ME5310: Incompressible Fluid Flow
Assignment - 1

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Due date: 21st August 2014, before the class begins.

Problem 1
By manipulating the symbols, show that the product of ϵ_{ijk} and S_{ij} is zero, where S_{ij} is any symmetric tensor and ϵ is the usual alternating tensor.

Problem 2
You have learnt in class that any tensor $\mathbf{T}$ can be decomposed as follows:

$$T_{ij} = Q_{ij} + R_{ij}$$

where $\mathbf{Q}$ is a symmetry tensor and $\mathbf{R}$ is an antisymmetric tensor. We can also construct a vector $\mathbf{d}$ as a product of a two tensors. If we define $\mathbf{d}$ as

$$d_i = \epsilon_{ijk} T_{jk},$$

then show the following relation:

$$T_{ij} = Q_{ij} + \frac{1}{2}\epsilon_{ijk} d_k.$$

Problem 3
If the second order tensor $\mathbf{T}$ is defined as

$$T_{ij} = v_k w_i S_{kj} + a\delta_{ij} + \epsilon_{ijk} w_k,$$

where a is a scalar, $\mathbf{v}$, $\mathbf{w}$ are vectors, $\mathbf{S}$ is an arbitrary second order tensor, ϵ and δ have their usual meaning.

The summation for the three indices $\{i, j, k\}$ goes from 1 to 3. Write down the expressions for the following tensor components:

(i) T_{11}

(ii) T_{12}

(iii) Contraction of T_{ij}: Make $i = j$ to obtain an expression for T_{ii}. Note that T_{ii} is equal to trace(T).

Problem 4
Using index notation, prove the following vector algebra identities between the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}$:

(i) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u})$.

(ii) $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{z}) = (\mathbf{u} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{z}) - (\mathbf{u} \cdot \mathbf{z})(\mathbf{v} \cdot \mathbf{w})$: Binet-Cauchy identity.

(iii) $(\mathbf{u} \times \mathbf{v}) \times (\mathbf{w} \times \mathbf{z}) = [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{z}]\mathbf{w} - [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}]\mathbf{z}$

(iv) $[(\mathbf{v} \times \mathbf{w}) \cdot (\mathbf{v} \times \mathbf{w})] + (\mathbf{v} \cdot \mathbf{w})^2 = v^2 w^2$, where v and w are the magnitudes of $\mathbf{v}$ and $\mathbf{w}$ respectively.

This identity is referred to as the Lagrange's identity* named after the great Italian mathematician Joseph-Louis Lagrange. The Lagrangian frame of reference is named after the same Lagrange.

Problem 5
Using index notation, prove the following vector calculus identities:

(i) $\boldsymbol{\nabla} \cdot \phi \mathbf{v} = \boldsymbol{\nabla} \phi \cdot \mathbf{v} + \phi(\boldsymbol{\nabla} \cdot \mathbf{u})$ where ϕ is a scalar.

(ii) $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \mathbf{u} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{u})$. LHS is simple $\boldsymbol{\nabla}^2 \mathbf{u}$, the vector Laplacian.

(iii) $\mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} = \frac{1}{2} \boldsymbol{\nabla}(\mathbf{u} \cdot \mathbf{u}) - [\mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{u})]$

(iv) $\boldsymbol{\nabla} \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\boldsymbol{\nabla} \cdot \mathbf{B}) - \mathbf{B}(\boldsymbol{\nabla} \cdot \mathbf{A}) + (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A} - (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B}$

Hint: In some cases, it may be useful to show that starting from RHS, we get LHS. I leave it to your good judgement to decide where you want to use this hint.

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*If we have a set of ordered pairs $\{a_1, a_2, a_3, ...\}$ and $\{b_1, b_2, b_3, ...\}$, then Lagrange's identity, as it appears in algebra, can also be written in the form

$$\left(\sum_{k=1}^{n} a_k^2\right) + \left(\sum_{k=1}^{n} b_k^2\right) - \left(\sum_{k=1}^{n} a_k b_k\right) = \left(\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (a_i b_j - a_j b_i)^2\right).$$

The LHS is simply a combination of magnitudes of vectors $\mathbf{a}$, $\mathbf{b}$ and scalar product $\mathbf{a} \cdot \mathbf{b}$, and RHS involves components of $\mathbf{a} \times \mathbf{b}$. For $n = 2$, this identity was first proved by Diophantus of Alexandria, considered the 'father of algebra', in 3rd century AD and later by the great Indian mathematician & astronomer Brahmagupta, in 7th century AD. Brahmagupta was also the first person to formulate rules to use zero in algebra. Now, the special case of $n = 2$ is referred to as Brahmagupta-Fibonacci identity or simply the Fibonacci identity. Notice that Lagrange's identity is a special case of Binet-Cauchy identity.

Cauchy, another great French mathematician, is well known for a wide range of contributions. More importantly for us, he extended the Euler's equations of inviscid fluids to more general class of fluids. This is referred to as the Cauchy momentum equation (written in vector form with a stress tensor σ and external forcing $\mathbf{f}$):

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u}\right] = \boldsymbol{\nabla} \cdot \sigma + \mathbf{f}$$

Incidentally, if we use a simpler form of the stress tensor by assuming it to be linearly proportional to the gradient of velocity with p as the pressure, i.e., if $\sigma = -\mathbf{pI} + \mu \boldsymbol{\nabla} \mathbf{u}$, then the Cauchy momentum equation reduces to the Navier-Stokes equation. We refer to the proportionality constant, μ, as viscosity.