Forces on a fluid

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We will now investigate the forces that act on a fluid element and the mathematical description for these forces. The analysis will be in broad terms and hence will be of a mathematical nature. The physical interpretation of these forces is left for later chapters.

1 Volume and surface forces on a fluid element

We have already seen that most fluids in normal circamstances can be treated like a continuum. By a continuum, we mean that the measured fluid property is constant for sensitive volumes small on the macroscopic scale but large on the microscopic scale.
For example, for most laboratory experiments, the linear dimension of the region occupied by the fluid is at least as large as $1cm$ and very little variations in fluid property occur even over distances as small as $10^{-3}cm$. This corresponds to a sensitive volume of about $10^{-9}cm^3$. Therefore, we are really seeing an average taken over a large number of molecules. Hence, it is possible to assign a definite meaning to the notion of value "at a point". We can also safely define forces at a point too. We will distinguish between two kinds of forces:

- **Body forces**: Long range forces like gravity, electromagnetic forces, fictitious forces such as centrifugal forces, etc.
- **Surface forces**: Short range forces such as viscosity, surface tension, etc.

1.1 Body forces

They vary (decrease) slowly with increasing distance between interacting elements. Example, the gravitational force per unit mass acting on an object is very well represented by a constant vector $\mathbf{g}$ ($|g| = 9.81m/s^2$)for distances from the earth's surface that are less than the order of magnitude of rate earth's radius ($\sim 6.4 \times 10^6 m$). A consequence of this is that the forces act equally on all the matter within a sufficiently small (infinitesimal) element of our fluid continuum and is proportional to the size of the volume element.

The body force per unit mass at a point $\mathbf{x}$ in the fluid at time t is $\mathbf{F}(\mathbf{x}, t)$. The total force on an infinitesimal element of volume δV is then

$$\mathbf{F}(\mathbf{x}, t)\rho\delta V.$$

For gravity, $\mathbf{F}(\mathbf{x}, t) = \mathbf{g}$ making the total gravitational force equal to $\rho\mathbf{g}\delta V$.
For a conducting fluid in a magnetic fleld $\mathbf{B}$, the total Lorenz force, $\mathbf{F}_L \propto \mathbf{i} \times \mathbf{B}$ where $\mathbf{i}$ is the current density.

1.2 Surface forces

The following are some characteristics of surface forces:

- Have a direct molecular origin - short range forces

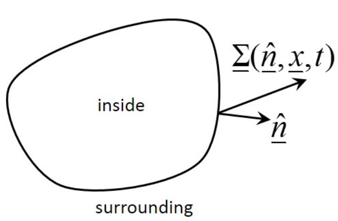
- Decrease extremely rapidly with distance
- Appreciable only when distance is of the order of separation of molecules of the liquid

Hence, these forces are negligible unless there is a direct mechanical contact between interacting elements. Strictly speaking, these contact forces are expected to act on a layer whose length is comparable to the mean free path, λ_{MFP}. But since $\lambda_{MFP} \ll V^{1/3}$, this layer can be assumed to be coincident with the surface itself. Hence short ranged forces manifest as surface forces that act to transport momentum across the boundaries of an infinitesimal element.
In gases, the momentum transport occurs due to molecules randomly crossing the boundary and thereby carrying momentum across in the appropriate directions. In liquids, the transport of momentum can occur without physical translation of the molecules via short-ranged forces acting between pairs of molecules on either side of the boundary and separated by a distance comparable to the inter-molecular potential. Therefore, the total effect of short ranged forces acting on a differential element is decided by its surface-area rather than its volume. We therefore consider a plane surface element in the fluid and specify the local short-ranged force as the total force exerted by the fluid on one side on the other side.
If δA is area of the element, then the total force across the element will be proportional to δA and its value at time t for an element at position $\mathbf{x}$ is the vector

