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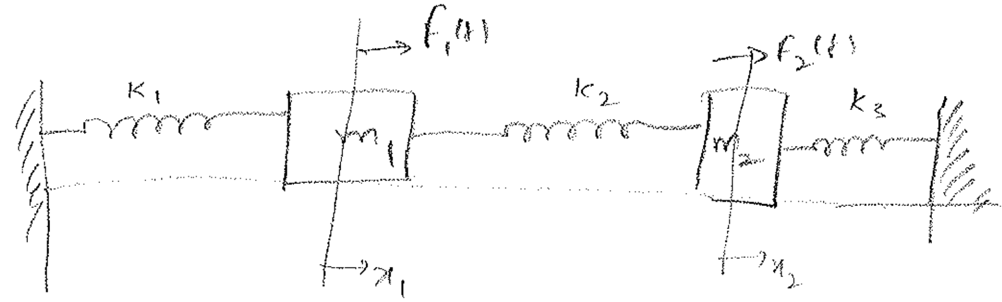
Systems of First Order Linear Equations



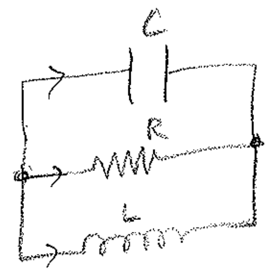
$$m \frac{d^2x_1}{dt^2} = -k_1x_1 + k_2(x_2-x_1) + f_1(t)$$

$$m \frac{d^2x_2}{dt^2} = -k_3x_2 - k_2(x_2-x_1) + f_2(t)$$

Electrical Circuit:

V: Voltage drop across C

I: Current through L



$$\frac{dI}{dt} = \frac{V}{L}$$

$$\frac{dV}{dt} = -\frac{I}{C} - \frac{V}{RC}$$

$V = IR$

$C\frac{dV}{dt} = I$

$L\frac{dI}{dt} = V$

Equations of Higher order can be reduced to systems of first order equations.

Ex. $y'' + y' + y = 0$

Let $y_1 = y$

$y_2 = y'$

$y_1' = y_2$

$y_2' = -y_2 - y_1$

Ex: $mx'' + \gamma x' + kx = F(t)$

Let $x_1 = x$
$x_2 = x'$

$\therefore x_1' = x' = x_2$

$m x_2' + \gamma x_2 + k x_1 = F(t)$

$$\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= -\left(\frac{k}{m}\right)x_1 - \left(\frac{\gamma}{m}\right)x_2 + \frac{F(t)}{m} \end{aligned}}$$

Ex: Transform into a single equation:

$y_1' = -y_1 + y_2$; $y_1(0) = 1$

$y_2' = 2y_1 - 3y_2$; $y_2(0) = 2$

$\Rightarrow y_2 = y_1' + y_1$

$\Rightarrow y_1'' + y_1' = 2y_1 - 3(y_1' + y_1)$

$\Rightarrow y_1'' + y_1' - 2y_1 + 3y_1' + 3y_1 = 0$

$\Rightarrow \boxed{y_1'' + 4y_1' + y_1 = 0}$

$y_1(0) = 1$

$y_1'(0) = 1$

$y_1'(0) + y_1(0) = 2$

$\Rightarrow y_1'(0) = 2 - 1$

$= 1$

Review of Matrices:

conjugate

$$A = \begin{bmatrix} 1+i & 2 \\ 3+2i & 4-5i \end{bmatrix} \qquad \bar{A} = \begin{bmatrix} 1-i & 2 \\ 3-2i & 4+5i \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1+i & 3+2i \\ 2 & 4-5i \end{bmatrix} \qquad A^* = \begin{bmatrix} 1-i & 3-2i \\ 2 & 4+5i \end{bmatrix} = (\bar{A})^T = \overline{(A^T)}$$

Transpose

Adjoint

Properties of Matrices:-

1. Equality : $A = B \Rightarrow a_{ij} = b_{ij}$
2. Zero : $A = 0 \Rightarrow a_{ij} = 0$
3. Addition : $A + B = a_{ij} + b_{ij} = B + A$
4. Multiplication: $AB \neq BC$ (in general)

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 0 & 1 \\ -1 & 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 1 & 3 \\ 2 & -1 & 2 \\ -2 & 3 & 4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3\times(-1) + 2\times 2 + (-1)\times(-2) & 3-2-3 & 9+4-4 \\ -1+0-2 & 1+0+3 & 3+0+4 \\ 1+4-6 & -1-2+9 & -3+4+12 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 & 9 \\ -3 & 4 & 7 \\ -1 & 6 & 13 \end{bmatrix}$$

