\epsilon_{ilm}v_lw_m, \\
&= (\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl})v_jw_kv_lw_m, \\
&= \delta_{jl}\delta_{km}v_jw_kv_lw_m-\delta_{jm}\delta_{kl}v_jw_kv_lw_m, \\
&= v_jw_kv_jw_k-v_jw_kv_kw_j, \\
&= (\mathbf{v}\cdot\mathbf{v})(\mathbf{w}\cdot\mathbf{w})-(\mathbf{v}\cdot\mathbf{w})(\mathbf{v}\cdot\mathbf{w}), \\
&= v^2w^2-(\mathbf{v}\cdot\mathbf{w})^2.
\end{aligned}$$

Problem 5

Using index notation, prove the following vector calculus identities:

(i) $\boldsymbol{\nabla}\cdot\phi\mathbf{v}=\boldsymbol{\nabla}\phi\cdot\mathbf{v}+\phi(\boldsymbol{\nabla}\cdot\mathbf{u})$ where ϕ is a scalar.

(ii) $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\mathbf{u}=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u})-\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{u})$. LHS is simple $\boldsymbol{\nabla}^2\mathbf{u}$, the vector Laplacian.

(iii) $\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}=\frac{1}{2}\boldsymbol{\nabla}(\mathbf{u}\cdot\mathbf{u})-[\mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{u})]$

(iv) $\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B})=\mathbf{A}(\boldsymbol{\nabla}\cdot\mathbf{B})-\mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{A})+(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A}-(\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B}$

Solution

(i) Consider the LHS:

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\phi\mathbf{v} &= \left(\sum_i \mathbf{e}_i\frac{\partial}{\partial x_i}\right)\cdot\left(\sum_j \phi\ \mathbf{e}_jv_j\right), \\
&= \sum_i\sum_j \delta_{ij}\left(\frac{\partial\phi}{\partial x_i}v_j+\phi\frac{\partial v_j}{\partial x_i}\right), \\
&= \sum_i\left(\frac{\partial\phi}{\partial x_i}v_i+\phi\frac{\partial v_i}{\partial x_i}\right), \\
&= \boldsymbol{\nabla}\phi\cdot\mathbf{v}+\phi(\boldsymbol{\nabla}\cdot\mathbf{v}) = \text{RHS}.
\end{aligned}$$

(ii) Consider the last term of RHS (I am omitting the $\sum$ symbol for simplicity):

$$\begin{aligned}
\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{u}) &= \epsilon_{ijk}\nabla_j(\boldsymbol{\nabla}\times\mathbf{u})_k, \\
&= \epsilon_{ijk}\frac{\partial}{\partial x_j}\epsilon_{klm}\frac{\partial u_m}{\partial x_l}, \\
&= \epsilon_{kij}\epsilon_{klm}\frac{\partial}{\partial x_j}\frac{\partial u_m}{\partial x_l}, \\
&= (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})\frac{\partial}{\partial x_j}\frac{\partial u_m}{\partial x_l}, \\
&= \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right)-\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right), \\
&= \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u})-\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\mathbf{u} = \text{RHS}.
\end{aligned}$$

(ii) Consider the last term of RHS (I am omitting the $\sum$ symbol for simplicity):

$$\begin{aligned}
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{u}) &= \epsilon_{ijk} \nabla_j (\boldsymbol{\nabla} \times \mathbf{u})_k, \\
&= \epsilon_{ijk} \frac{\partial}{\partial x_j} \epsilon_{klm} \frac{\partial u_m}{\partial x_l}, \\
&= \epsilon_{kij} \epsilon_{klm} \frac{\partial}{\partial x_j} \frac{\partial u_m}{\partial x_l}, \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \frac{\partial}{\partial x_j} \frac{\partial u_m}{\partial x_l}, \\
&= \frac{\partial}{\partial x_i} \left(\frac{\partial u_j}{\partial x_j} \right) - \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} \right), \\
&= \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) - \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \mathbf{u} = \text{RHS}.
\end{aligned}$$

(iii) Consider the last term of RHS (I am omitting the $\sum$ symbol for simplicity):

$$\begin{aligned}
\mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{u}) &= \epsilon_{ijk} u_j (\boldsymbol{\nabla} \times \mathbf{u})_k, \\
&= \epsilon_{ijk} u_j \epsilon_{klm} \frac{\partial u_m}{\partial x_l}, \\
&= \epsilon_{kij} \epsilon_{klm} u_j \frac{\partial u_m}{\partial x_l}, \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) u_j \frac{\partial u_m}{\partial x_l}, \\
&= u_j \left(\frac{\partial u_j}{\partial x_i} \right) - u_j \left(\frac{\partial u_i}{\partial x_j} \right), \\
&= \frac{\partial}{\partial x_i} \left(\frac{1}{2} u_j^2 \right) - u_j \left(\frac{\partial u_i}{\partial x_j} \right), \\
&= \frac{1}{2} \boldsymbol{\nabla}(\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} = \text{RHS}.
\end{aligned}$$

(iv) Consider the LHS:

$$\begin{aligned}
\boldsymbol{\nabla} \times (\mathbf{A} \times \mathbf{B}) &= \epsilon_{ijk} \frac{\partial}{\partial x_j} (\mathbf{A} \times \mathbf{B})_k, \\
&= \epsilon_{ijk} \frac{\partial}{\partial x_j} \epsilon_{klm} A_l B_m, \\
&= \epsilon_{kij} \epsilon_{klm} \frac{\partial (A_l B_m)}{\partial x_j}, \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \left(B_m \frac{\partial A_l}{\partial x_j} + A_l \frac{\partial B_m}{\partial x_j} \right), \\
&= B_j \frac{\partial A_i}{\partial x_j} + A_i \frac{\partial B_j}{\partial x_j} - B_i \frac{\partial A_j}{\partial x_j} - A_j \frac{\partial B_i}{\partial x_j}, \\
&= \mathbf{B} \cdot \boldsymbol{\nabla} \mathbf{A} + \mathbf{A}(\boldsymbol{\nabla} \cdot \mathbf{B}) - \mathbf{B}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \mathbf{A} \cdot \boldsymbol{\nabla} \mathbf{B}, \\
&= \text{RHS}.
\end{aligned}$$