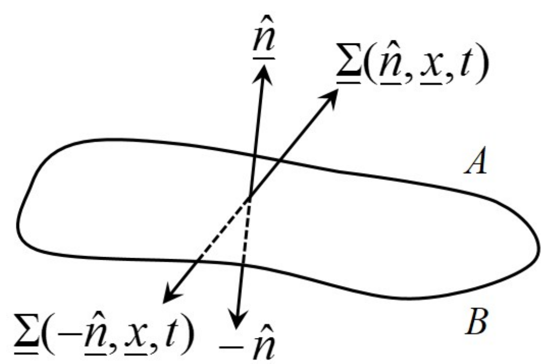
$$\mathbf{\Sigma}(\mathbf{n}, \mathbf{x}, t)\delta A,$$

where $\mathbf{n}$ is the unit normal to the element and $\mathbf{\Sigma}$ is the force per unit area (or simply the stress vector). The $\mathbf{n}$ dependence is expected since the force is expected to depend on the orientation of the surface element.

Convention: $\mathbf{\Sigma}$ is the stress exerted by the fluid on the side of the surface element to which $\mathbf{n}$ points, on the fluid on the side to which $\mathbf{n}$ points away from.



(a)

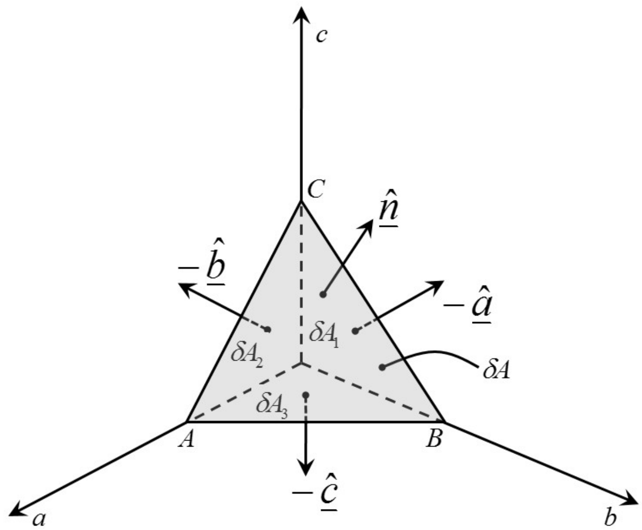


(b)

Hence tension will become a positive force.

$\mathbf{\Sigma}$ is the stress exerted by the A-side of the fluid surface on the B-side. Now, we show that a *stress tensor rather than a stress vector* $\mathbf{\Sigma}$ *is the more fundamental quantity.* Working with a stress tensor is necessary because the resulting governing equations have to be coordinate independent, nature does not care for how we define our coordinate system. In order to do this, we need to extract the $\mathbf{n}$ dependence of $\mathbf{\Sigma}$.

Since $\mathbf{\Sigma}(\mathbf{n}, \mathbf{x}, t)$ is the force exerted by the fluid on the side to which $\mathbf{n}$ points, that exerted by the fluid on the opposite side should be equal and opposite, since there can be no net force on a surface element (which



has zero mass). In our notation, this same force can be written in terms of normal, $-\mathbf{n}$, as $\mathbf{\Sigma}(-\mathbf{n}, \mathbf{x}, t)$. Thus

$$\mathbf{\Sigma}(-\mathbf{n}, \mathbf{x}, t) = -\mathbf{\Sigma}(\mathbf{n}, \mathbf{x}, t). \tag{1}$$

Hence stress vector, $\mathbf{\Sigma}$ is an odd function of $\mathbf{n}$. To deduce the specific $\mathbf{n}$ dependence, we consider a tetrahedral volume element as shown in the figure.
The three orthogonal faces have areas δA_1, δA_2 and δA_3 and unit (outward) normals $-\mathbf{a}$, $-\mathbf{b}$ and $-\mathbf{c}$ respectively. The fourth inclined face has area δA and unit outward normal $\mathbf{n}$.