5. Multiplication of Vectors:

$\vec{x} = (x)_{n\times 1}$

Dot Product: $\vec{x}\cdot\vec{y} = x^T y$

$$= \sum_{i=1}^{n} x_i y_i$$

Inner Product: $(\vec{x},\vec{y}) = \sum_{i=1}^{n} x_i \bar{y}_i$

$$\vec{x} = \begin{pmatrix} i \\ 2 \\ 1+i \end{pmatrix} \qquad \vec{y} = \begin{pmatrix} 2-i \\ i \\ 3 \end{pmatrix}$$

$$\vec{x}\cdot\vec{y} = \vec{x}^T\vec{y} = \begin{bmatrix} i & 2 & 1+i \end{bmatrix}_{1\times 3} \begin{bmatrix} 2-i \\ i \\ 3 \end{bmatrix}_{3\times 1}$$

$$= (i)(2-i) + 2\times i + (1+i)\times 3$$

$$= 4+3i$$

$$(x,y) = \begin{bmatrix} i & 2 & 1+i \end{bmatrix} \begin{bmatrix} 2+i \\ -i \\ 3 \end{bmatrix} = 2+7i$$

6. Identity $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$AI = IA = A$$

9. Inverse: A is nonsingular or invertible if we can find a B such that

$$AB = I$$

$$B = A^{-1}$$

Gaussian Elimination:

Ex: $A = \begin{bmatrix} 1 & -1 & -1 \\ 3 & -1 & 2 \\ 2 & 2 & 3 \end{bmatrix}$

$A|I = \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$

↓ Augmented matrix

(a) $R_2 \longrightarrow R_2 - 3R_1$
$R_3 \longrightarrow R_3 - 2R_1$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 5 & -3 & 1 & 0 \\ 0 & 4 & 5 & -2 & 0 & 1 \end{array}\right]$$

(b) $R_2 \to R_2/2$: $\left[\begin{array}{ccc|ccc} 1 & -1 & [illegible] & 1 & 0 & 0 \\ 0 & 1 & 5/2 & -3/2 & 1/2 & 0 \\ 0 & 4 & 5 & -2 & 0 & 1 \end{array}\right]$

(c) $R_3 \to R_3 - 4R_2$; $R_1 \to R_1 + R_2$

$$\begin{bmatrix} 1 & \cancel{0} & \cancel{0} \end{bmatrix}$$

$$\left[\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I} \;\middle|\; \underbrace{\begin{matrix} \frac{7}{10} & \frac{-1}{10} & \frac{3}{10} \\ \frac{1}{2} & -1/2 & 1/2 \\ -4/5 & 2/5 & -1/5 \end{matrix}}_{A^{-1}}\right]$$

Ex: $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

$A|I = \left[\begin{matrix} 2 & 3 \\ 4 & 5 \end{matrix} \;\middle|\; \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right]$

(a) $R_1 \to R_1/2$: $\left[\begin{matrix} 1 & 3/2 \\ 4 & 5 \end{matrix} \;\middle|\; \begin{matrix} 1/2 & 0 \\ 0 & 1 \end{matrix}\right]$

(b) $R_2 \to R_2 - 4R_1$

$$\left[\begin{array}{cc|cc} 1 & 3/2 & 1/2 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right]$$

(c) $R_3 \to R_3/(-1)$

$$\left[\begin{array}{cc|cc} 1 & 3/2 & 1/2 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

(d) $R_1 \to R_1 - \frac{3}{2} R_2$

$$\left[\begin{array}{cc|cc} 1 & 0 & -5/2 & \frac{3}{2} \\ 0 & 1 & 2 & -1 \end{array}\right]$$

$$\underbrace{\begin{matrix} -5/2 & \frac{3}{2} \\ 2 & -1 \end{matrix}}_{A^{-1}}$$

$$|A| = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -2$$

$$\Rightarrow A^{-1} = \frac{1}{|A|}\begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -5/2 & 3/2 \\ 2 & -1 \end{bmatrix}$$