Surface forces will act on the fluid in the tetrahedron across each of the four faces and their sum is

$$\mathbf{\Sigma}(\mathbf{n}, \mathbf{x}, t)\delta A + \mathbf{\Sigma}(-\mathbf{a}, \mathbf{x}, t)\delta A_1 + \mathbf{\Sigma}(-\mathbf{b}, \mathbf{x}, t)\delta A_2 + \mathbf{\Sigma}(-\mathbf{c}, \mathbf{x}, t)\delta A_3. \tag{2}$$

We have assumed that the volume element is small enough such that the position vector $\mathbf{x}$ is the same for all the faces. From the figure, it is clear that area of each of the orthogonal faces is related to the area of the inclined face through the relation $\delta A_1 = \mathbf{a} \cdot \mathbf{n}\delta A$, $\delta A_2 = \mathbf{b} \cdot \mathbf{n}\delta A$ and $\delta A_3 = \mathbf{c} \cdot \mathbf{n}\delta A$. Suppressing $\mathbf{x}$ and t in the expression, total surface force becomes

$$\{\mathbf{\Sigma}(\mathbf{n}) - \mathbf{\Sigma}(\mathbf{a})\ (\mathbf{a} \cdot \mathbf{n}) - \mathbf{\Sigma}(\mathbf{b})\ (\mathbf{b} \cdot \mathbf{n}) - \mathbf{\Sigma}(\mathbf{c})\ (\mathbf{c} \cdot \mathbf{n})\}\, \delta A \tag{3}$$

where we have used $\mathbf{\Sigma}(\mathbf{n}) = -\mathbf{\Sigma}(\mathbf{n})$.

The total surface force is proportional to δA, whereas the total body force is proportional to δV, which is smaller than δA when the size of the fluid element becomes small. Also, the mass of the fluid in the tetrahedron too is proportional to δV. The general force balance for the element becomes

$$\underbrace{F_{\text{surface force}}}_{O(\delta A)} + \underbrace{F_{\text{body force}}}_{O(\delta V)} = \underbrace{ma}_{O(\delta V)},$$

where we have assumed the accelerations are finite. Hence if we make the linear dimension of the tetrahedron go to zero without change in its shape, we expect the total surface force to vanish. This can only happen if the coefficient of δA in the above expression vanishes. We therefore have

$$\begin{aligned} & \mathbf{\Sigma}(\mathbf{n}) - \mathbf{\Sigma}(\mathbf{a})\ (\mathbf{a} \cdot \mathbf{n}) - \mathbf{\Sigma}(\mathbf{b})\ (\mathbf{b} \cdot \mathbf{n}) - \mathbf{\Sigma}(\mathbf{c})\ (\mathbf{c} \cdot \mathbf{n}) = 0, \\ \Longrightarrow \quad & \mathbf{\Sigma}(\mathbf{n}) = \{\mathbf{\Sigma}(\mathbf{a})\ \mathbf{a} + \mathbf{\Sigma}(\mathbf{b})\ \mathbf{b} + \mathbf{\Sigma}(\mathbf{c})\ \mathbf{c}\} \cdot \mathbf{n}, \end{aligned} \tag{4}$$

or in index notation

$$\Sigma_i(\mathbf{n}) = \{\Sigma_i(\mathbf{a})a_j + \Sigma_i(\mathbf{b})b_j + \Sigma_i(\mathbf{c})c_j\}\, n_j. \tag{5}$$

Since the vector $\mathbf{n}$ and $\mathbf{\Sigma}$ do not in anyway depend on the choice of axes of reference, the expression within the curly brackets in the RHS that relates $\mathbf{\Sigma}$ and $\mathbf{n}$ must likewise be independent on the particular choice of the axes.
In other words, for any given i and j, this expression much correspond to the ij^{th} component of a second-order tensor which we denote by $\boldsymbol{\sigma}$. The above relation therefore reduces to

$$\mathbf{\Sigma}_i(\mathbf{n}) = \sigma_{ij} n_j. \tag{6}$$

σ_{ij} is the ij^{th} component of the force per unit area in the i^{th} direction exerted across a plane surface element normal to the j^{th} direction.

$\boldsymbol{\sigma}$ is called the stress tensor.