Derivative and Integral of Matrices:

$$A = \begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) & \cdots \\ & & & \\ a_{n1}(t) & \cdots & & a_{nn}(t) \end{bmatrix}$$

$$\frac{dA}{dt} = \frac{d}{dt}(a_{ij})$$

$$\int_a^b A\,dt = \left[\int_a^b a_{ij}\,dt \right]$$

Ex: $A(t) = \begin{bmatrix} \sin t & t \\ 1 & \cos t \end{bmatrix}$

$$\int_0^{\pi} \sin t\,dt = -\cos t \Big]_0^{\pi} = -[-1-1] = 2$$

$$\frac{dA}{dt} = \begin{bmatrix} \cos t & 1 \\ 0 & -\sin t \end{bmatrix}$$

$$\int_0^{\pi} A(t)\,dt = \begin{bmatrix} 2 & \pi^2/2 \\ \pi & 0 \end{bmatrix}$$

$\frac{d}{dt}$ Other Properties:

(a) C : constant matrix $\Rightarrow$ $\frac{d}{dt}(CA) = C\frac{dA}{dt}$

(b) $\frac{d}{dt}(A+B) = \frac{dA}{dt} + \frac{dB}{dt}$; (c) $\frac{d}{dt}(AB) = A\frac{dB}{dt} + \frac{dA}{dt}B$.

Ex: Verify that $\vec{x} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} e^{2t}$ satisfies

$$\vec{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \vec{x}$$

$$\vec{x}' = \begin{bmatrix} 8 \\ 4 \end{bmatrix} e^{2t}$$

$$\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} e^{2t} = \begin{bmatrix} 12-4 \\ 8-4 \end{bmatrix} e^{2t} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} e^{2t}$$

Eigenvalues / Eigenvectors

$AX = B$

If $B = 0$; then we have $\boxed{AX = 0}$

↳ Homogeneous system of equations.

If $|A| \neq 0$, then $AX = B$ has a unique solution

$X = A^{-1}B$

$\therefore B = 0$, the only unique solution is then $X = 0$

→ If $|A| = 0$: (Singular matrix)

→ The homogeneous eqn $AX = 0$ has infinitely many solutions

→ $AX = B$ has no solution unless B satisfies the condition $(B, y) = 0$ for all vectors $A^* y = 0$ where A^* is the adjoint of A.

If this condition is met, then $AX = B$ has infinitely many solutions.

General solution.

$$x = \underbrace{x^{(0)}}_{\text{particular solution of } AX = B} + \underbrace{\xi}_{\text{General solution of } AX = 0}$$

Ex:

$$x_1 - 2x_2 = 3$$
$$3x_1 - 4x_2 = 5$$

$$A|B = \left[\begin{array}{cc|c} 1 & -2 & 3 \\ 3 & -4 & 5 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$

$$\Rightarrow A|B = \left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 2 & -4 \end{array}\right]$$

$$\begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$x_1 - 2x_2 = 3$$
$$2x_2 = -4 \implies x_2 = -2$$
$$x_1 = 3 + 2x_2 = -1$$

Ex:
$$x_1 - 2x_2 + 3x_3 = 7$$
$$-x_1 + x_2 - 2x_3 = -5$$
$$2x_1 - x_2 - x_3 = 4$$

After row operations

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 7 \\ 0 & \underline{1} & -1 & -2 \\ 0 & 0 & \underline{1} & 1 \end{array}\right] \qquad \vec{x} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Ex:
$$x_1 - 2x_2 + 3x_3 = b_1$$
$$-x_1 + x_2 - 2x_3 = b_2$$
$$2x_1 - x_2 + 3x_3 = b_3$$

$$A|B = \left[\begin{array}{ccc|c} 1 & -2 & 3 & b_1 \\ -1 & 1 & -2 & b_2 \\ 2 & -1 & 3 & b_3 \end{array}\right] \quad \text{———} \quad (*)$$

$R_2 \to R_2 + R_1 \quad ; \quad R_3 \to R_3 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & b_1 \\ 0 & -1 & 1 & b_2 + b_1 \\ 0 & 3 & -3 & b_3 - 2b_1 \end{array}\right]$$

$R_3 \to R_3 + \cancel{B}\, 3R_2$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & b_1 \\ 0 & -1 & 1 & b_2 + b_1 \\ 0 & 0 & 0 & b_3 - 2b_1 + 3(b_2 + b_1) \end{array}\right]$$

$$b_1 + 3b_2 + b_3 = 0$$

This equation has no solution unless

$$b_1 + 3b_2 + b_3 = 0$$

LINEAR INDEPENDENCE: Vectors are linearly dependent if $\exists$ a set of constants C's at least one of which is non-zero.