1.3 On the structure of the stress tensor

Consider the i^{th} component of the total moment about a point O within the volume, exerted by the surface forces at the boundary. The total moment integrated over the surface becomes

$$\int (\mathbf{r} \times \mathbf{\Sigma})\mathbf{n} dA = \int \epsilon_{ijk} r_j \sigma_{kl} n_l dA, \tag{7}$$

where we have used the relation between $\mathbf{\Sigma}$ and $\boldsymbol{\sigma}$, $\mathbf{r}$ is the position vector of any point on the surface element where the normal vector is $\mathbf{n}$ relative to O. Using divergence theorem, we can reduce this to a volume integral, i.e.

$$\begin{aligned} \int \epsilon_{ijk} r_j \sigma_{kl} n_l dA &= \int \epsilon_{ijk} \frac{\partial}{\partial r_j}(r_j \sigma_{kl}) dV, & (8)\\ &= \int \epsilon_{ijk} \left(\sigma_{kl}\delta_{jl} + r_j \frac{\partial \sigma_{kl}}{\partial x_l}\right) dV, & (9)\\ &= \int \epsilon_{ijk} \left(\sigma_{kj} + r_j \frac{\partial \sigma_{kl}}{\partial x_l}\right) dV. & (10) \end{aligned}$$

If the volume $V \to 0$ such that the shape of the element remains unchanged, then the first term in RHS goes to zero as $O(V)$ whereas the second term goes to zero as $O(V^{4/3})$. Thus

$$\int \epsilon_{ijk} \sigma_{kj} dV \gg \int \epsilon_{ijk} r_j \frac{\partial \sigma_{kl}}{\partial x_l} dV \qquad \text{as} \quad V \to 0. \tag{11}$$

The total moment in the limit of $V \to 0$ then becomes

$$\int \epsilon_{ijk} \sigma_{kj} dV.$$

Setting this to zero, we require

$$\epsilon_{ijk} \sigma_{kj} = 0. \tag{12}$$

Since ϵ_{ijk} is an anti-symmetric tensor, the only way the above product is always zero for an arbitrary σjk is when $\boldsymbol{\sigma}$ is a symmetric tensor. This ensures that the product of $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$ always vanishes. Therefore

$$\sigma_{ij} = \sigma_{ji}. \tag{13}$$

Hence $\boldsymbol{\sigma}$ has only six independent components. We can write $\boldsymbol{\sigma}$ in matrix form as

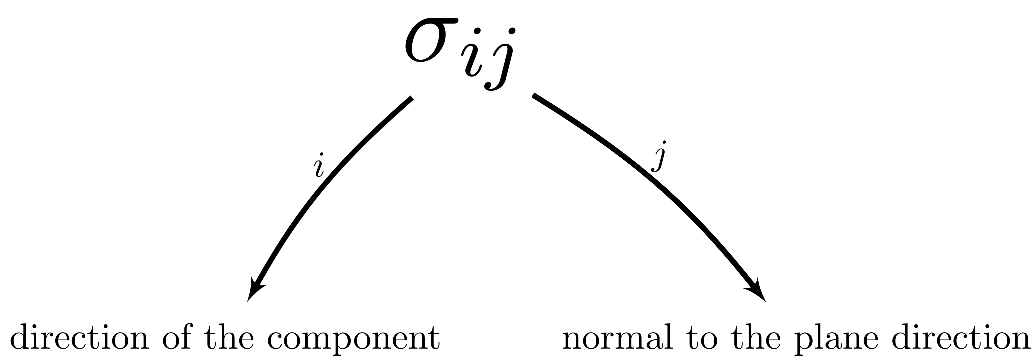
$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix},$$

where $\sigma_{12} = \sigma_{21}$, $\sigma_{13} = \sigma_{31}$ and $\sigma_{23} = \sigma_{32}$.

The diagonal components of $\boldsymbol{\sigma}$ are called ***normal stresses***, i.e., they give the normal component of surface force acting across the plane surface element parallel to one of the coordinate planes.

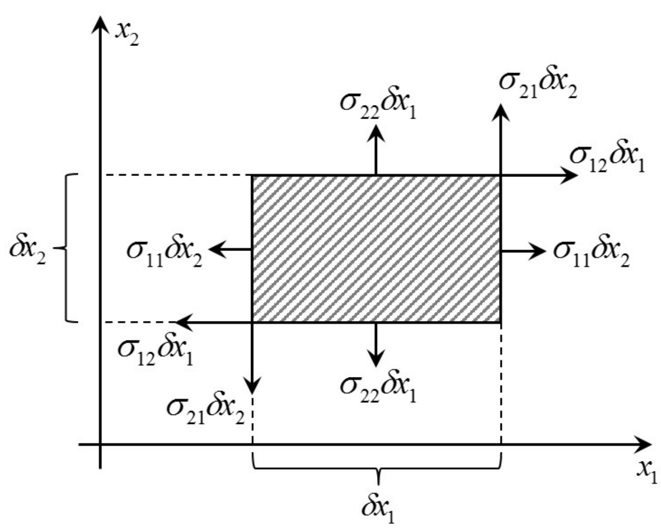
The off-diagonal components of $\boldsymbol{\sigma}$ are called ***tangential stresses*** or ***shearing stresses***.

1.4 Convention



1.5 Stresses in the $(x_1 - x_2)$ plane

Based on the above convention for σ_{ij}, let us take a look at all the surface forces that exist on a cuboidal element of size $\delta x_1 \times \delta x_2 \times 1$. We have assumed here that $\delta x_3 = 1$. The normal force on the x_2=constant plane is equal to $\sigma_{11}\delta x_2$. Similarly, the tangential force on the x_1=constant plane in the x_2 direction



is $\sigma_{21}\delta x_2$, and that on the x_2=constant plane in the the x_1 direction is $\sigma_{12}\delta x_1$. The direction of these components is identically to the convention that we followed for the unit dyads, where the first unit vector gives the direction of the force and the second unit vector gives the direction of the normal vector of the plane on which the force is acting.

1.6 Normal and Principal stresses

We can always choose the directions of the orthogonal axes so that all the off-diagonal elements of the symmetrical second-order tensor, σ_{ij}, become zero. These directions are the *principal axes* of the stress tensor σ_{ij}.
Such a transformation can be achieved by the usual rotation transformation, i.e. from x_i to x'_i as defined by

$$\sigma'_{ij} = \frac{\partial r_k}{\partial x'_i}\frac{\partial r_l}{\partial x'_j}\sigma_{kl}. \tag{14}$$

In this rotated axes, the *principal axes*, the diagonal element of the stress tensor σ_{ij} at a given point $\mathbf{x}$ become the *principal stresses*, σ'_{11}, σ'_{22}, σ'_{33}. It is well known that changes to the directions of the orthogonal axes do not change the sum of the diagonal elements, i.e.

$$tr(\boldsymbol{\sigma}) = \sigma'_{11} + \sigma'_{22} + \sigma'_{33} = \sigma_{ii}. \tag{15}$$

Now the components of the force per unit area relative to the new principal axes acting across an element with normals (n'_1, n'_2, n'_3) are

$$\sigma'_{11}n'_1, \quad \sigma'_{22}n'_2, \quad \sigma'_{33}n'_3.$$

Recall that from our convention, σ'_{11} is the normal stress acting on the $2-3$ face in the 1-direction, and similarly for σ'_{22} and σ'_{33}. If $\sigma'_{11} > 0$, the surface is said to be in tension and if $\sigma'_{11} < 0$, it is said to be in compression.

***Corollary*:** Since σ'_{11}, σ'_{22}, σ'_{33} are the tensions (or compressions) on the respective planes, in general, the state of a fluid near a given point can be regarded as a superpositions of tensions in three orthogonal directions.

2 The stress tensor in a fluid at rest

A fluid by definition is a material which is unable to withstand any tendency by applied forces to deform it without change in volume. This definition has consequences for the form of the stress tensor.