$$\vec{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} ; \quad \vec{x}^{(2)} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} ; \quad \vec{x}^{(3)} = \begin{pmatrix} -4 \\ 1 \\ -11 \end{pmatrix}$$

~~Obj~~ (b) When $c_1 \vec{x}^{(1)} + c_2 \vec{x}^{(2)} + c_3 \vec{x}^{(3)} = 0$; if we can find c_1, c_2, c_3, then $\vec{x}$ are linearly dependent.

$$\begin{bmatrix} 1 & 2 & -4 \\ 2 & 1 & 1 \\ -1 & 3 & 11 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 2 & 1 & 1 & 0 \\ -1 & 3 & 11 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$c_1 + 2c_2 - 4c_3 = 0$$
$$c_2 - 3c_3 = 0$$

c_3 can be taken to be non-zero.

$$c_2 = 3c_3$$
$$c_1 = -2c_3$$

If $c_3 = -1$; $\boxed{2\vec{x}^{(1)} - 3\vec{x}^{(2)} - \vec{x}^{(3)} = 0}$

Linearly dependent.

$$|\vec{X}| = \begin{vmatrix} 1 & 2 & -4 \\ 2 & 1 & 1 \\ -1 & 3 & -11 \end{vmatrix} = 0$$

Eigenvalues / Eigenvectors

$$\cancel{A = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}}$$

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

$$A X = \lambda X$$

$$\Rightarrow (A - \lambda I) X = 0$$

$$\begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} = 0 \Rightarrow (1-\lambda)^2 - 4 = 0$$

$$\Rightarrow 1 + \lambda^2 - 2\lambda - 4 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda - 3 = 0$$

$$\Rightarrow \lambda_1 = 3 ; \quad \lambda_2 = -1$$

Eigenvectors: $\lambda_1 = 3$:

$$\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

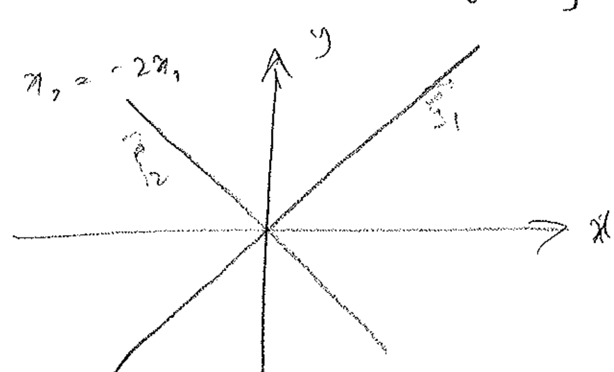
$$\Rightarrow \begin{array}{l} -2x_1 + x_2 = 0 \\ 4x_1 - 2x_2 = 0 \end{array} \qquad \begin{array}{l} \Rightarrow x_2 = 2x_1 \\ \Rightarrow x_2 = 2x_1 \end{array}$$

$$\vec{\xi}^{(1)} = c \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\lambda_2 = -1$:

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \vec{\xi}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$2x_1 + x_2 = 0 \quad \Rightarrow x_2 = -2x_1$$



Homogeneous Equations:

Ex. $\vec{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{x}$ $\qquad A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

$X' = AX$

Let $\vec{x} = \vec{\xi} e^{rt}$

$X' = \vec{\xi} r e^{rt}$

$r\vec{\xi} e^{rt} = A \vec{\xi} e^{rt}$

$\Rightarrow A\vec{\xi} - r\vec{\xi} = 0$

$\Rightarrow (A - rI)\vec{\xi} = 0$

$$\begin{bmatrix} 1-r & 1 \\ 4 & 1-r \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$r_1 = 3 \; ; \quad \vec{\xi}_1^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$r_2 = -1 \; ; \quad \vec{\xi}_2^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\therefore \vec{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} \; ; \quad \vec{x}^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$

$W[\vec{x}^{(1)}, \vec{x}^{(2)}](t) = \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -4e^{2t}$

$$\vec{x} = c_1 \vec{x}^{(1)}(t) + c_2 \vec{x}^{(2)}(t)$$

$$= c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

$y_1 = y$
$y_2 = y'$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$y_1' = y_1 + y_2$
$y_2' = 4y_1 + y_2$

$y_1'' = y_1' + y_2'$
$= y_1' + (4y_1 + y_2)$
$\Rightarrow y'' = y' + 4y + y'$
$\Rightarrow \boxed{y'' - 2y' - 4y = 0}$