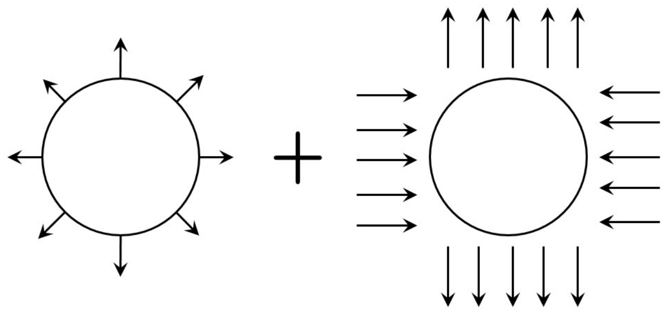
Consider surface forces exerted on a fluid within a sphere by the surrounding fluid. We choose the axes (locally) to coincide with the principal axes such that only the diagonal entries of $\boldsymbol{\sigma}$ are non-zero. We further decompose the stress tensor into a two tensors, the first one being isotropic or having a spherical symmetry, and the second one have a zero trace.

Mathematically, this can be written as

$$\boldsymbol{\sigma} = \begin{pmatrix} \frac{1}{3}\sigma_{ii} & 0 & 0 \\ 0 & \frac{1}{3}\sigma_{ii} & 0 \\ 0 & 0 & \frac{1}{3}\sigma_{ii} \end{pmatrix} + \begin{pmatrix} \sigma'_{11} - \frac{1}{3}\sigma_{ii} & 0 & 0 \\ 0 & \sigma'_{22} - \frac{1}{3}\sigma_{ii} & 0 \\ 0 & 0 & \sigma'_{33} - \frac{1}{3}\sigma_{ii} \end{pmatrix}$$

.

This decomposition is shown schematically in the figure below. In the first term, the force per unit area at a point where normal is $\mathbf{n}$ is $\dfrac{1}{3}\sigma_{ii}\mathbf{n}$ (usually negative). The second matrix on the RHS is the departure



of the shear stress from an isotropic view. Since $\sigma'_{11} + \sigma'_{22} + \sigma'_{33} = \sigma_{ii}$, the diagonal of the matrix has zero sum. Thus, we at least one normal stress to be compressive and one in tension.
Physically, the first contribution tends to compress or expand a sphere into a smaller or bigger sphere respectively, whereas the second contribution tends to deform a sphere into an ellipsoid. The deformation into an ellipsoid is actually a "flow", as a result of non-zero values of the force components, and is not compatible with the state of rest. Hence the only possibility is for the principal stress to be and same in all directions and equal to $\frac{1}{3}\sigma_{ii}$, i.e.

$$\sigma'_{11} = \sigma'_{22} = \sigma'_{33} = \frac{1}{3}\sigma_{ii}. \tag{16}$$

at all points in a fluid at rest.

Hence the stress tensor in a fluid at rest is everywhere isotropic.

Fluids at rest are normally in a state of compression and it is therefore convenient to write the stress tensor in a fluid at rest as

$$\sigma_{ij} = -p\delta_{ij} \tag{17}$$

where $p = -\frac{1}{3}\sigma_{ii}$ is called the static pressure or the hydrostatic pressure.
The compressive interpretation of the stress in a fluid at rest is consistent with the inability of simple fluids to sustain tensile stresses.

3 Static fluid with body forces

We now consider the force balance under the action of body forces and surface forces. This will tell us the constraints required on the body forces in order for there to be no motion.
For a fluid at rest, i.e., with no acceleration, sum of body and surface forces has to vanish.

$$\int \rho F_i(x_i, t) dV + \int \sigma_{ij} n_j dA = 0. \tag{18}$$

Using $\sigma_{ij} = -p\delta_{ij}$, we have

$$\int \rho F_i(x_i, t) dV - \int p n_i dA = 0. \tag{19}$$

Using divergence theorem,

$$\int \rho F_i(x_i, t) dV - \int \frac{\partial p}{\partial x_i} dV = 0, \qquad \Longrightarrow \qquad \frac{\partial p}{\partial x_i} = \rho F_i. \tag{20}$$

Let us now consider a static fluid in a gravitational field like in the case of atmosphere[1]. In this case $F_i = g_i$. We therefore get

$$\frac{\partial p}{\partial x_i} = \rho g_i \qquad \text{i.e.} \qquad \boldsymbol{\nabla} p = \rho \mathbf{g}. \tag{21}$$

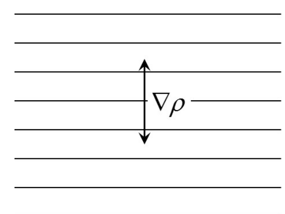
Taking curl on both sides, we have

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} p = \boldsymbol{\nabla} \times \rho \mathbf{g}. \tag{22}$$

Since LHS = 0, we have

$$\epsilon_{ijk} \frac{\partial}{\partial x_j} (\rho g_i) = 0 \qquad \Longrightarrow \qquad \epsilon_{ijk} \frac{\partial \rho}{\partial x_i} g_i = 0. \tag{23}$$

Since $\boldsymbol{\nabla}\rho$ is normal to constant density surfaces as shown in the figure, the above relation requires that constant density surfaces must be perpendicular to gravity. Hence, it is possible for a fluid to remain at rest and with varying density if density varies only in a direction perpendicular to gravity. If the density lines are tilted, then $\epsilon_{ijk} \dfrac{\partial \rho}{\partial x_i} g_i \neq 0$ and fluid motion has to occur.



Consider the simple case of a container half filled with water with gravity acting vertically downwards. In this case, density changes only at one location, at the interface separating water and air. When this interface is horizontal, the fluid is at rest at all times. But if we instantaneously tilt the container, then for a brief moment, the interface is not perpendicular to gravity. This results in a motion in the container. Assuming that the container is not tilted to such an extent that the fluid spills, this motion tends to make the fluid oscillate eventually reaching a state of rest. In this rest state, again the interface becomes horizontal, exactly perpendicular to gravity.

Returning to our previous relation, $\boldsymbol{\nabla} p = \rho \mathbf{g}$, we have

$$\frac{dp}{dz} = -g\rho(z), \tag{24}$$

where $\mathbf{g} = -g\mathbf{e}_z$ and $\rho(z)$ is taken to vary with height. When the density of the fluid is constant, we can readily integrate this equation to obtain

$$p = p_0 - \rho g z, \tag{25}$$

where p_0 is the pressure at the ground level, $z = 0$. In the case earth's atmosphere, the ρ decreases with decrease of pressure owing to the compressibility of air. Though temperature varies with altitude, for simplicity, we can assume this to be constant. In other words, the main cause of density is not temperature but pressure. In this approximation, we can assume

$$\frac{p}{\rho} = \text{constant}, = gH(\text{say}), \tag{26}$$

[1] In reality, the atmosphere is never in a static state. It is continuously in motion as result of energy supplied by the sun causing convection currents and due to rotation of earth leading to coriolis forces. The motion in general is too complicated and strictly speaking, a static fluid approximation should not work. But the atmosphere is vertically stratified making the motions 2D-like and motion is either localized (like in the case of hurricanes/cyclones) or is confined to narrow regions like the gulf stream. Of course, by localized, we are comparing the scale of the phenomena to the size of the earth itself. In this very large scale, treating the atmosphere as a static fluid is a good first approximation. Such an approximation misses out all the finer details such as weather patterns, etc., but is a good indicator of how certain quantities like temperature, pressure, etc. vary with altitude. Given that we are treating the entire atmosphere as a whole and not a localized phenomenon, it is reasonable to expect any predictions obtained with such an analysis to be valid all over the earth.

corresponding to Boyle's law for an isothermal ideal gas. Using this relation, we have

$$\frac{dp}{dz} = -\frac{p}{H}, \qquad \Longrightarrow \qquad p = p_0 e^{-z/H}. \tag{27}$$

The height H is the 'scale height' at which both p and ρ decrease by a factor of e from their corresponding ground level values. For air at $0°C$, $H \approx 8.0kms$.